Research Article
On a New Efficient Steffensen-Like Iterative Class by Applying a Suitable Self-Accelerator Parameter

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Received 28 August 2013; Accepted 5 December 2013; Published 3 March 2014

Academic Editors: L. Acedo and P. K. Papadopoulos

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It is attempted to present an efficient and free derivative class of Steffensen-like methods for solving nonlinear equations. To this end, firstly, we construct an optimal eighth-order three-step uniparameter without memory of iterative methods. Then the self-accelerator parameter is estimated using Newton’s interpolation in such a way that it improves its convergence order from 8 to 12 without any extra function evaluation. Therefore, its efficiency index is increased from $8^{1/4}$ to $12^{1/4}$ which is the main feature of this class. To show applicability of the proposed methods, some numerical illustrations are presented.

1. Introduction

Kung and Traub are pioneers in constructing optimal general multistep methods without memory. They devised two general $n$-step methods based on interpolation. Moreover, they conjectured any $n$-step methods without memory using $n + 1$ function evaluations may reach the convergence order at most $2^n$ [1]. Accordingly, many authors during the last years, specially the four past years, are attempted to construct iterative methods without memory which support this conjecture with optimal order [1–22].

Although construction of optimal methods without memory is still an active field, however, much attention has not been paid for developing methods with memory. Based on our best knowledge, Traub in his book introduces the first method with memory. The main feature of these methods is that they improve convergence order as well as efficiency index without any new function evaluations. Indeed, Traub changed Steffensen’s method slightly as follows (see [18, pp. 185-187]):

$$ N_1(x) = f(x_n) + (x - x_n)f[x_n, w_n], $$

$$ y_{n+1} = -\frac{1}{N'_1(x_n)}, $$

$$ w_{n+1} = x_{n+1} + y_{n+1}f(x_{n+1}). $$

The parameter $y_n$ is called self-accelerator and method (1) has convergence order 2.41. It is still possible to increase the convergence order using better self-accelerator parameter based on better Newton interpolation. Free-derivative can be considered as another virtue of (1).

In this work, motivated by Traub’s work (1), we construct a new class of methods with memory. To this end, we first try to devise a new optimal free-derivative three-step without memory of iterative methods with eight order of convergence and using merely four function evaluations per step. In other words, our first step is the same as Traub’s method (1). The second and third steps use combination Steffensen-like methods and weight function idea so that we achieve an optimal class of methods without memory. Finally, we apply a self-accelerator parameter to extend it to with memory case. We remember two main properties of this work: increasing
efficiency index without any new functional evaluations and nonusing derivatives of a given function.

We use the symbols $\to$, $O$, and $\sim$ according to the following conventions [18]: if $\lim_{x \to \infty} g(x_0) = C$, we write $g(x_0) \to C \text{ or } g \to C$. If $\lim_{x \to a} g(x) = C$, we write $g(x) \to C \text{ or } g \to C$. If $\frac{f}{g} \to C$, where $C$ is a nonzero constant, we write $f \sim C g$. Let $f(x)$ be a function defined on an interval $I$, where $I$ is the smallest interval containing $k + 1$ distinct nodes $x_1, x_2, \ldots, x_k$. The divided difference $f[x_0, x_1, \ldots, x_k]$ with $k$th-order is defined as follows: $f[x_0] = f(x_0)

\begin{align*}
\frac{f[x_0, x_1, \ldots, x_k]}{x_k - x_0} = f[x_1, x_2, \ldots, x_k] - f[x_0, x_1, \ldots, x_k-1].
\end{align*}

Moreover, we recall the definition of efficiency index (EI) as $E = p^{1/n}$, where $p$ is the order of convergence and $n$ is the total number of function evaluations per iteration.

This work is organized as follows: Section 2 present construction and error analysis of optimal three-step class of without memory class. Section 3 is devoted to with memory extension. Numerical results are demonstrated in Section 5. We sum up this work in Section 5.

2. Derivative-Free Three-Point Method

This section concerns construction a new class of three-step free-derivative methods without memory for solving nonlinear equations. In the next section, it is extended to its with memory cases. To this end, let us first start with the following three-step Steffensen-type [23] initiative:

\begin{equation}
y_n = x_n - \frac{f(x_n)}{f[x_n, w_n]}, \quad w_n = x_n + y_n f(x_n), \quad 0 \neq y_n \in R, \ n = 0, 1, 2, \ldots,
\end{equation}

\begin{equation}
z_n = y_n - \frac{f(y_n)}{f[y_n, w_n]}, \quad x_{n+1} = z_n - \frac{f(z_n)}{f[z_n, w_n]}.
\end{equation}

This scheme is not optimal in the sense of Kung and Traub [1] as it is of fourth-order convergence using four functions evaluations per iteration. In other words, its error equation has the form

\begin{equation}
e_{n+1} = (1 + f(\alpha)) \gamma^3 e_n^4 + O(e_n^5).
\end{equation}

If the initial approximation $x_0$ is sufficiently close to the zero $\alpha$ of a function $f$, then the convergence order of the family (5) is eight.

Proof. Let $e_n = x_n - \alpha, e_{y_n} = y_n - \alpha, e_{w_n} = z_n - \alpha, e_{w_n} = e_n + y_n f(x_n)$, and $e_n^{(h)}(\alpha) = f^{(h)}(\alpha)/n! f'(\alpha), n = 1, 2, \ldots$ Using Taylor’s expansion and taking into account $f(\alpha) = 0$, we have

\begin{equation}
f(x_n) = f'(\alpha) \left[ e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + O(e_n^7) \right],
\end{equation}

where $c_2 = 1/2, c_3 = 1/6, c_4 = 1/24, c_5 = 1/120, c_6 = 1/720.$
\[ f [x_n, w_n] = f'(\alpha) \left[ 1 + c_2 \left( 2 + \gamma f'(\alpha) \right) e_n \right] + (c_2^2 \gamma f'(\alpha) + c_3 \left( 3 + \gamma f'(\alpha) \right) (3 + \gamma f'(\alpha))) e_n^2 + \cdots + O(e_n^8). \]

(7)

Substituting these into the first step of (5) gives

\[
e_n = e_n - \frac{f(x_n)}{f[x_n, w_n]} = c_2 \left( 1 + \gamma f'(\alpha) \right) e_n + (c_3 \left( 1 + \gamma f'(\alpha) \right) (2 + \gamma f'(\alpha)) - c_2^2 \left( 2 + \gamma f'(\alpha) \right) (2 + \gamma f'(\alpha))) e_n^3 + (c_4 \left( 1 + \gamma f'(\alpha) \right) (3 + \gamma f'(\alpha) (3 + \gamma f'(\alpha)) + c_5 \left( 4 + \gamma f'(\alpha) (5 + \gamma f'(\alpha) (3 + \gamma f'(\alpha))) - c_6 c_3 \left( 7 + \gamma f'(\alpha) (10 + \gamma f'(\alpha) \times (7 + 2 \gamma f'(\alpha))) \right) e_n^4 + \cdots + O(e_n^8). \]

(8)

Set \( t_n = f(y_n)/f(x_n), u_n = f(w_n)/f(x_n), \) and \( H_{i,j} = \partial H(t_n, u_n)/\partial t_i \partial u_j, \) and expanding \( H(t_n, u_n) \) about \((0, 0)\), yields

\[
H(t_n, u_n) = H_{0,0} + H_{1,0} t_n + H_{0,1} u_n + \frac{1}{2} \left( H_{2,0} t_n^2 + 2 H_{1,1} t_n u_n + H_{0,2} u_n^2 \right) + \cdots. \]

(9)

Substituting (9) into (5), we can assert that

\[
e_n = y_n - H(t_n, u_n) \frac{f(y_n)}{f[y_n, w_n]} = k_1 e_n^2 + k_2 e_n^3 + k_4 e_n^4 + k_5 e_n^5 + k_6 e_n^6 + k_7 e_n^7 + O(e_n^8), \]

(10)

where

\[
k_1 = -\frac{1}{6} \left( 2 (1 + y_n f'(\alpha)) \times \left( -6 + 6H_{0,0} + 3H_{0,2}(1 + y_n f'(\alpha))^2 + H_{0,3}(1 + y_n f'(\alpha))^3 + 6 (H_{0,1} + H_{0,1} y_n f'(\alpha)) \right) \right) \]

(11)

\[
k_2 = \left( -2c_2^2 + 2c_3 - 2c_2 y_n f'(\alpha) + 3c_3 y_n f'(\alpha) - c_2^2 y_n f'(\alpha) + c_3 y_n f'(\alpha)^2 - \frac{1}{2} c_2^2 \left( 1 + y_n f'(\alpha) \right) \times \left( 2H_{1,1} y_n f'(\alpha)^2 + H_{1,2} y_n f'(\alpha)^3 \right) + 2 \left( H_{1,0} + H_{1,0} y_n f'(\alpha) \right) \gamma_n f'(\alpha) + y_n f'(\alpha) (2 + y_n f'(\alpha)) \times (2H_{0,1} + (1 + y_n f'(\alpha)) \times (2H_{0,2} + H_{0,3} + H_{0,3} y_n f'(\alpha)) \right) \right) + \frac{1}{6} \left( c_3 \left( 1 + y_n f'(\alpha) \right) (2 + y_n f'(\alpha)) + c_5 \left( 3 + 2y_n f'(\alpha) (2 + y_n f'(\alpha)) \right) \times (6H_{0,0} + (1 + y_n f'(\alpha)) \times (6H_{0,1} + (1 + y_n f'(\alpha)) \times (3H_{0,2} + H_{0,3} + H_{0,3} y_n f'(\alpha)) \right) \right) e_n^4. \]

(12)

To achieve the fourth-order methods in the first two steps of (5), we attempt to vanish the coefficients of \( e_n^2, e_n^3 \) in (10). For this purpose, it suffices to set

\[
H_{0,0} = H_{1,0} = 1, \quad H_{0,1} = H_{0,3} = H_{0,2} = H_{1,1} = H_{1,2} = 0. \]

(13)

Define \( s_n = f(z_n)/f(y_n), v_n = f(z_n)/f(x_n), G_{i,j} = \partial G(t_n, s_n)/\partial t_i \partial s_j, \) and \( W_{i,j} = (\partial W(y_n, s_n))/\partial y_i \partial s_j, \) \( i, j = 1, 2, \ldots \). Taylor's series for \( G(t_n, s_n), W(v_n, s_n) \) about \((0, 0)\) are

\[
G(t_n, s_n) = G_{0,0} + G_{1,0} t_n + G_{0,1} s_n + \frac{1}{2} \left( G_{2,0} t_n^2 + 2G_{1,1} t_n s_n + G_{0,2} s_n^2 \right) + \cdots, \]

(14)

\[
W(v_n, s_n) = W_{0,0} + W_{1,0} v_n + W_{0,1} s_n + \frac{1}{2} \left( W_{2,0} v_n^2 + 2W_{1,1} v_n s_n + W_{0,2} s_n^2 \right) + \cdots. \]
Under the conditions stated above (13) and substituting these Taylor’s series into the third step of (5), we obtain

$$e_{n+1} = z_n - G(t_n, s_n) \cdot W(v_n, s_n) \cdot \frac{f(z_n)}{f(z_n)} = R_4 e_n^4 + R_5 e_n^5 + R_6 e_n^6 + R_7 e_n^7 + O(e_n^8),$$

where

$$R_4 = \frac{1}{2} c_2 (1 + G_{0,0} W_{0,0}) \left(1 + \gamma_n f'(\alpha)\right)^2$$

$$\times \left(2c_3 + c_2^2 \left( (H_{2,0} + \gamma_n f'(\alpha)) H_{2,0} + H_{2,1} + \gamma_n f'(\alpha))^2 \right) - 2 \left(3 + \gamma_n f'(\alpha))^2 \right) \right),$$

and to get $R_5$ = 0, it is sufficient to put $G_{1,0} = 1$.

In the same manner, we can see that the coefficient of $e_n^6$ is

$$R_6 = -\frac{1}{4} c_2 (1 + \gamma_n f'(\alpha))^3$$

To vanish the coefficient of $e_n^6$, set $G_{0,1} = 1$, $W_{0,1} = H_{2,0} = H_{2,1} = G_{2,0} = 0$, and we conclude similarly that

$$R_7 = \frac{1}{6} c_2 (1 + \gamma_n f'(\alpha))^4$$

$$\times \left[ (-c_3 + c_2^2 (3 + \gamma_n f'(\alpha))) \times (-6c_3 (-2 + G_{1,1} + W_{1,0}) + c_2^2 (-24 + 18G_{1,1} + G_{3,0} - H_{3,0}) + 18W_{1,0} + \gamma_n f'(\alpha) \times (-6 + 6G_{1,1} + G_{3,0} - H_{3,0} + 6W_{1,0})) \right].$$

As in the above cases, choosing $G_{1,1} = 2$, $W_{1,0} = 0$, and $G_{3,0} = H_{3,0} - 6 - 6/(1 + \gamma_n f(x_n, w_n))$ gives $R_7 = 0$.

On account of the above conditions, we see that

$$e_{n+1} = -\frac{1}{6} c_2 (1 + \gamma_n f'(\alpha))^4$$

$$\times \left[ (-c_3 + c_2^2 (3 + \gamma_n f'(\alpha))) \times (-6c_3 (-2 + G_{0,2} + W_{0,2}) - 3c_2^2 (-22 + 6G_{0,2} + G_{2,1} + 6W_{0,2}) \times (-6 + 2G_{0,2} + G_{2,1} + 2W_{0,2})) + c_2^4 \left( -H_{3,0} (1 + \gamma_n f'(\alpha))^2 \right.$$

$$+ 3G_{2,1} (1 + \gamma_n f'(\alpha) (3 + \gamma_n f'(\alpha)) + 3G_{0,2} (3 + \gamma_n f'(\alpha))^2 + 3 \left( W_{0,2} (3 + \gamma_n f'(\alpha))^2 \right.$$

$$- 2 \left(13 + \gamma_n f'(\alpha) (7 + \gamma_n f'(\alpha)) \right)) \right] \times e_n^8 + O(e_n^9).$$

Some simple but efficient weight functions satisfying the conditions of Theorem 1 are

$$H_1(t_n, u_n) = 1 + t_n,$$

$$H_2(t_n, u_n) = \frac{3 + t_n + \sin \left( \frac{2t_n}{(-2/9) t_n^2 + 3} \right)}{t_n},$$

$$G_1(t_n, s_n) = 1 + t_n + s_n + 2t_n s_n + (-1 - \phi_n) t_n^3,$$

$$G_2(t_n, s_n) = \frac{1}{(1 + \phi_n)} \frac{(1 + t_n + s_n + 2t_n s_n) + t_n^2}{1/(1 + \phi_n) + t_n^2},$$

where $\phi_n = 1/(1 + \gamma_n f(x_n, w_n))$. 

\[ \square \]
Consider
\[
W_1(s_n, v_n) = 1 + s_n^2 + v_n^2,
\]
\[
W_2(s_n, v_n) = 1 + s_n^2 \frac{v_n^2}{v_n^2 + 1}.
\] (22)

In the next section we introduce a new three-step method with memory. The efficiency index of the optimal class (5) is 
\[ E = 8^{1/4}, \] we extent proposed class (5) to its with memory version, using an accelerator parameter, which improves the efficiency index to 12^{1/4}.

3. A New Method with Memory

Looking at the error equation (20) of the class (5) reveals that we can increase the convergence order of this class if the crucial element \( 1 + y_n f'(\alpha) \) vanishes. This can be done if \( y_n = -1/f'(\alpha) \). Although this is true theoretically, it is not possible practically since \( \alpha \) is unknown. Fortunately, during the iterative process (5), finer approximations to \( \alpha \) are generated by the sequence \( \{x_n\} \), and therefore we try to obtain a good approximate for \( f'(\alpha) \). Each iteration, \( x_n, w_n, y_n, z_n, \) and \( x_{n+1} \), are accessible, except at the initial step. Hence, we can interpolate \( f'(\alpha) \) using these nodes. It is natural that we estimate the best interpolator, and as a result we consider Newton interpolating polynomial as follows:

\[
N_4'(x_n) = \left[ \frac{d}{dt}N_4(t; x_{n-1}, w_{n-1}, y_{n-1}, z_{n-1}, x_n) \right]_{t=x_n} \\
= \left[ \frac{d}{dt} \left( f(x_n) + f(x_{n-1}, z_{n-1}, y_{n-1}, x_{n-1}, x_n) (t - x_n) \\
+ f(x_{n-1}, z_{n-1}, y_{n-1}, x_{n-1}, x_n) (t - z_{n-1}) \\
+ f(x_{n-1}, z_{n-1}, y_{n-1}, x_{n-1}, x_{n-1}) (t - y_{n-1}) \\
\times (t - z_{n-1}) (t - y_{n-1}) \\
+ f(x_{n-1}, z_{n-1}, y_{n-1}, x_{n-1}, w_{n-1}, x_{n-1}) (t - x_n) \\
\times (t - z_{n-1}) (t - y_{n-1}) (t - x_{n-1}) \right) \right]_{t=x_n} \\
= f(x_n, z_{n-1}) + f(x_{n-1}, z_{n-1}, y_{n-1}) (x_n - z_{n-1}) \\
+ f(x_{n-1}, z_{n-1}, y_{n-1}, x_{n-1}) (x_n - y_{n-1}) \\
+ f(x_{n-1}, z_{n-1}, y_{n-1}, x_{n-1}, w_{n-1}) (x_n - y_{n-1}) \\
\times (x_n - z_{n-1}) (x_n - y_{n-1}) (x_n - x_{n-1}). 
\] (23)

In the next theorem we prove that if \( y_n = -1/N_4'(x_n) \), then convergence order of the proposed class in Theorem 1 improves to 12.

**Theorem 2.** Suppose that \( x_0 \) is an approximation to a simple zero \( \alpha \) of \( f \), then the R-order of convergence of the three-point method (5) is at least 12.

**Proof.** Suppose that an iterative method generates a sequence \( \{x_n\} \) approximating a zero \( \alpha \) of \( f \) and \( C_1 \) tends to the asymptotic error constant \( D_2 \) when \( n \to \infty \), so

\[
e_{n+1} \sim D_n e_n^r, \quad e_n = x_n - \alpha.
\] (24)

Assume that the iterative sequences \( \{w_n\}, \{y_n\}, \) and \( \{z_n\} \) have the R-order \( p, q, \) and \( s \), respectively; that is,

\[
e_{n,w} \sim A_n w^p = A_n(D_{n-1} e_{n-1})^p = A_n D_{n-1} e_{n-1}^p,
\]
\[
e_{n,y} \sim B_n y^q = B_n(D_{n-1} e_{n-1})^q = B_n D_{n-1} e_{n-1}^q,
\]
\[
e_{n,z} \sim C_n z^s = C_n(D_{n-1} e_{n-1})^s = C_n D_{n-1} e_{n-1}^s,
\]
\[
e_{n+1} \sim D_n(D_{n-1} e_{n-1})^r = D_n D_{n-1} e_{n-1}^r.
\] (25)

On the other hand, based on error analysis of the Theorem 1, we have

\[
e_{w_n} \sim (1 + y_n f'(\alpha)) e_n,
\]
\[
e_{y_n} \sim c_2 (1 + y_n f'(\alpha))^2 e_n,
\]
\[
e_{z_n} \sim k_4 (1 + y_n f'(\alpha))^4 e_n
\]
\[
e_{n+1} \sim k_4 (1 + y_n f'(\alpha))^4 e_n
\]

where \( k_4 = -(1/2)c_2c_5 + \frac{c_1^2}{2} (H_{2,0}(1 + y_n f'(\alpha)) + H_{2,1}(1 + y_n f'(\alpha)) - 3 + y_n f'(\alpha))) \) and \( c_4, c_5 \) are explicit from (20) and depend on iteration index since \( y_k \) is recalculated in each step.

By (23) and the order of interpolatory iteration function, see Section 4.2 in [18], we can also conclude that

\[
N_4'(x_n) = f'(\alpha) \left( 1 + c_4 e_{n-1} w_{n-1} e_{n-1} e_{n-1} + \cdots \right).
\] (27)

Since \( y_n = -1/N_4'(x_n) \), then

\[
1 + y_n f'(\alpha) \sim c_4 e_{n-1} w_{n-1} e_{n-1} e_{n-1}.
\] (28)

Combining (26) with (28), we infer that

\[
e_{w_n} \sim c_5 e_{n-1} w_{n-1} e_{n-1} e_{n-1} e_{n-1} e_{n-1}
\]
\[
\sim c_5 A_{n-1} B_{n-1} C_{n-1} D_{n-1} e_{n-1}^{r+p+q+s+1},
\]
\[
e_{y_n} \sim c_5 e_{n-1} w_{n-1} e_{n-1} e_{n-1} e_{n-1}^2
\]
\[
\sim c_5 c_2 A_{n-1} B_{n-1} C_{n-1} D_{n-1} e_{n-1}^{2r+p+q+s+1},
\]
\[
e_{z_n} \sim k_4 c_5 e_{n-1} w_{n-1} e_{n-1} e_{n-1} e_{n-1}^2
\]
\[
\sim k_4 k_2 A_{n-1} B_{n-1} C_{n-1} D_{n-1} e_{n-1}^{4r+2p+2q+2s+2},
\]
\[
e_{n+1} \sim k_4 A_{n-1} B_{n-1} C_{n-1} D_{n-1} e_{n-1}^{8r+4p+4q+4s+4}.
\] (29)
Equating powers on right-hand-side of relations (25) and (29), correspondingly, we form the following system of equations:

\[
\begin{align*}
rq - r - s - p - q - 1 &= 0, \\
rp - 2r - s - p - q - 1 &= 0, \\
rs - 4r - 2s - 2p - 2q - 2 &= 0, \\
r^2 - 8r - 4s - 4p - 4q - 4 &= 0.
\end{align*}
\]

Nontrivial solution of this system is \( q = 2, p = 3, s = 6, \) and \( r = 12. \) Therefore, the \( R \)-order of the methods with memory (5) under assumptions of Theorem 1, when \( y_n = 1/N'_4(x_n), \) is at least 12.

Remark 3. If we use lower Newton interpolation, we achieve lower \( R \)-order.

4. Numerical Results

In this section, we test our proposed methods and compare their results with some other methods of the same order of convergence. First, we introduce some concrete methods based on the proposed class in this work.

Considering weight functions (21)-(22), we have

Concrete method 1

\[
\begin{align*}
x_0, w_0, y_0 & \text{ are given suitably,} \\
y_n &= x_n - \frac{f(x_n)}{f[x_n, w_n]}, \quad n = 0, 1, 2, \ldots, \\
z_n &= y_n - (1 + t_n) \cdot \frac{f(y_n)}{f[y_n, w_n]}, \\
x_{n+1} &= z_n - \left(1 + t_n + s_n + 2t_n s_n + (-1 - \phi_n) t_n^3\right) \\
\times \left(1 + s_n^2 + v_n^2\right) \frac{f(z_n)}{f[z_n, w_n]}, \\
y_{n+1} &= -\frac{1}{N'_4(x_n)}, \\
w_{n+1} &= x_{n+1} + y_{n+1} f(x_{n+1}).
\end{align*}
\]

Concrete method 2

\[
\begin{align*}
x_0, w_0, y_0 & \text{ are given suitably,} \\
y_n &= x_n - \frac{f(x_n)}{f[x_n, w_n]}, \quad n = 0, 1, 2, \ldots, \\
z_n &= y_n - (1 + t_n) \cdot \frac{f(y_n)}{f[y_n, w_n]}, \\
x_{n+1} &= z_n - \left(1 + t_n + s_n + 2t_n s_n + (-1 - \phi_n) t_n^3\right) \\
\times \left(1 + s_n^2 + v_n^2\right) \frac{f(z_n)}{f[z_n, w_n]}, \\
y_{n+1} &= -\frac{1}{N'_4(x_n)}, \\
w_{n+1} &= x_{n+1} + y_{n+1} f(x_{n+1}).
\end{align*}
\]

Concrete method 3

\[
\begin{align*}
x_0, w_0, y_0 & \text{ are given suitably,} \\
y_n &= x_n - \frac{f(x_n)}{f[x_n, w_n]}, \quad n = 0, 1, 2, \ldots, \\
z_n &= y_n - \left(3 + t_n + \sin\left(\frac{2t_n}{(2/9)t_n^2 + 3}\right)\right) \frac{f(y_n)}{f[y_n, w_n]}, \\
x_{n+1} &= z_n - \left(1 + t_n + s_n + 2t_n s_n + t_n^2\right) \\
\times \left(1 + s_n^2 + v_n^2\right) \frac{f(z_n)}{f[z_n, w_n]}, \\
y_{n+1} &= -\frac{1}{N'_4(x_n)}, \\
w_{n+1} &= x_{n+1} + y_{n+1} f(x_{n+1}).
\end{align*}
\]

Concrete method 4

\[
\begin{align*}
x_0, w_0, y_0 & \text{ are given suitably,} \\
y_n &= x_n - \frac{f(x_n)}{f[x_n, w_n]}, \quad n = 0, 1, 2, \ldots, \\
z_n &= y_n - \left(3 + t_n + \sin\left(\frac{2t_n}{(2/9)t_n^2 + 3}\right)\right) \frac{f(y_n)}{f[y_n, w_n]}, \\
x_{n+1} &= z_n - \left(1 + t_n + s_n + 2t_n s_n + (-1 - \phi_n) t_n^3\right) \\
\times \left(1 + s_n^2 + v_n^2\right) \frac{f(z_n)}{f[z_n, w_n]}, \\
y_{n+1} &= -\frac{1}{N'_4(x_n)}, \\
w_{n+1} &= x_{n+1} + y_{n+1} f(x_{n+1}).
\end{align*}
\]
Table 1: \( f(x) = \exp(x^2 - 3x) \sin(x) + \log(x^2 + 1) \), \( x_0 = 0.35, \alpha = 0, \gamma = 1 \).

| Methods             | \( |x_1 - \alpha| \)  | \( |x_2 - \alpha| \)  | \( |x_3 - \alpha| \)  | COC (39) |
|---------------------|-----------------|-----------------|-----------------|----------|
| New Method (31)     | 0.61569 (–3)    | 0.23067 (–21)   | 0.91264 (–169)  | 7.999    |
| New Method (32)     | 0.56124 (–3)    | 0.32323 (–21)   | 0.38634 (–167)  | 8.000    |
| New Method (33)     | 0.55232 (–3)    | 0.28429 (–21)   | 0.13833 (–167)  | 8.000    |
| New Method (34)     | 0.62453 (–3)    | 0.25853 (–21)   | 0.22480 (–168)  | 7.999    |
| Method (35)         | 0.19676 (–4)    | 0.44197 (–33)   | 0.28657 (–262)  | 8.000    |
| Method (36)         | 0.85597 (–4)    | 0.28686 (–29)   | 0.45644 (–233)  | 8.000    |
| Method (37)         | 0.17236 (–4)    | 0.32121 (–35)   | 0.46744 (–281)  | 8.000    |
| Method (38)         | 0.34824 (–4)    | 0.36172 (–33)   | 0.48972 (–265)  | 8.000    |

Table 2: \( f(x) = \exp(x^2 + x \cos(x) - 1) \sin(\pi x) + x \log(x \sin(x) + 1) \), \( x_0 = 0.6, \alpha = 0, \gamma = -1 \).

| Methods             | \( |x_1 - \alpha| \)  | \( |x_2 - \alpha| \)  | \( |x_3 - \alpha| \)  | COC (39) |
|---------------------|-----------------|-----------------|-----------------|----------|
| New Method (31)     | 0.57578 (–3)    | 0.71057 (–29)   | 0.38797 (–236)  | 7.999    |
| New Method (32)     | 0.45807 (–3)    | 0.52955 (–31)   | 0.35165 (–254)  | 7.999    |
| New Method (33)     | 0.46574 (–3)    | 0.65400 (–31)   | 0.10135 (–253)  | 7.999    |
| New Method (34)     | 0.56806 (–3)    | 0.63795 (–29)   | 0.16377 (–236)  | 7.999    |
| Method (35)         | 0.48202 (–3)    | 0.27805 (–30)   | 0.34404 (–248)  | 7.999    |
| Method (36)         | 0.31009 (–3)    | 0.26712 (–31)   | 0.84195 (–256)  | 7.999    |
| Method (37)         | 0.23448 (–3)    | 0.10417 (–32)   | 0.15929 (–267)  | 7.999    |
| Method (38)         | 0.46334 (–3)    | 0.23577 (–30)   | 0.10713 (–248)  | 7.999    |

For comparison purposes, we consider the following methods:

Three-point by Sharma et al. [20]:

\[
x_{n+1} = z_n - f(z_n)
\]

where \( u_n = f(y_n)/f(x_n) \) and \( v_n = f(y_n)/f(w_n) \).

Three-point by Kung and Traub [1]:

\[
x_{n+1} = z_n - f(z_n)
\]

Three-point by Zheng et al. [21]:

\[
x_{n+1} = z_n - f(z_n)
\]
Table 3: \( f(x) = \exp(x^2 - 3x) \sin(x) + \log(x^2 + 1), \ x_0 = 0.35, \alpha = 0, \gamma_0 = 0.01. \)

| Methods       | \( |x_1 - \alpha| \)     | \( |x_2 - \alpha| \)     | \( |x_3 - \alpha| \)     | COC (39) |
|---------------|----------------|----------------|----------------|---------|
| New Method (31)| 0.91937 (−4)   | 0.18790 (−44) | 0.11705 (−532) | 11.998  |
| New Method (32)| 0.11053 (−3)   | 0.77050 (−44) | 0.31600 (−525) | 11.988  |
| New Method (33)| 0.11306 (−3)   | 0.99434 (−44) | 0.67431 (−524) | 11.987  |
| New Method (34)| 0.89413 (−4)   | 0.13787 (−44) | 0.28524 (−534) | 11.998  |
| Method (35)    | 0.14850 (−5)   | 0.17577 (−61) | 0.48167 (−738) | 12.097  |
| Method (36)    | 0.84533 (−4)   | 0.39381 (−45) | 0.10032 (−540) | 11.991  |
| Method (37)    | 0.30874 (−6)   | 0.17978 (−67) | 0.12617 (−812) | 12.169  |
| Method (38)    | 0.16768 (−4)   | 0.15361 (−56) | 0.21643 (−680) | 11.988  |

Table 4: \( f(x) = \exp(x^2 + x \cos(x) - 1) \sin(\pi x) + x \log(x \sin(x) + 1), \ x_0 = 0.6, \alpha = 0, \gamma_0 = -0.1. \)

| Methods       | \( |x_1 - \alpha| \)     | \( |x_2 - \alpha| \)     | \( |x_3 - \alpha| \)     | COC (39) |
|---------------|----------------|----------------|----------------|---------|
| New Method (31)| 0.71066 (−4)   | 0.20396 (−49) | 0.49715 (−596) | 12.002  |
| New Method (32)| 0.80715 (−4)   | 0.15495 (−49) | 0.65738 (−595) | 11.929  |
| New Method (33)| 0.78950 (−4)   | 0.14520 (−49) | 0.30139 (−596) | 11.931  |
| New Method (34)| 0.72833 (−4)   | 0.22472 (−49) | 0.15905 (−595) | 12.000  |
| Method (35)    | 0.64946 (−4)   | 0.48258 (−50) | 0.11725 (−600) | 11.936  |
| Method (36)    | 0.60478 (−4)   | 0.17480 (−48) | 0.27838 (−582) | 11.985  |
| Method (37)    | 0.65693 (−4)   | 0.27775 (−50) | 0.42521 (−607) | 11.993  |
| Method (38)    | 0.86612 (−4)   | 0.81112 (−64) | 0.23877 (−765) | 12.089  |

Three-point by Soleymani et al. [22]:

\[ x_0, w_0, y_0 \] are given suitably,

\[ y_n = x_n - \frac{f(x_n)}{f[w_n, x_n]}, \quad n = 0, 1, 2, \ldots, \]

\[ z_n = y_n - f(y_n) \]

\[ \times (f[y_n, x_n] + f[w_n, x_n, y_n] (y_n - x_n) + (y_n - x_n)(y_n - w_n)^{-1}, \]

\[ x_{n+1} = z_n - f(z_n) \]

\[ \times (f[x_n, z_n] + f[w_n, x_n, y_n] - f[w_n, x_n, z_n] - f[y_n, x_n, z_n]) (x_n - z_n)^{-1}, \]

\[ + (z_n - x_n) (z_n - w_n) (z_n - y_n)^{-1}, \]

\[ y_{n+1} = \frac{1}{N'_1(x_n)}, \]

\[ w_{n+1} = x_{n+1} + y_{n+1} f(x_{n+1}). \quad (38) \]

By \( |x_n - \alpha| \) we denote approximations to the zero \( \alpha \), \( b(-\alpha) \) stands for \( b \times 10^{-a} \), and the computational order of convergence (COC). Here, COC is defined by [16]:

\[ \text{COC} = \frac{\ln(|x_{n+1} - \alpha| / |x_n - \alpha|)}{\ln(|x_n - \alpha| / |x_{n-1} - \alpha|)}. \quad (39) \]

Also the following functions are used:

\[ f(x) = \exp(x^2 - 3x) \sin(x) + \log(x^2 + 1), \ x_0 = 0.35, \alpha = 0, \gamma_0 = 0.01. \quad (40) \]

Tables 1 and 2 show numerical results for various optimal without memory methods (31)–(38). It is clear that all these methods behave very well practically and confirm their relevant theories.

Tables 3 and 4 present numerical results for various with memory methods (31)–(38). It is also clear that all these methods behave very well practically and confirm their relevant theories. They all provide 12th-order of convergence asymptotically without any new function evaluations.

5. Conclusions

In this work we proposed a new optimal class of methods without and with memory for computing simple root of a nonlinear equation. Its without and with memory methods attain 8 and 12 orders of convergence, respectively, using only four function evaluations per iterations. This class is free-derivative which can be considered as another virtue for it. All together, we managed to increase efficiency index of methods without memory from \( 8^{1/4} \) to \( 12^{1/4} \) using a very suitable self-accelerator parameter based on Newton interpolation.
Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

First of all, the authors express their sincere appreciation to the referees for their valuable comments. This research was supported by Islamic Azad University, Hamedan Branch, as a research plan entitled "A new class of iterative methods with and without memory."

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