Research Article

$H_\infty$ Cluster Synchronization for a Class of Neutral Complex Dynamical Networks with Markovian Switching

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Received 29 August 2013; Accepted 21 November 2013; Published 6 April 2014

Academic Editors: L. Acedo, M. Bruzón, and J. S. Cánovas

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$H_\infty$ cluster synchronization problem for a class of neutral complex dynamical networks (NCDNs) with Markovian switching is investigated in this paper. Both the retarded and neutral delays are considered to be interval mode dependent and time varying. The concept of $H_\infty$ cluster synchronization is proposed to quantify the attenuation level of synchronization error dynamics against the exogenous disturbance of the NCDNs. Based on a novel Lyapunov functional, by employing some integral inequalities and the nature of convex combination, mode delay-range-dependent $H_\infty$ cluster synchronization criteria are derived in the form of linear matrix inequalities which depend not only on the disturbance attenuation but also on the initial values of the NCDNs. Finally, numerical examples are given to demonstrate the feasibility and effectiveness of the proposed theoretical results.

1. Introduction

During the past decades, the research on the complex dynamical networks (CDNs) has attracted extensive attention of scientific and engineering researchers in all fields domestic and overseas since the pioneering work of Watts and Strogatz [1]. One of the reasons is that the complex networks have extensively existed in many practical applications, such as ecosystems, the Internet, scientific citation web, biological neural networks, and large scale robotic system; see, for example, [2–4]. It should be noted that the synchronization phenomena of CDNs have been paid more attention to and intensively have been investigated in various different fields; please refer to [5–10] and references therein for more details.

Since time delay inevitably exists and has become an important issue in studying the CDNs, synchronization problems for complex networks with time delays have gained increasing research attention and considerable progress has been made; see, for example, [5–16] and references therein for more details. However, in some practical applications, past change rate of the state variables affects the dynamics of nodes in the networks. This kind of complex dynamical network is termed as neutral complex dynamical network (NCDN), which contains delays both in its states and in the derivatives of its states. There are some results about the synchronization design problem for neutral systems [17–21]. In these works, [18, 19] had studied the synchronization control for a kind of master-response setup and further extended to the case of neutral-type neural networks with stochastic perturbation. References [17, 20] had researched the synchronization problem for a class of complex networks with neutral-type coupling delays. Reference [21] had investigated the robust global exponential synchronization problem for an array of neutral-type neural networks. However, much fewer results have been proposed for neutral complex dynamical networks (NCDNs) compared with the rich results for CDNs with only discrete delays.

Recently, as a special synchronization on CDNs, cluster synchronization has been observed in biological science, distributed computation, and social contact networks. Because most of these networks have the clustering characteristic, many individuals maintain close contact with others in a same cluster, while only a few individuals link with an outside cluster. Hence, the individuals are synchronized inside the same cluster, but there is no synchronization among the clusters. Many researchers have made a lot of progress on the cluster synchronization problem; see, for example, [22–26]. In [24], cluster synchronization criteria are proposed for
the coupled Josephson equation by constructing different coupling schemes. Then, in [26], a coupling scheme with cooperative and competitive weight couplings is used to realize cluster synchronization for connected chaotic networks. In [22], cluster synchronization in an array of hybrid coupled neural networks with delays has been investigated and a new method is proposed to realize cluster synchronization by constructing a special coupling matrix. Besides, in the latest two years, cluster synchronization is considered for an array of coupled stochastic delayed neural networks by using the pinning control strategy in [23]. Linear pinning control schemes are given for cluster mixed synchronization of complex networks with community structure and nonidentical nodes in [25]. However, most of the research results in general complex networks ensure global or asymptotical synchronization, but the external disturbance is always existent, which may cause complex networks to diverge or oscillate. Therefore it is imperative to enhance the anti-interference ability of the system. To our knowledge, not much has been done for $H_{\infty}$ cluster synchronization for continuous-time complex dynamical networks with neutral time delays and Markovian switching.

The purpose of this paper is to minimize this gap. In addition, due to the complexity of high-order and large-scale networks, network mode switching is also a universal phenomenon in CDNs of the actual systems, and sometimes the network has finite modes that switch from one to another with certain transition rate; then such switching can be governed by a Markovian chain. The stability and synchronization problem of complex networks and neural networks with Markovian jump parameters and delays are investigated in [15, 27–30] and references therein. Motivated by the above analysis, the $H_{\infty}$ cluster synchronization problem for a class of NCDNs with Markovian switching and mode-dependent time-varying delays is investigated in this paper. The addressed NCDNs consist of $M$ modes and the networks switch from one mode to another according to a Markovian chain.

In this paper, $H_{\infty}$ cluster synchronization of the NCDNs with Markovian jump parameters is studied for the first time, which is first introduced to quantify the attenuation level of synchronization error dynamics against the exogenous disturbance of NCDNs with Markovian switching. It is assumed that the neutral and retarded delays are interval mode dependent and time varying. By utilizing a new augmented Lyapunov functional, $H_{\infty}$ cluster synchronization criteria, which depend on interval mode-dependent delays, disturbance attenuation lever, and the initial values of NCDNs, are derived based on the Lyapunov stability theory, integral matrix inequalities, and convex combination. All the proposed results are in terms of LMIs that can be solved numerically, which are proved to be effective in numerical examples.

The remainder of the paper is organized as follows. Section 2 presents the problem and preliminaries. Section 3 gives the main results, which are then verified by numerical examples in Section 4. The paper is concluded in Section 5.

**Notations.** The following notations are used throughout the paper. $\mathbb{R}^n$ denotes the $n$ dimensional Euclidean space and $\mathbb{R}^{m \times n}$ is the set of all $m \times n$ matrices. $X < Y$ ($X > Y$), where $X$ and $Y$ are both symmetric matrices, meaning that $X - Y$ is negative (positive) definite. $I$ is the identity matrix with proper dimensions. For a symmetric block matrix, we use $*$ to denote the terms introduced by symmetry. $\mathbb{E}$ stands for the mathematical expectation, $\|v\|$ is the Euclidean norm of vector $v$, and $\|A\|$ is spectral norm of matrix $A$. $\mathcal{P}$ denotes the set of all matrices of proper dimensions. For a symmetric block matrix, we use $*$ to denote the terms introduced by symmetry. $\mathbb{E}$ stands for the mathematical expectation, $\|v\|$ is the Euclidean norm of vector $v$, and $\|A\|$ is spectral norm of matrix $A$. $\mathcal{P}$ denotes the set of all matrices of proper dimensions. For a symmetric block matrix, we use $*$ to denote the terms introduced by symmetry. $\mathbb{E}$ stands for the mathematical expectation, $\|v\|$ is the Euclidean norm of vector $v$, and $\|A\|$ is spectral norm of matrix $A$. $\mathcal{P}$ denotes the set of all matrices of proper dimensions. For a symmetric block matrix, we use $*$ to denote the terms introduced by symmetry. $\mathbb{E}$ stands for the mathematical expectation, $\|v\|$ is the Euclidean norm of vector $v$, and $\|A\|$ is spectral norm of matrix $A$. $\mathcal{P}$ denotes the set of all matrices of proper dimensions. For a symmetric block matrix, we use $*$ to denote the terms introduced by symmetry. $\mathbb{E}$ stands for the mathematical expectation, $\|v\|$ is the Euclidean norm of vector $v$, and $\|A\|$ is spectral norm of matrix $A$. $\mathcal{P}$ denotes the set of all matrices of proper dimensions. For a symmetric block matrix, we use $*$ to denote the terms introduced by symmetry. $\mathbb{E}$ stands for the mathematical expectation, $\|v\|$ is the Euclidean norm of vector $v$, and $\|A\|$ is spectral norm of matrix $A$. $\mathcal{P}$ denotes the set of all matrices of proper dimensions. For a symmetric block matrix, we use $*$ to denote the terms introduced by symmetry. $\mathbb{E}$ stands for the mathematical expectation, $\|v\|$ is the Euclidean norm of vector $v$, and $\|A\|$ is spectral norm of matrix $A$. $\mathcal{P}$ denotes the set of all matrices of proper dimensions. For a symmetric block matrix, we use $*$ to denote the terms introduced by symmetry. $\mathbb{E}$ stands for the mathematical expectation, $\|v\|$ is the Euclidean norm of vector $v$, and $\|A\|$ is spectral norm of matrix $A$. $\mathcal{P}$ denotes the set of all matrices of proper dimensions. For a symmetric block matrix, we use $*$ to denote the terms introduced by symmetry. $\mathbb{E}$ stands for the mathematical expectation, $\|v\|$ is the Euclidean norm of vector $v$, and $\|A\|$ is spectral norm of matrix $A$. $\mathcal{P}$ denotes the set of all matrices of proper dimensions. For a symmetric block matrix, we use $*$ to denote the terms introduced by symmetry. $\mathbb{E}$ stands for the mathematical expectation, $\|v\|$ is the Euclidean norm of vector $v$, and $\|A\|$ is spectral norm of matrix $A$. $\mathcal{P}$ denotes the set of all matrices of proper dimensions. For a symmetric block matrix, we use $*$ to denote the terms introduced by symmetry. $\mathbb{E}$ stands for the mathematical expectation, $\|v\|$ is the Euclidean norm of vector $v$, and $\|A\|$ is spectral norm of matrix $A$. $\mathcal{P}$ denotes the set of all matrices of proper dimensions. For a symmetric block matrix, we use $*$ to denote the terms introduced by symmetry. $\mathbb{E}$ stands for the mathematical expectation, $\|v\|$ is the Euclidean norm of vector $v
\[ z_k(t) = L(r(t)) x_k(t), \quad (4) \]

where \( x_k(t) = (x_{k1}(t), x_{k2}(t), \ldots, x_{kn}(t))^T \in \mathbb{R}^n \) and \( z_k(t) = (z_{k1}(t), z_{k2}(t), \ldots, z_{kn}(t))^T \in \mathbb{R}^n \) are state variable and the controlled output of the node \( k \in \{1, 2, \ldots, N\} \), respectively. \( \omega_k(t) \in \mathbb{R} \) is the exogenous disturbance input. \( r(t) \) describes the evolution of the mode. \( A(r(t)), B(r(t)), C(r(t)), D(r(t)), E(r(t)), F(r(t)) \in \mathbb{R}^{m \times n} \) represent the connection weight matrices and the delayed connection weight matrices with real values in mode \( r(t) \). \( H_k(r(t)) \in \mathbb{R}^n (k = 1, 2, \ldots, N) \) is the disturbance matrix in mode \( r(t) \). \( L(r(t)) \in \mathbb{R}^{m \times n} \) is a parametric matrix in mode \( r(t) \). In this paper, these parametric matrices of NCDN (3) and (4) are known constant matrices in certain mode \( r(t) \). \( f_1, f_2, f_3 : \mathbb{R}^n \rightarrow \mathbb{R}^n \) are continuously nonlinear vector functions which are with respect to the current state \( x(t) \), the delayed state \( x(t - d(t, r(t))) \), and the neutral delay state \( x_k(t - \tau_k(t, r(t))) \).\( \Gamma_i(r(t)) \in \mathbb{R}^{m \times n} \) are state variable and the delayed state variable in mode \( r(t) \). \( L(r(t)) \in \mathbb{R}^{m \times n} \) is a parametric matrix in mode \( r(t) \).\( \omega(t) = \text{col} \{\omega_1(t), \omega_2(t), \ldots, \omega_N(t)\} \)

\[ z(t) = \text{col} \{z_1(t), z_2(t), \ldots, z_N(t)\} \]

\[ w(t) = \text{col} \{w_1(t), w_2(t), \ldots, w_N(t)\} \]

where \( W_i^{(l)} \) and \( W_j^{(l)} \), \( i = 1, 2, 3, \) are two constant matrices with \( W_i^{(l)} - W_j^{(l)} \geq 0 \). Such a description of nonlinear functions has been exploited in [32–34] and is more general than the commonly used Lipschitz conditions, which would be possible to reduce the conservativeness of the main results caused by quantifying the nonlinear functions via a matrix inequality technique.

For simplicity of notations, we denote \( A(r(t)), B(r(t)), C(r(t)), D(r(t)), E(r(t)), F(r(t)), G^{(m)}(r(t)), \Gamma_m(r(t)), (m = 1, 2, 3), H_k(r(t)), \) and \( L(r(t)) \) by \( A_1, B_1, C_1, D_1, E_1, F_1, G_1^{(m)}, \Gamma_m, \) \( (m = 1, 2, 3), H_k, \) and \( L_i \) for \( r(t) = i \in S \). By utilizing the Kronecker product of the matrices, (3) and (4) can be written in a more compact form as

\[ \dot{x}(t) = A_i x(t) + B_i x(t - d_i(t)) + C_i \dot{x}(t - \tau_i(t)) + D_i F_1(x(t)) + E_i F_2(x(t - d_i(t))) + F_i F_3(\dot{x}(t - \tau_i(t))) + \mathbb{B}_i \omega(t) \]

\[ z(t) = \mathbb{L}_i x(t) \]

\[ x(t) = \text{col} \{x_1(t), x_2(t), \ldots, x_N(t)\}, \]

\[ x(t - d_i(t)) = \text{col} \{x_1(t - d_i(t)), x_2(t - d_i(t)), \ldots, x_N(t - d_i(t))\}, \]

\[ \dot{x}(t - \tau_i(t)) = \text{col} \{\dot{x}_1(t - \tau_i(t)), \dot{x}_2(t - \tau_i(t)), \ldots, \dot{x}_N(t - \tau_i(t))\}, \]

\[ F_1(x(t)) = \text{col} \{f_1(x_1(t)), f_1(x_2(t)), \ldots, f_1(x_N(t))\}, \]

\[ F_2(x(t - d_i(t))) = \text{col} \{f_2(x_1(t - d_i(t))), f_2(x_2(t - d_i(t))), \ldots, f_2(x_N(t - d_i(t)))\}, \]

\[ F_3(\dot{x}(t - \tau_i(t))) = \text{col} \{f_3(\dot{x}_1(t - \tau_i(t))), f_3(\dot{x}_2(t - \tau_i(t))), \ldots, f_3(\dot{x}_N(t - \tau_i(t)))\}, \]

\[ \omega(t) = \text{col} \{\omega_1(t), \omega_2(t), \ldots, \omega_N(t)\} \]

\[ z(t) = \text{col} \{z_1(t), z_2(t), \ldots, z_N(t)\} \]

(10)
Assumption 1 (see [22]). The coupling matrix $G^{(m)}_i$ can be expressed in the following form:

$$G^{(m)}_i = \begin{bmatrix}
N^{(m)}_{i11} & N^{(m)}_{i12} & \cdots & N^{(m)}_{i1k} \\
N^{(m)}_{i21} & N^{(m)}_{i22} & \cdots & N^{(m)}_{i2k} \\
\vdots & \vdots & \ddots & \vdots \\
N^{(m)}_{ik1} & N^{(m)}_{ik2} & \cdots & N^{(m)}_{ikk}
\end{bmatrix}, \ m = 1, 2, 3. \quad (11)$$

It should be especially emphasized that we do not assume that the coupling matrix is symmetric or diagonal. However, most of the former works about network synchronization are based on symmetric or diagonal coupling matrix.

Before moving onto the main results, some definitions and lemmas are introduced below.

Definition 2 (see [35]). Define operator $\mathcal{D} : C([-\zeta, 0], \mathbb{R}^n) \to \mathbb{R}^n$ by $\mathcal{D}(x_t) = x(t) - Cx(t - \tau)$. $\mathcal{D}$ is said to be stable if the homogeneous difference equation

$$\mathcal{D}(x_t) = 0, \ t \geq 0,$$

$$x_0 = \psi \in \{\phi \in C([-\zeta, 0], \mathbb{R}^n) : \mathcal{D}\phi = 0\}$$

is uniformly asymptotically stable. In this paper, that is, $\|I_N \otimes G_i + G^{(3)}_i \otimes C\| < 1$.

Definition 3 (see [36]). Define the stochastic Lyapunov-Krasovskii function of the NCDNs (3) and (4) as $V(x(t), r(t) = i, t > 0) = V(x(t), i, t)$ where its infinitesimal generator is defined as

$$\Gamma V(x(t), i, t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} [E[V(x(t + \Delta t), r(t + \Delta t)), t + \Delta t)] - V(x(t), r(t)]$$

$$=\frac{\partial}{\partial t} V(x(t), i, t) + \frac{\partial}{\partial x} V(x(t), i, t) \dot{x}(t)$$

$$+ \sum_{j=1}^{N} p_{ij} V(x(t), j, t). \quad (13)$$

Definition 4 (see [26]). A network with $N$ nodes realizes cluster synchronization if the $N$ nodes are split into several clusters, such as $\{(1, 2, \ldots, m_1), (m_1 + 1, m_1 + 2, \ldots, m_2), \ldots, (m_{k-1} + 1, m_{k-1} + 2, \ldots, m_k), m_0 = 0, m_k = N, m_{k-1} < m_1\}$, and the nodes in the same cluster synchronize with one another (i.e., for the states $x_i(t)$ and $x_j(t)$ of arbitrary nodes $i$ and $j$ in the same cluster, $\lim_{\tau \to \infty} \|x_i(t) - x_j(t)\| = 0$ holds). The set

$$\mathcal{S} = \{x = (x_1(s), x_2(s), \ldots, x_N(s)) : x_1(s) = x_2(s) = \cdots = x_{m_1}(s), x_{m_1+1}(s) = x_{m_1+2}(s) = \cdots = x_{m_2}(s), \ldots, x_{m_{k-1}+1}(s) = x_{m_{k-1}+2}(s) = \cdots = x_{m_k}(s)\} \quad (14)$$

is called the cluster synchronization manifold.

Lemma 5 (see [37]). Let $G$ be an $N \times N$ matrix in the set $T(\mathbb{R}, K)$, where $\mathbb{R}$ denotes a ring and $T(\mathbb{R}, K) =$ \{the set of matrices with entries $\mathbb{R}$ such that the sum of the entries in each row is equal to $K$ for some $K \in \mathbb{R}$. Then the $(N-1) \times (N-1)$ matrix $X$ satisfies $MG = XM$, where $X = MG_I$,

$$M = \begin{bmatrix}
1 & -1 & 0 & \cdots & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \cdots & 0 & -1 & \cdots & 0 \\
0 & -1 & 2 & \cdots & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -1 & 2 & -1 & \cdots & 0 \\
0 & 0 & \cdots & 0 & -1 & 2 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & -1 & 2
\end{bmatrix}_{(N-1) \times N} \quad (15)$$

$$J = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
0 & 1 & 1 & \cdots & 1 \\
0 & 0 & 1 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0 \\
N & (N-1)
\end{bmatrix} \quad (16)$$

Furthermore, the matrix $X$ can be rewritten explicitly as follows:

$$X_{pq} = \sum_{k=1}^{q} (G_{p,k} - G_{p+1,k}), \text{ for } p, q \in \{1, 2, \ldots, N - 1\}. \quad (17)$$

Lemma 6. Under Assumption 1, the $(N-k) \times (N-k)$ matrix $X^{(m)}_i$ satisfies $\bar{M}G^{(m)}_i = X^{(m)}_i \bar{M}, m = 1, 2, 3$,

$$\bar{M} = \begin{bmatrix}
M_1 & \cdots & 0 \\
0 & \ddots & \vdots \\
0 & \cdots & M_k
\end{bmatrix}_{(N-k) \times N}, \quad (18)$$

$$J = \begin{bmatrix}
J_1 \\
J_2 \\
\vdots \\
J_k
\end{bmatrix}_{N \times (N-k)} \quad (19)$$

And $X^{(m)}_i = \bar{M}N^{(m)}_i J, N^{(m)}_p \in \mathbb{R}^{m_p \times m_p}, M_p \in \mathbb{R}^{(m_p-1) \times m_p}, J_p \in \mathbb{R}^{m_p \times (m_p-1)}$, and $p = 1, 2, \ldots, k$.

Proof. From Assumption 1 and Lemma 5, it can be easily obtained that

$$\bar{M}G^{(m)}_i = \begin{bmatrix}
M_1 & \cdots & 0 \\
0 & \ddots & \vdots \\
0 & \cdots & M_k
\end{bmatrix} \times \begin{bmatrix}
X^{(m)}_{i11} & \cdots & X^{(m)}_{i1k} \\
X^{(m)}_{i21} & \cdots & X^{(m)}_{i2k} \\
\vdots & \vdots & \vdots \\
X^{(m)}_{ik1} & \cdots & X^{(m)}_{ikk}
\end{bmatrix}$$
The neutral complex dynamical networks (3) and (4) are:

\[
\begin{align*}
\Theta_i &< 0, \\
\Omega_i &+ 1
\end{align*}
\]

\[
\begin{align*}
\Pi_i &< 0, \\
\Pi_i &+ 1
\end{align*}
\]

\[
\begin{align*}
(\Omega_i) &< 0, \\
\Pi_i &+ 1
\end{align*}
\]

\[
\begin{align*}
(\Pi_i) &< 0
\end{align*}
\]

(25)

Lemma 7 (see [22]). \( x \in S \) if and only if \( \exists t_0 \) such that \( \|Mx(t)\|_2 \) is well defined, then

\[
\begin{align*}
\|Mx(t)\|_2 &> \|x(0)\|_2 \\
&> \|x(\infty)\|_2
\end{align*}
\]

Lemma 9 (see [38]). Given matrices \( A, B, C, \) and \( D \) with appropriate dimensions and scalar \( \alpha \), by the definition of the Kronecker product, the following properties hold:

\[
\begin{align*}
(aA) \otimes B & = A \otimes (aB), \\
(A + B) \otimes C & = A \otimes C + B \otimes C, \\
(A \otimes B) (C \otimes D) & = (AC) \otimes (BD), \\
(A \otimes B)^T & = A^T \otimes B^T
\end{align*}
\]

Lemma 10 (see [39, 40]). For any constant matrix \( H = H^T > 0 \) and scalars \( \tau_0 > 0 \), \( \tau_2 > \tau_1 > 0 \) such that the following integrations are well defined, then

\[
\begin{align*}
-a (\tau_2 - \tau_1) &\int_{\tau_2}^{\tau_1} x^T(s) H x(s) \, ds \\
&\leq -\left( \int_{\tau_2}^{\tau_1} x^T(s) H x(s) \, ds \right) H \left( \int_{\tau_2}^{\tau_1} x(s) \, ds \right)
\end{align*}
\]

(22)

Lemma 11 (see [41]). Supposing that \( 0 < \tau_m \leq \tau(t) \leq \tau_M \), \( \Xi_1, \Xi_2, \) and \( \Omega \) are constant matrices of appropriate dimensions, then

\[
(\tau(t) - \tau_m) \Xi_1 + (\tau_M - \tau(t)) \Xi_2 + \Omega < 0
\]

(24)

3. Main Results

In this section, sufficient conditions are presented to ensure \( H_{\infty} \), cluster synchronization for the neutral complex dynamical network (NCDN) (3) and (4).

3.1. \( H_{\infty} \) Cluster Synchronization Analysis

Theorem 12. Given the transition rate matrix \( Y \), the initial positive definite matrix \( Y = Y^T > 0 \), constant scalars \( \tau_{ij}, \tau_{2j}, \tau_{3j}, \) and \( \tau_{mi}, d_{mi}, d_{mi} \) satisfying \( \tau_{ij} < \tau_{mi} < \tau_{2i}, d_{ij} < d_{mi} < d_{2j} \), respectively, the NCDN systems (3) and (4) with sector-bounded condition (7) are \( H_{\infty} \) cluster synchronization with a disturbance attenuation lever \( \delta \) if \( \|x(0)\|_2 \leq 1 \) and there exist \( (N-k)X \times (N-k) \) symmetric positive matrices \( P_i > 0 \) (\( i \in S \)), \( Q_j > 0 \) (\( j = 1, 2, \ldots, 6 \)), \( R_k > 0 \) (\( k = 1, 2, \ldots, 7 \)), \( T_l > 0 \) (\( l, m, n = 1, 2, \ldots, 6 \)) for any scalars \( \varepsilon_1, \varepsilon_2, \varepsilon_3 > 0 \) such that the following linear matrix inequalities hold:

\[
\begin{align*}
\Omega_{i1} + \frac{1}{2} \Theta_{i0} &< 0, \\
\Omega_{i2} + \frac{1}{2} \Theta_{i0} &< 0, \\
\Omega_{i3} + \frac{1}{2} \Theta_{i0} &< 0, \\
\Pi_{i1} + \frac{1}{2} \Theta_{i0} &< 0, \\
\Pi_{i2} + \frac{1}{2} \Theta_{i0} &< 0, \\
\Pi_{i3} + \frac{1}{2} \Theta_{i0} &< 0,
\end{align*}
\]

(25)
\[ V(0) < \delta^2 x^T(0) y_x(0), \quad (26) \]

where

\[
\begin{align*}
\Theta_{i0} &= \sum_{m=1}^{30} E_m \Phi_m E_m^T + \mathcal{L}(A) + A^T \Lambda A - (E_1 - E_3) \\
&\times U_1 \left( E_{1}^T - E_{13}^T \right) - (E_1 - E_{16}) U_4 \left( E_{2}^T - E_{16}^T \right) \\
&- (r_{i2} - r_{i1}) U_2 \left( E_{1}^T - E_{2}^T \right) \\
&- \left[ (r_{ia} - r_{ib}) E_1 - E_{13} \right] V_2 \left[ (r_{ia} - r_{ib}) E_1^T - E_{13}^T \right]
\end{align*}
\]

\[ E_3 = [0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0]. \quad (28) \]

\[ \mathcal{L} \text{ is a linear operator on real square matrices by} \]

\[
\mathcal{L}(A) = FA + A^T, \quad \forall A \in \mathbb{R}^{n \times n},
\]

\[
J = \eta(T - r_{i1}) R_1 + R_2 + R_3 + \tau_{i1}^2 U_1 + d_{i1}^2 U_4 \\
+ (r_{ia} - r_{ib}) \tau_{i2}^2 U_2 + (r_{ia} - r_{ib}) \tau_{i1}^2 U_3 \\
+ (d_{ia} - d_{ib}) \tau_{i5}^2 U_5 + (d_{ia} - d_{ib}) \tau_{i4}^2 U_6 \\
+ d_{i1}^4 V_2 + \frac{3}{4} \tau_{i1}^2 \tau_{i2}^2 V_2 + \frac{3}{4} \tau_{i1}^2 \tau_{i2}^2 V_3 \\
+ \frac{3}{4} \tau_{i2}^2 \tau_{i1}^2 V_5 + \frac{3}{4} \tau_{i2}^2 \tau_{i1}^2 V_6,
\]

\[
\Lambda = \left( A_i^\theta + X_i^{(1)} \right) E_1^T + \left( C_i^\theta + X_i^{(3)} \right) E_6^T \\
+ \left( B_i^\theta + X_i^{(2)} \right) E_{15}^T + D_i^\theta E_{27} + E_i^\theta E_{28}
\]

\[
\Phi = E_1 \left( P_i D_i^\theta + \epsilon_i W_1^{(1)} + \epsilon_i W_2^{(1)} \right) E_{27} \\
+ E_1 P_i D_i^\theta E_{29} + E_1 P_i D_i^\theta E_{29} + E_1 P_i D_i^\theta E_{29} + E_1 P_i D_i^\theta E_{29} + E_1 P_i D_i^\theta E_{29} + E_1 P_i D_i^\theta E_{29} + E_1 P_i D_i^\theta E_{29} + E_1 P_i D_i^\theta E_{29}
\]

\[
\Phi_1 = \mathcal{L} \left[ P_i \left( A_i^\theta + B_i^\theta + C_i^\theta \right) + X_i^{(1)} + X_i^{(2)} + X_i^{(3)} \right] \\
- \epsilon_i W_1^{(1)} W_2^{(1)} + \sum_{j \in S} y_{j1} P_j + Q_1 + Q_4 \\
+ \tau_{i1}^2 T_1 + d_{i1}^2 T_4 + (r_{ia} - r_{ib}) \tau_{i2}^2 T_2 + (r_{ia} - r_{ib}) \tau_{i1}^2 T_3 \\
+ (d_{ia} - d_{ib}) \tau_{i4}^2 T_5 + (d_{ia} - d_{ib}) \tau_{i4}^2 T_6 + L_i^\theta L_i^\theta, \]

where \( E_{i} \) \((i = 1, 2, \ldots, 30)\) are block entry matrices; that is,

\[
\Phi_3 = Q_2 - R_1, \quad \Phi_4 = Q_4 - R_2, \\
\Phi_5 = -Q_3, \quad \Phi_6 = -(1 - \eta_i) R_1 - \epsilon_i W_1^{(3)} W_2^{(3)} \\
\Phi_7 = R_1 + R_3 - R_2, \quad \Phi_8 = R_4 - R_3 \\
\Phi_9 = -Q_4, \quad \Phi_{10} = -T_1, \\
\Phi_{15} = -\epsilon_i W_1^{(2)} W_2^{(2)}, \quad \Phi_{16} = Q_3 - Q_4, \\
\Phi_{17} = Q_6 - Q_5, \quad \Phi_{18} = -Q_6, \\
\Phi_{19} = R_6 - R_5, \quad \Phi_{20} = R_7 - R_6, \\
\Phi_{21} = -R_7, \quad \Phi_{27} = -2\epsilon_i I, \\
\Phi_{28} = -2\epsilon_i I, \quad \Phi_{29} = -2\epsilon_i I, \\
\Phi_{30} = -\delta^2 I, \\
\Phi_m = 0, \quad (m = 2, 11, 12, 13, 14, 23, 24, 25, 26), \\
A_i^\theta = I_{N-k} \bigotimes A_i, \quad B_i^\theta = I_{N-k} \bigotimes B_i, \\
C_i^\theta = I_{N-k} \bigotimes C_i, \quad D_i^\theta = I_{N-k} \bigotimes D_i, \\
E_i^\theta = I_{N-k} \bigotimes E_i, \quad F_i^\theta = I_{N-k} \bigotimes F_i, \\
L_i^\theta = I_{N-k} \bigotimes L_i, \\
X_i^{(m)} = X_i^{(m)} \bigotimes \Gamma_{mi}, \quad (m = 1, 2, 3), \\
\Omega_{11} = -E_{11} T_2 E_{11}^T - 2 \left( E_{13} - E_{11} \right) T_2 \left( E_{13}^T - E_{11}^T \right) \\
- \left( E_3 - E_2 \right) U_2 \left( E_{11}^T - E_{11}^T \right) - 2 \left( E_2 - E_3 \right) \\
\times U_2 \left( E_{11}^T - E_{11}^T \right) - E_{11} T_3 E_{14}^T \\
- \left( E_4 - E_3 \right) U_3 \left( E_{12}^T - E_{12}^T \right).
\[
\Omega_2 = -2E_{11}T_2^E - (E_{13} - E_{11})T_2 (E_{13}^T - E_{11}^T) \\
- 2(E_3 - E_2)U_2 (E_3^T - E_2^T) - (E_2 - E_4) \\
\times U_2 (E_2^T - E_1^T) - E_{14}T_3 E_{14}^T \\
- (E_4 - E_3)U_3 (E_4^T - E_3^T), \]
\[
\Omega_3 = -2E_{12}T_3^E - (E_{14} - E_{12})T_3 (E_{14}^T - E_{12}^T) \\
- (E_4 - E_3)U_3 (E_4^T - E_3^T) - 2(E_2 - E_5) \\
\times U_3 (E_2^T - E_5^T) - E_{13}T_2 E_{13}^T \\
- (E_5 - E_4)U_2 (E_5^T - E_4^T), \]
\[
\Omega_{14} = -E_{12}T_3^E - 2(E_{14} - E_{12})T_3 (E_{14}^T - E_{12}^T) \\
- 2(E_4 - E_2)U_3 (E_4^T - E_2^T) \\
- (E_3 - E_4)U_2 (E_3^T - E_4^T), \]
\[
\Pi_1 = -E_{23}T_3^E - 2(E_{25} - E_{23})T_5 (E_{25}^T - E_{23}^T) \\
- (E_{16} - E_{15})U_5 (E_{16}^T - E_{15}^T) \\
- 2(E_{15} - E_{17})U_5 (E_{15}^T - E_{17}^T) \\
- E_{26}T_6 E_{26}^T - (E_{17} - E_{18})U_6 (E_{17}^T - E_{18}^T), \]
\[
\Pi_2 = -E_{25}T_5^E - (E_{25} - E_{23})T_5 (E_{25}^T - E_{23}^T) \\
- 2(E_{16} - E_{15})U_5 (E_{16}^T - E_{15}^T) \\
- (E_{15} - E_{17})U_5 (E_{15}^T - E_{17}^T) \\
- E_{25}T_6 E_{25}^T - (E_{17} - E_{18})U_6 (E_{17}^T - E_{18}^T), \]
\[
\Pi_{13} = -E_{26}T_6^E - (E_{26} - E_{24})T_6 (E_{26}^T - E_{24}^T) \\
- (E_{17} - E_{15})U_6 (E_{17}^T - E_{15}^T) \\
- 2(E_{15} - E_{19})U_6 (E_{15}^T - E_{18}^T) - E_{25}T_5 E_{25}^T \\
- (E_{16} - E_{15})U_7 (E_{16}^T - E_{17}^T), \]
\[
\Pi_{14} = -E_{26}T_6^E - 2(E_{26} - E_{24})T_6 (E_{26}^T - E_{24}^T) \\
- 2(E_{17} - E_{15})U_6 (E_{17}^T - E_{15}^T) \\
- (E_{15} - E_{18})U_6 (E_{15}^T - E_{18}^T) - E_{25}T_5 E_{25}^T \\
- (E_{16} - E_{17})U_7 (E_{16}^T - E_{17}^T), \]
\[
V(0) = x^T(0)M^T P M x(0) + \sum_{k=2}^{6} V_k(0), \]
\[
V_2(0) = \int_{-\tau_u}^{0} x^T(s)M^T Q_1 M x(s) \, ds \\
+ \int_{-\tau_u}^{-\tau_m} x^T(s)M^T Q_2 M x(s) \, ds \\
+ \int_{-\tau_m}^{-\tau_m} x^T(s)M^T Q_3 M x(s) \, ds, \]
\[
V_3(0) = \int_{-\tau_{u(0)}}^{0} x^T(s)M^T R_1 M \dot{x}(s) \, ds \\
+ \int_{-\tau_u}^{0} x^T(s)M^T R_2 M \dot{x}(s) \, ds \\
+ \int_{-\tau_m}^{-\tau_m} x^T(s)M^T R_3 M x(s) \, ds \\
+ \int_{-\tau_m}^{-\tau_m} x^T(s)M^T R_4 M \dot{x}(s) \, ds, \]
\[
V_4(0) = \int_{-\tau_u}^{0} \int_{0}^{0} \tau_u x^T(s)M^T T_1 M x(s) \, ds \, d\theta \\
+ \int_{-\tau_u}^{0} \int_{0}^{-\tau_m} \tau_{u} x^T(s)M^T T_2 M x(s) \, ds \, d\theta \\
+ \int_{-\tau_m}^{0} \int_{0}^{-\tau_m} \tau_{m} x^T(s)M^T T_3 M x(s) \, ds \, d\theta \\
+ \int_{-\tau_m}^{0} \int_{0}^{-\tau_m} \tau_{m} x^T(s)M^T T_4 M x(s) \, ds \, d\theta \\
+ \int_{-\tau_{m(0)}}^{0} d_{u_{1}} x^T(s)M^T T_5 M x(s) \, ds \, d\theta \\
+ \int_{-\tau_u}^{0} \int_{0}^{0} d_{u_{1}} x^T(s)M^T T_6 M x(s) \, ds \, d\theta, \]
\[ V_5 (0) \]
\[ = \int_{-\tau_{1i}}^{0} \int_{0}^{0} \tau_{1i} x^T(s) M^T U_1 \dot{M} x(s) \, ds \, d\theta \]
\[ + \int_{-\tau_{mi}}^{-\tau_{1i}} \int_{0}^{0} \left( \tau_{mi} - \tau_{1i} \right) x^T(s) M^T U_2 \dot{M} x(s) \, ds \, d\theta \]
\[ + \int_{-d_{1i}}^{-d_{mi}} \int_{0}^{0} d_{1i} x^T(s) M^T U_4 \dot{M} x(s) \, ds \, d\theta \]
\[ + \int_{-d_{mi}}^{-d_{2i}} \int_{0}^{0} d_{2i} x^T(s) M^T U_6 \dot{M} x(s) \, ds \, d\theta \]
\[ + \int_{-\tau_{1i}}^{0} \int_{0}^{0} \eta x^T(s) M^T R_1 \dot{M} x(s) \, ds \, d\theta, \]

\[ V_6 (0) \]
\[ = \int_{-\tau_{1i}}^{0} \int_{0}^{0} \frac{\tau_{1i}^2}{2} x^T(s) M^T V_1 \dot{M} x(s) \, ds \, d\lambda \, d\theta \]
\[ + \int_{-\tau_{mi}}^{-\tau_{1i}} \int_{0}^{0} \frac{\tau_{mi}^2}{2} x^T(s) M^T V_2 \dot{M} x(s) \, ds \, d\lambda \, d\theta \]
\[ + \int_{-d_{1i}}^{-d_{mi}} \int_{0}^{0} \frac{d_{1i}^2}{2} x^T(s) M^T V_4 \dot{M} x(s) \, ds \, d\lambda \, d\theta \]
\[ + \int_{-d_{mi}}^{-d_{2i}} \int_{0}^{0} \frac{d_{2i}^2}{2} x^T(s) M^T V_6 \dot{M} x(s) \, ds \, d\lambda \, d\theta, \]

(29)

**Proof.** Construct the Lyapunov functional candidate as follows:

\[ V(x(t), i, t) = \sum_{k=1}^{6} V_k(x(t), i, t), \]

where

\[ V_1(x(t), i, t) = x^T(t) M^T P_i M x(t), \]

\[ V_2(x(t), i, t) = \int_{t-t_{1i}}^{t} x^T(s) M^T Q_1 M x(s) \, ds \]
\[ + \int_{t-t_{mi}}^{t} x^T(s) M^T Q_2 M x(s) \, ds \]
\[ + \int_{t-t_{2i}}^{t} x^T(s) M^T Q_3 M x(s) \, ds \]
\[ + \int_{t-t_{2i}}^{t} x^T(s) M^T Q_4 M x(s) \, ds \]

\[ V_3(x(t), i, t) = \int_{t-t_{1i}}^{t} x^T(s) M^T R_1 M x(s) \, ds \]
\[ + \int_{t-t_{mi}}^{t} x^T(s) M^T R_2 M x(s) \, ds \]
\[ + \int_{t-t_{2i}}^{t} x^T(s) M^T R_3 M x(s) \, ds \]
\[ + \int_{t-t_{2i}}^{t} x^T(s) M^T R_4 M x(s) \, ds \]

\[ V_4(x(t), i, t) \]
\[ = \int_{t-t_{1i}}^{t} \int_{t+\theta}^{t} \tau_{1i} x^T(s) M^T T_1 M x(s) \, ds \, d\theta \]
\[ + \int_{t-t_{mi}}^{t} \int_{t+\theta}^{t} \left( \tau_{mi} - \tau_{1i} \right) x^T(s) M^T T_2 M x(s) \, ds \, d\theta \]
\[ + \int_{t-t_{2i}}^{t} \int_{t+\theta}^{t} \left( \tau_{2i} - \tau_{mi} \right) x^T(s) M^T T_3 M x(s) \, ds \, d\theta \]
\[ + \int_{t-t_{2i}}^{t} \int_{t+\theta}^{t} d_{1i} x^T(s) M^T T_4 M x(s) \, ds \, d\theta \]
\[ + \int_{t-t_{mi}}^{t} \int_{t+\theta}^{t} d_{2i} x^T(s) M^T T_5 M x(s) \, ds \, d\theta \]
\[ + \int_{t-t_{mi}}^{t} \int_{t+\theta}^{t} d_{2i} x^T(s) M^T T_6 M x(s) \, ds \, d\theta, \]
\[ V_2(x(t), i, t) = \int_{-\tau_1}^{0} \int_{t+\theta}^{t} \tau_1 \dot{x}^T(s) M^T U_1 M \dot{x}(s) ds d\theta 
+ \int_{-\tau_m}^{0} \int_{t+\theta}^{t} (\tau_m - \tau_i) \dot{x}^T(s) M^T U_2 M \dot{x}(s) ds d\theta 
+ \int_{-\tau_2}^{0} \int_{t+\theta}^{t} (\tau_2 - \tau_m) \dot{x}^T(s) M^T U_3 M \dot{x}(s) ds d\theta 
+ \int_{-d_1}^{0} \int_{t+\theta}^{t} d_1 \dot{x}^T(s) M^T U_4 M \dot{x}(s) ds d\theta 
+ \int_{-d_m}^{0} \int_{t+\theta}^{t} (d_m - d_1) \dot{x}^T(s) M^T U_5 M \dot{x}(s) ds d\theta 
+ \int_{-d_2}^{0} \int_{t+\theta}^{t} (d_2 - d_m) \dot{x}^T(s) M^T U_6 M \dot{x}(s) ds d\theta 
+ \int_{-d_i}^{0} \int_{t+\theta}^{t} d_i \dot{x}^T(s) M^T U_7 M \dot{x}(s) ds d\theta, \]

\[ V_6(x(t), i, t) = \int_{-\tau_1}^{0} \int_{t+\theta}^{t} \frac{d^2_0}{2} \dot{x}^T(s) M^T V_1 M \dot{x}(s) ds d\lambda d\theta 
+ \int_{-\tau_m}^{0} \int_{t+\theta}^{t} \frac{d^2_m}{2} \dot{x}^T(s) M^T V_2 M \dot{x}(s) ds d\lambda d\theta 
+ \int_{-d_2}^{0} \int_{t+\theta}^{t} \frac{d^2_2}{2} \dot{x}^T(s) M^T V_3 M \dot{x}(s) ds d\lambda d\theta. \]

By the structure of \( M \) and by Lemmas 6 and 9, we obtain the following equalities:

\[ M(I_N \otimes A_i) = (I_{N-k} \otimes A_i) M = A^\mu_i M, \]
\[ M(I_N \otimes B_i) = (I_{N-k} \otimes B_i) M = B^\mu_i M, \]
\[ M(I_N \otimes C_i) = (I_{N-k} \otimes C_i) M = C^\mu_i M, \]
\[ M(I_N \otimes D_i) = (I_{N-k} \otimes D_i) M = D^\mu_i M, \]
\[ M(I_N \otimes E_i) = (I_{N-k} \otimes E_i) M = E^\mu_i M, \]
\[ M(I_N \otimes F_i) = (I_{N-k} \otimes F_i) M = F^\mu_i M, \]
\[ \Gamma V_2(x(t), i, t) = (\mu \otimes I_N) (G^{(m)} \otimes \Gamma_{mi}) \]
\[ + (\mu G_i^{(m)}) \otimes \Gamma_{mi} = (X^{(m)} \otimes \bar{M}) \otimes \Gamma_{mi} \]
\[ = (X^{(m)} \otimes \Gamma_{mi}) (\bar{M} \otimes I_N) \]
\[ = X^{(m)} \otimes \bar{M}, \]
\[ m = 1, 2, 3, \]
\[ M(I_N \otimes L_i) = (I_{N-k} \otimes L_i) M = L^\mu_i M. \]

(32)

Taking \( \Gamma \) as its infinitesimal generator along the trajectory of (8), we obtain the following from Definition 3 and (30)–(31):

\[ \Gamma V(x(t), i, t) = 6 \sum_{k=1}^{\mu} \Gamma V_k(x(t), i, t) = 2 \dot{x}^T(t) M^T P_j M \]
\[ \times [A_i \dot{x}(t) + B_i \dot{\chi}(t - d_i(t)) \]
\[ + C_i \dot{x}(t - \tau_i(t)) + D_i F_i (x(t)) \]
\[ + E_i F_i (x(t - d_i(t))) \]
\[ + F_i F_i (x(t - \tau_i(t))) + H_i \dot{\omega}(t)] \]
\[ + \sum_{j \in S} \gamma_j \dot{x}^T(t) M^T P_j M \dot{x}(t) \]
\[ = 2(M \dot{x}(t))^T P_1 \left(A^\mu_i + \dot{X}^{(m)}_i\right)(M \dot{x}(t)) \]
\[ + 2(M \dot{x}(t))^T P_1 \left(B^\mu_i + \dot{X}^{(2)}_i\right)(M \dot{x}(t)) \]
\[ + 2(M \dot{x}(t))^T P_1 \left(C^\mu_i + \dot{X}^{(3)}_i\right)(M \dot{x}(t)) \]
\[ + 2(M \dot{x}(t))^T P_i D^\mu_i (M F_3 (x(t))) \]
\[ + 2(M \dot{x}(t))^T P_i E^\mu_i (M F_3 (\dot{x}(t - \tau_i(t)))) \]
\[ + 2(M \dot{x}(t))^T P_i F^\mu_i (M F_3 (x(t - d_i(t)))) \]
\[ + 2(M \dot{x}(t))^T P_i H_i \dot{\omega}(t) \]
\[ + (M \dot{x}(t))^T \left(\sum_{j \in S} \gamma_j P_j\right) (M \dot{x}(t)), \]
\[ \Gamma V_2(x(t), i, t) = (M \dot{x}(t))^T \left[Q_1 + Q_4\right] (M \dot{x}(t)) \]
\[ + (M \dot{x}(t - \tau_1))^T \left[Q_2 - Q_1\right] (M \dot{x}(t - \tau_1)) \]
\[ + (M \dot{x}(t - \tau_m))^T \left[Q_3 - Q_2\right] (M \dot{x}(t - \tau_m)) \]
\[ - (M \dot{x}(t - \tau_2))^T Q_3 (M \dot{x}(t - \tau_2)) \]
\[ + (M \dot{x}(t - d_1))^T \left[Q_5 - Q_4\right] (M \dot{x}(t - d_1)) \]
\[ + (M \dot{x}(t - d_m))^T \left[Q_6 - Q_5\right] (M \dot{x}(t - d_m)) \]
\[ - (M \dot{x}(d_2))^T Q_6 (M \dot{x}(t - d_2)), \]
\[
\Gamma V_5 (x(t), i, t) \leq (M \dot{x}(t))^T [R_2 + R_5] (M \dot{x}(t)) \\
+ (M \dot{x}(t - \tau_i))^T [R_1 + R_3 - R_2] (M \dot{x}(t - \tau_i)) \\
- (1 - \tau_i (t)) (M \dot{x}(t - \tau_i (t)))^T R_1 (M \dot{x}(t - \tau_i (t))) \\
- (M \dot{x}(t - \tau_2))^T R_4 (M \dot{x}(t - \tau_2)) \\
+ (M \dot{x}(t - \tau_{mi}))^T [R_4 - R_3] (M \dot{x}(t - \tau_{mi})) \\
+ (M \dot{x}(t - d_{i1}))^T [R_6 - R_5] (M \dot{x}(t - d_{i1})) \\
+ (M \dot{x}(t - d_{mi}))^T [R_7 - R_6] (M \dot{x}(t - d_{mi})) \\
- (M \dot{x}(t - d_{2i}))^T R_7 (M \dot{x}(t - d_{2i})) \\
+ \sum_{j \in S} \int_{t - \tau_j (t)}^{t - \tau_i (t)} \dot{x}^T (s) M^T R_1 M \dot{x}(s) ds \\
\leq (M \dot{x}(t))^T [R_2 + R_5] (M \dot{x}(t)) \\
+ (M \dot{x}(t - \tau_i))^T [R_1 + R_3 - R_2] (M \dot{x}(t - \tau_i)) \\
- (1 - \tau_i (t)) (M \dot{x}(t - \tau_i (t)))^T R_1 (M \dot{x}(t - \tau_i (t))) \\
- (M \dot{x}(t - \tau_2))^T R_4 (M \dot{x}(t - \tau_2)) \\
+ (M \dot{x}(t - \tau_{mi}))^T [R_4 - R_3] (M \dot{x}(t - \tau_{mi})) \\
+ (M \dot{x}(t - d_{i1}))^T [R_6 - R_5] (M \dot{x}(t - d_{i1})) \\
+ (M \dot{x}(t - d_{mi}))^T [R_7 - R_6] (M \dot{x}(t - d_{mi})) \\
- (M \dot{x}(t - d_{2i}))^T R_7 (M \dot{x}(t - d_{2i})) \\
- \int_{t - \tau_{mi}}^{t - \tau_i} \dot{x}^T (s) M^T R_1 M \dot{x}(s) ds \\
\leq (M \dot{x}(t))^T [R_2 + R_5] (M \dot{x}(t)) \\
+ (M \dot{x}(t - \tau_i))^T [R_1 + R_3 - R_2] (M \dot{x}(t - \tau_i)) \\
- (1 - \tau_i (t)) (M \dot{x}(t - \tau_i (t)))^T R_1 (M \dot{x}(t - \tau_i (t))) \\
- (M \dot{x}(t - \tau_2))^T R_4 (M \dot{x}(t - \tau_2)) \\
+ (M \dot{x}(t - \tau_{mi}))^T [R_4 - R_3] (M \dot{x}(t - \tau_{mi})) \\
+ (M \dot{x}(t - d_{i1}))^T [R_6 - R_5] (M \dot{x}(t - d_{i1})) \\
+ (M \dot{x}(t - d_{mi}))^T [R_7 - R_6] (M \dot{x}(t - d_{mi})) \\
- (M \dot{x}(t - d_{2i}))^T R_7 (M \dot{x}(t - d_{2i})) \\
+ \eta \int_{t - \tau}^{t - \tau_i} \dot{x}^T (s) M^T R_1 M \dot{x}(s) ds,
\]

\[
\Gamma V_4 (x(t), i, t) = (M x(t))^T \left[ r_{11} T_1 + d_i^2 T_4 + (\tau_{mi} - \tau_i)^2 T_2 \right] \\
+ (\tau_{2i} - \tau_{mi})^2 T_3 + (d_{mi} - d_{i1})^2 T_5 \\
+ (d_{2i} - d_{mi})^2 T_6 (M x(t)) \\
- \int_{t - \tau_{mi}}^{t - \tau_i} \tau_i x^T (s) M^T T_1 M x(s) ds \\
- \int_{t - \tau_{mi}}^{t - \tau_i} (\tau_{mi} - \tau_i) \dot{x}^T (s) M^T T_2 M x(s) ds \\
- \int_{t - \tau_{mi}}^{t - \tau_i} (\tau_{2i} - \tau_{mi}) \dot{x}^T (s) M^T T_3 M x(s) ds \\
- \int_{t - \tau_{mi}}^{t - \tau_i} (d_{2i} - d_{mi}) \dot{x}^T (s) M^T T_4 M x(s) ds \\
\Gamma V_5 (x(t), i, t) = (M x(t))^T \left[ r_{11} U_1 + d_i^2 U_4 + (\tau_{mi} - \tau_i)^2 U_2 \right] \\
+ (\tau_{2i} - \tau_{mi})^2 U_3 + (d_{mi} - d_{i1})^2 U_5 \\
+ (d_{2i} - d_{mi})^2 U_6 + \eta (\tau - \tau_{mi}) R_1 (M x(t)) \\
- \int_{t - \tau_{mi}}^{t - \tau_i} \tau_i x^T (s) M^T U_1 M x(s) ds \\
- \int_{t - \tau_{mi}}^{t - \tau_i} (\tau_{mi} - \tau_i) \dot{x}^T (s) M^T U_2 M x(s) ds \\
- \int_{t - \tau_{mi}}^{t - \tau_i} (\tau_{2i} - \tau_{mi}) \dot{x}^T (s) M^T U_3 M x(s) ds \\
- \int_{t - \tau_{mi}}^{t - \tau_i} (d_{2i} - d_{mi}) \dot{x}^T (s) M^T U_4 M x(s) ds \\
- \int_{t - \tau_{mi}}^{t - \tau_i} (d_{mi} - d_{i1}) \dot{x}^T (s) M^T U_5 M x(s) ds \\
- \eta \int_{t - \tau}^{t - \tau_i} \dot{x}^T (s) M^T R_1 M x(s) ds,
\]
\[ \Gamma V_6(x(t), i, t) = (M\ddot{x}(t))^T \left[ \begin{array}{l} \frac{\tau_1^4}{4} V_1 + \frac{d_{11}^4}{4} V_4 + \frac{(\tau_{mi}^2 - \tau_{11}^2)^2}{4} V_2 \\
\frac{(\tau_{mi}^2 - \tau_{11}^2)^2}{4} V_3 + \frac{(d_{mi}^2 - d_{11}^2)^2}{4} V_5 \\
\frac{(d_{mi}^2 - d_{11}^2)^2}{4} \end{array} \right] (M\ddot{x}(t)) \]

\[ - \int_{-\tau_i}^{0} \int_{t+\theta}^{t} \frac{\tau_{mi}^2 - \tau_{11}^2}{2} x^T(s) M^T V_1 M\ddot{x}(s) ds d\theta \]

\[ - \int_{-\tau_i}^{0} \int_{t+\theta}^{t} \frac{\tau_{mi}^2 - \tau_{11}^2}{2} x^T(s) M^T V_2 M\ddot{x}(s) ds d\theta \]

\[ - \int_{-\tau_i}^{0} \int_{t+\theta}^{t} \frac{\tau_{mi}^2 - \tau_{11}^2}{2} x^T(s) M^T V_3 M\ddot{x}(s) ds d\theta \]

\[ - \int_{-d_{ti}}^{0} \int_{t+\theta}^{t} \frac{d_{mi}^2 - d_{11}^2}{2} x^T(s) M^T V_4 M\ddot{x}(s) ds d\theta \]

\[ - \int_{-d_{ti}}^{0} \int_{t+\theta}^{t} \frac{d_{mi}^2 - d_{11}^2}{2} x^T(s) M^T V_5 M\ddot{x}(s) ds d\theta \]

\[ > 0 \]

From (33) and (36), we have

\[ \leq e_1 \left[ M F_1(x(t)) \right]^T W_2^{(1)} \]

\[ \times \left[ f_1(x(t)) - f_1(x_{j+1}(t)) \right] \]

\[ \leq e_1 \left[ M F_1(x(t)) \right]^T W_1^{(1)} \left[ f_1(x(t)) - f_1(x_{j+1}(t)) \right] \]

\[ \times \left[ W_1^{(1)} \right] \]

\[ \leq e_1 \left[ M F_1(x(t)) \right]^T [M x(t)] \]

\[ + e_2 \left[ M(x(t)) \right]^T W_1^{(1)} \left[ M F_1(x(t)) \right] \]

\[ - e_1 \left[ M(x(t)) \right]^T W_1^{(1)} [M x(t)] , \]

\[ e_2 [M F_2(x(t-d_i(t)))]^T [M F_2(x(t-d_i(t)))] \]

\[ \leq e_2 \left[ M F_2(x(t-d_i(t))) \right]^T W_2^{(2)} [M x(t-d_i(t))] \]

\[ + e_2 \left[ M(x(t-d_i(t))) \right]^T W_1^{(2)} \left[ M F_2(x(t-d_i(t))) \right] \]

\[ - e_2 \left[ M(x(t-d_i(t))) \right]^T W_1^{(2)} W_2^{(2)} \times \left[ M(x(t-d_i(t))) \right] , \]

\[ e_3 \left[ M F_3(x(t-d_i(t))) \right]^T [M F_3(x(t-d_i(t)))] \]

\[ \leq e_3 \left[ M F_3(x(t-d_i(t))) \right]^T W_2^{(3)} [M x(t-d_i(t))] \]

\[ + e_3 \left[ M(x(t-d_i(t))) \right]^T W_1^{(3)} \left[ M F_3(x(t-d_i(t))) \right] \]

\[ - e_3 \left[ M(x(t-d_i(t))) \right]^T W_1^{(3)} W_2^{(3)} \left[ M(x(t-d_i(t))) \right] . \]

From (33) and (36), we have

\[ \Gamma V(x(t), i, t) \]

\[ \leq \sum_{k=1}^{6} \Gamma V_k(x(t), i, t) \]

\[ - 2e_1 [M F_1(x(t))]^T [M F_1(x(t))] \]

\[ + 2e_1 [M F_1(x(t))]^T W_2^{(1)} (M x(t)) \]
Noticing (a) of Lemma 10, then
\[
- \int_{t-d_1}^{t} \tau_1 x^T(s) \mathbf{M}^T \mathbf{T}_1 \mathbf{M} x(s) \, ds \\
\leq -\xi^T(t) (E_1 - E_3) U_1 \left( E_1 - E_3 \right) \xi(t),
\]
\[
- \int_{t-d_1}^{t} d_{11} x^T(s) \mathbf{M}^T \mathbf{T}_1 \mathbf{M} x(s) \, ds \\
\leq -\xi^T(t) (E_1 - E_{16}) U_4 \left( E_1 - E_{16} \right) \xi(t).
\]
Noticing (b) of Lemma 10, then
\[
- \int_{t-\tau_m}^{t} \frac{\tau_{21}^2 - \tau_{11}^2}{2} x^T(s) \mathbf{M}^T \mathbf{V}_1 \mathbf{M} x(s) \, ds \, d\theta \\
\leq -\xi^T(t) \left( \tau_{11} E_1 - E_{10} \right) V_2 \left( \tau_{11} E_1 - E_{10} \right) \xi(t),
\]
\[
- \int_{t-\tau_m}^{t} \frac{\tau_{21}^2 - \tau_{11}^2}{2} x^T(s) \mathbf{M}^T \mathbf{V}_2 \mathbf{M} x(s) \, ds \, d\theta \\
\leq -\xi^T(t) \left( \tau_{11} - \tau_{11} \right) E_1 - E_{13} \right) \times V_2 \left( \tau_{11} - \tau_{11} \right) E_1 \left( E_1 - E_{13} \right) \xi(t),
\]
\[
- \int_{t-\tau_m}^{t} \tau_{21}^2 x^T(s) \mathbf{M}^T \mathbf{V}_3 \mathbf{M} x(s) \, ds \, d\theta \\
\leq -\xi^T(t) \left[ \left( \tau_{11} - \tau_{11} \right) E_1 - E_{13} \right] \times V_3 \left( \tau_{11} - \tau_{11} \right) E_1 \left( E_1 - E_{13} \right) \xi(t),
\]
\[
- \int_{t-\tau_m}^{t} \tau_{21}^2 \mathbf{x}^T(s) \mathbf{M}^T \mathbf{V}_4 \mathbf{M} \mathbf{x}(s) \, ds \, d\theta \\
\leq -\xi^T(t) \left[ \left( \tau_{11} - \tau_{11} \right) E_1 - E_{14} \right] \times V_5 \left( \tau_{11} - \tau_{11} \right) E_1 \left( E_1 - E_{14} \right) \xi(t),
\]
\[
- \int_{t-\tau_m}^{t} \tau_{21}^2 \mathbf{x}^T(s) \mathbf{M}^T \mathbf{V}_5 \mathbf{M} \mathbf{x}(s) \, ds \, d\theta \\
\leq -\xi^T(t) \left[ \left( \tau_{11} - \tau_{11} \right) E_1 - E_{25} \right] \times V_6 \left( \tau_{11} - \tau_{11} \right) E_1 \left( E_1 - E_{25} \right) \xi(t).
\]
If \( \tau_i(t) \in [\tau_{ji}, \tau_{mi}] \) and \( d_i(t) \in [d_{ji}, d_{mi}] \), let
\[
\lambda_{ji}(t) = \frac{\tau_{ji}(t) - \tau_{ji}}{\tau_{mi} - \tau_{ji}} \quad \kappa_{ji}(t) = \frac{d_{ji} - d_{ji}}{d_{mi} - d_{ji}}.
\]
Then the following is held from (a) of Lemma 10:
\[
- \int_{t-\tau_m}^{t} \tau_{ji} x^T(s) \mathbf{M}^T \mathbf{T}_2 \mathbf{M} x(s) \, ds \\
\leq -\xi^T(t) \left[ \left( \tau_{11} - \tau_{11} \right) E_1 - E_{25} \right] \times V_6 \left( \tau_{11} - \tau_{11} \right) E_1 \left( E_1 - E_{25} \right) \xi(t).
\]
(39)
(40)
Similarly,
\[ -\int_{t_{\tau_{mi}}}^{t_{\tau_{mi}}} (\tau_{mi} - \tau_{ni}) \tilde{x}(s) M^T T_3 M \tilde{x}(s) \, ds \]
\[ = -\left( \int_{t_{\tau_{mi}}}^{t_{\tau_{ni}}} + \int_{t_{\tau_{mi}}}^{t_{\tau_{ni}}} \right) \left( \tau_{mi} - \tau_{ni} \right) \tilde{x}(s) M^T T_2 M \tilde{x}(s) \, ds \]
\[ \leq -\xi^T(t) (E_3 - E_2) U_2 \left( E_3^T - E_2^T \right) \xi(t) \]
\[ - \left( 1 - \lambda_{1i}(t) \right) \tilde{x}(t) (E_3 - E_2) U_2 \left( E_3^T - E_2^T \right) \xi(t) \]
\[ - \xi^T(t) (E_2 - E_4) U_2 \left( E_2^T - E_4^T \right) \xi(t) \]
\[ - \lambda_{1i}(t) \tilde{x}(t) (E_2 - E_4) U_2 \left( E_2^T - E_4^T \right) \xi(t) , \]
\[ -\int_{t_{d_{mi}}}^{t_{d_{mi}}} (d_{mi} - d_{ni}) \tilde{x}(s) M^T T_3 M \tilde{x}(s) \, ds \]
\[ = -\left( \int_{t_{d_{mi}}}^{t_{d_{ni}}} + \int_{t_{d_{mi}}}^{t_{d_{ni}}} \right) \left( d_{mi} - d_{ni} \right) \tilde{x}(s) M^T T_3 M \tilde{x}(s) \, ds \]
\[ \leq -\xi^T(t) (E_{12} - E_{15}) U_5 \left( E_{15}^T - E_{12}^T \right) \xi(t) \]
\[ - \left( 1 - \kappa_{1i}(t) \right) \tilde{x}(t) (E_{12} - E_{15}) U_5 \left( E_{15}^T - E_{12}^T \right) \xi(t) \]
\[ - \xi^T(t) (E_{15} - E_{17}) U_5 \left( E_{17}^T - E_{15}^T \right) \xi(t) , \]
\[ -\lambda_{1i}(t) \tilde{x}(t) (E_{15} - E_{17}) U_5 \left( E_{17}^T - E_{15}^T \right) \xi(t) \]
\[ \times (E_{15} - E_{17}) U_5 \left( E_{17}^T - E_{15}^T \right) \xi(t) . \]
(42)

Considering
\[ -\int_{t_{\tau_{mi}}}^{t_{\tau_{mi}}} (\tau_{mi} - \tau_{ma}) \tilde{x}(s) M^T T_3 M \tilde{x}(s) \, ds, \]
\[ -\int_{t_{\tau_{mi}}}^{t_{\tau_{mi}}} (\tau_{ma} - \tau_{ma}) \tilde{x}(s) M^T U_3 M \tilde{x}(s) \, ds, \]
\[ -\int_{t_{d_{mi}}}^{t_{d_{mi}}} (d_{ma} - d_{ma}) \tilde{x}(s) M^T T_3 M \tilde{x}(s) \, ds, \]
\[ -\int_{t_{d_{mi}}}^{t_{d_{mi}}} (d_{ma} - d_{ma}) \tilde{x}(s) M^T U_3 M \tilde{x}(s) \, ds, \]
we have
\[ -\int_{t_{\tau_{mi}}}^{t_{\tau_{mi}}} (\tau_{mi} - \tau_{ma}) \tilde{x}(s) M^T T_3 M \tilde{x}(s) \, ds \]
\[ \leq -\xi^T(t) E_{14} T_3 E_{14} \xi(t) , \]
\[ -\int_{t_{\tau_{mi}}}^{t_{\tau_{mi}}} (\tau_{ma} - \tau_{ma}) \tilde{x}(s) M^T U_3 M \tilde{x}(s) \, ds \]
\[ \leq -\xi^T(t) E_{44} E_{14} \xi(t) , \]
\[ -\int_{t_{d_{mi}}}^{t_{d_{mi}}} (d_{ma} - d_{ma}) \tilde{x}(s) M^T T_3 M \tilde{x}(s) \, ds \]
\[ \leq -\xi^T(t) E_{24} E_{24} \xi(t) , \]
\[ -\int_{t_{d_{mi}}}^{t_{d_{mi}}} (d_{ma} - d_{ma}) \tilde{x}(s) M^T U_3 M \tilde{x}(s) \, ds \]
\[ \leq -\xi^T(t) E_{18} E_{18} \xi(t) . \]
(43)

In addition, according to (8), we know that \( M \tilde{x}(t) = \Lambda \xi(t) \) and
\[ \left[ M \tilde{x}(t) \right]^T J \left[ M \tilde{x}(t) \right] = \xi(t) \Lambda^T \Lambda \xi(t) , \]
(45)
where \( \Lambda \) and \( J \) have been defined in Theorem 12.
From (3.1) and (37)–(45), we obtain
\[ \Gamma V(x(t), i, t) + \|Mz(t)\|^2 - \delta^2 \|\omega(t)\|^2 \]
\[ \leq \sum_{k=1}^{n} \Gamma V_k(x(t), i, t) - 2\epsilon_1 [M F_1 (x(t))]^T [M F_1 (x(t))] + 2\epsilon_1 [M F_1 (x(t))]^T W_1 (Mz(t)) + 2\epsilon_1 [Mz(t)]^T W_1 (Mz(t)) \]
\[ \times W_1 (Mz(t)) = [M F_2 (x(t - d_i(t)))]^T [M F_2 (x(t - d_i(t)))] + 2\epsilon_2 [M F_2 (x(t - d_i(t)))]^T W_1 (Mz(t) - d_i(t)) \]
\[ + 2\epsilon_2 (Mx(t - d_i(t)))^T W_1 (Mx(t - d_i(t))) + 2\epsilon_2 (Mx(t - d_i(t)))^T W_1 (Mx(t - d_i(t))) \]
\[ \times Mx(t - d_i(t)) - 2\epsilon_2 [MF_2 (x(t - d_i(t)))]^T \]
\[ - 2\epsilon_2 [Mz(t - d_i(t))] W_2 (Mx(t - d_i(t))) \]
\[ - 2\epsilon_2 [Mz(t - d_i(t))] W_2 (Mx(t - d_i(t))) \]
\[ \leq \xi^T(t) \left[ \lambda_{1i}(t) \Omega_{1i} + (1 - \lambda_{1i}(t)) \Omega_{1i} + \Theta_1 \right] \xi(t) , \]
\[ + \xi^T(t) \left[ \kappa_{1i}(t) \Pi_{1i} + (1 - \kappa_{1i}(t)) \Pi_{1i} + \Theta_1 \right] \xi(t) . \]
(46)
For \( \tau_i(t) \in [\tau_{mi}, \tau_2] \) and \( d_i(t) \in [d_{mi}, d_2] \), let
\[
\lambda_2(t) = \frac{\tau_i(t) - \tau_{mi}}{\tau_2 - \tau_{mi}}, \quad \kappa_2(t) = \frac{d_i(t) - d_{mi}}{d_2 - d_{mi}}.
\] (47)

Then, following the above procedure, we can obtain
\[
\Gamma V(x(t), i, t) + \|Mz(t)\|^2 - \delta^2 \|\omega(t)\|^2
\leq \xi^T(t) \left[ \lambda_2(t) \Omega_1 + (1 - \lambda_2(t)) \Omega_4 + \frac{\Theta_0}{2} \right] \xi(t)
+ \xi^T(t) \left[ \kappa_2(t) \Pi_1 + (1 - \kappa_2(t)) \Pi_2 + \frac{\Theta_2}{2} \right] \xi(t).
\] (48)

For other situations, where \( \tau_i(t) \in [\tau_{mi}, \tau_2] \), \( d_i(t) \in [d_{mi}, d_2] \), and \( \tau_i(t) \in [\tau_{mi}, \tau_2] \), \( d_i(t) \in [d_{mi}, d_2] \), we derive (49) and (50), respectively, as
\[
\Gamma V(x(t), i, t) + \|Mz(t)\|^2 - \delta^2 \|\omega(t)\|^2
\leq \xi^T(t) \left[ \lambda_2(t) \Omega_1 + (1 - \lambda_2(t)) \Omega_4 + \frac{\Theta_0}{2} \right] \xi(t)
+ \xi^T(t) \left[ \kappa_2(t) \Pi_1 + (1 - \kappa_2(t)) \Pi_2 + \frac{\Theta_2}{2} \right] \xi(t).
\] (49)

Therefore, with (46), (48), (49), and (50), by Lemma II, the following inequality (51) is held for \( \tau_i(t) \in [\tau_{mi}, \tau_2] \) and \( d_i(t) \in [d_{mi}, d_2] \), if (25) is satisfied:
\[
\Gamma V(x(t), i, t) + \|Mz(t)\|^2 - \delta^2 \|\omega(t)\|^2 < 0.
\] (51)

If (26) is held, integrating the function in (51) from 0 to \( \infty \), we have
\[
\int_0^\infty \|Mz(t)\|^2 dt < \delta^2 \int_0^\infty \|\omega(t)\|^2 dt + \mathcal{V}(0)
\leq \delta^2 \left( \int_0^\infty \|\omega(t)\|^2 dt + x^T(0) \mathcal{Y} x(0) \right).
\] (52)

By Definition 8, the NCDNs (3) and (4) can reach \( H_{\infty} \) cluster synchronization with a disturbance attenuation \( \delta \). This completes the proof.

Remark 13. It should be mentioned that the proposed Lyapunov functional contains some triple-integral terms. Compared with the existing ones, [39, 42] have shown that such a Lyapunov functional type is very effective in the reduction of conservatism. Besides, the information on the lower bound of the delay is sufficiently used by introducing the integral terms on \([t - \tau_i(t), t - \tau_{ti}], [t - \tau_i(t), t - \tau_{mi}], [t - \tau_{mi}, t - \tau_{ti}]\) and \([t - d_i(t), t - d_{ti}], [t - d_i(t), t - d_{mi}], [t - d_{mi}, t - d_{ti}]\).
\[ \bar{\Lambda} = \left( A_\theta + X_1^{(1)} \right) E_1^T + \left( B_\theta + X_2^{(2)} \right) E_2^T + D_\theta^T E_4^T \]
\[ + E_7^T E_{15} + M \otimes E_{16}, \]
\[ \bar{\pi} = E_1 \left( p_1 D_\theta^\phi + e_1 W_1^{(1)\phi} + e_1 W_2^{(2)\phi} \right) E_{14}^T + E_1 p_1 \left( D_\theta^\phi E_{15}^T \right) \]
\[ + E_1 p_1 \otimes E_{16} + E_2 \left( e_2 W_1^{(1)\phi} + e_2 W_2^{(2)\phi} \right) E_{15}^T, \]
\[ \bar{\mathcal{V}}_1 = \mathcal{L} \left[ p_1 \left( A_\theta^\phi + B_\theta^\phi + X_1^{(1)\phi} + X_2^{(2)\phi} \right) - e_1 W_1^{(1)\phi} W_2^{(2)\phi} \right] \]
\[ + \sum_{j \in S} y_j p_j + Q_4 + d_{1j}^2 T_4 + (d_{mj} - d_{1j}) T_5 \]
\[ + (d_{2j} - d_{mj}) T_6 + k_{ij}^\phi L_i^\phi, \]
\[ \bar{\Phi}_4 = -e_2 W_1^{(2)\phi} W_2^{(2)\phi}, \quad \bar{\Phi}_3 = Q_5 - Q_4, \]
\[ \bar{\Phi}_4 = Q_6 - Q_5, \quad \bar{\Phi}_5 = -Q_6, \]
\[ \bar{\Phi}_6 = R_6 - R_5, \quad \bar{\Phi}_7 = R_7 - R_6, \quad \bar{\Phi}_8 = -R_7, \]
\[ \bar{\Phi}_{14} = -2e_1 I, \quad \bar{\Phi}_{15} = -2e_2 I, \]
\[ \bar{\Phi}_{16} = -d_2 I, \quad \bar{\Phi}_m = 0, \quad (m = 9, 10, 11, 12, 13), \]
\[ \bar{\Pi}_1 = -E_{10} T_5 E_{10}^T - 2 \left( E_{12} - E_{10} \right) T_5 \left( E_{12}^T - E_{10}^T \right) \]
\[ - \left( E_3 - E_2 \right) U_5 \left( E_3^T - E_2^T \right) - 2 \left( E_2 - E_4 \right) \]
\[ \times U_5 \left( E_4^T - E_5 \right) - E_{13} T_6 E_{13}^T - \left( E_4 - E_5 \right) \]
\[ \times U_6 \left( E_6^T - E_5 \right), \]
\[ \bar{\Pi}_2 = -2E_{10} T_5 E_{10}^T - \left( E_{12} - E_{10} \right) T_5 \left( E_{12}^T - E_{10}^T \right) \]
\[ - 2 \left( E_3 - E_2 \right) U_5 \left( E_3^T - E_2^T \right) - \left( E_2 - E_4 \right) \]
\[ \times U_5 \left( E_4^T - E_5 \right) - E_{13} T_6 E_{13}^T - \left( E_4 - E_5 \right) \]
\[ \times U_6 \left( E_6^T - E_5 \right), \]
\[ \bar{\Pi}_3 = -2E_{10} T_6 E_{10}^T - \left( E_{13} - E_{11} \right) T_6 \left( E_{13}^T - E_{11}^T \right) \]
\[ - \left( E_4 - E_2 \right) U_6 \left( E_4^T - E_2^T \right) - 2 \left( E_2 - E_5 \right) \]
\[ \times U_6 \left( E_5^T - E_6 \right) - E_{12} T_5 E_{12}^T - \left( E_5 - E_4 \right) \]
\[ \times U_5 \left( E_5^T - E_4 \right), \]
\[ \bar{\Pi}_4 = -E_{11} T_6 E_{11}^T - 2 \left( E_{13} - E_{11} \right) T_6 \left( E_{13}^T - E_{11}^T \right) \]
\[ - 2 \left( E_4 - E_2 \right) U_6 \left( E_4^T - E_2^T \right) - \left( E_2 - E_5 \right) \]
\[ \times U_6 \left( E_5^T - E_6 \right) - E_{15} T_6 E_{15}^T - \left( E_5 - E_4 \right) \]
\[ \times U_5 \left( E_5^T - E_4 \right). \]

Other notations are the same as those in Theorem 12.

**Proof.** Since \( \tau_i(t) \equiv 0 \), we choose the Lyapunov functional as follows:

\[ \nabla \left( x(t), i, t \right) = V_1 \left( x(t), i, t \right) + \sum_{k=2} \nabla_k \left( x(t), i, t \right), \]

where

\[ \nabla_2 \left( x(t), i, t \right) = \int_{t-d_i}^t x^T \left( s \right) M^T Q_4 M x \left( s \right) ds \]

\[ + \int_{t-d_{1i}}^{t-d_i} x^T \left( s \right) M^T Q_5 M x \left( s \right) ds \]

\[ + \int_{t-d_{mi}}^{t-d_{mi}} x^T \left( s \right) M^T Q_6 M x \left( s \right) ds, \]

\[ \nabla_3 \left( x(t), i, t \right) = \int_{t-d_i}^t \dot{x}^T \left( s \right) M^T R_3 M \dot{x} \left( s \right) ds \]

\[ + \int_{t-d_{1i}}^{t-d_i} \dot{x}^T \left( s \right) M^T R_5 M \dot{x} \left( s \right) ds \]

\[ + \int_{t-d_{mi}}^{t-d_{mi}} \dot{x}^T \left( s \right) M^T R_7 M \dot{x} \left( s \right) ds, \]

\[ \nabla_4 \left( x(t), i, t \right) = \int_{-d_{1i}}^0 \int_{t+\theta}^t d_{1i} x^T \left( s \right) M^T T_3 M x \left( s \right) ds d\theta \]

\[ + \int_{-d_{mi}}^{0} \int_{t+\theta}^t \left( d_{mi} - d_{1i} \right) x^T \left( s \right) M^T \]

\[ \times T_3 M x \left( s \right) ds d\theta \]

\[ + \int_{-d_{mi}}^{0} \int_{t+\theta}^t \left( d_{mi} - d_{1i} \right) x^T \left( s \right) M^T \]

\[ \times T_6 M x \left( s \right) ds d\theta, \]
\[ V_5 (\alpha (t), i, t) = \int_{-d_{1i}}^{0} \int_{t+\theta}^{t} \left[ \left( d_{mi} - d_{1i} \right) \dot{x}(s) \right] M^T \bar{U}_4 \dot{\bar{M}}(s) \, ds \, d\theta \]
\[ + \int_{-d_{mi}}^{-d_{1i}} \int_{t+\theta}^{t} \left[ \left( d_{2i} - d_{mi} \right) \dot{x}(s) \right] M^T \times U_4 \dot{\bar{M}}(s) \, ds \, d\theta, \]
\[ \nabla_6 (\alpha (t), i, t) = \int_{-d_{1i}}^{0} \int_{t+\lambda}^{t} \left[ \left( d_{2i} - d_{mi} \right) \dot{x}(s) \right] M^T \times U_6 \dot{\bar{M}}(s) \, ds \, d\lambda \, d\theta. \]

(58)

And we define
\[ \bar{\xi} (t) = \text{col} \left\{ \bar{M} \dot{x} (t), \bar{M} \dot{x} (t - d_i (t)), \bar{M} \dot{x} (t - d_{1i}), \bar{M} \dot{x} (t - d_{mi}), \bar{M} \dot{x} (t - d_{2i}), M \int_{t-d_{1i}}^{t-d_{1i}} x(s) \, ds M \int_{t-d_{mi}}^{t-d_{mi}} x(s) \, ds \right\}. \]

(59)

Then we follow a similar line as in proof of Theorem 12 and obtain the result.

\[ \Gamma_{1i} = \Gamma_{2i} = \Gamma_{3i} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad L_i = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad i \in S = \{1, 2\}, \]

(60)

4. Numerical Examples

In this section, numerical examples are presented to demonstrate the effectiveness of the developed design on \( H_{\infty} \) cluster synchronization.

Example 1. A four-node NCDN (3) and (4) with Markovian switching between two modes is taken into consideration;

\[ A_1 = \begin{bmatrix} -0.40 & -0.15 \\ 0.10 & -0.60 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.30 & 0.09 \\ 0.20 & -0.40 \end{bmatrix}, \]
\[ B_1 = \begin{bmatrix} 0.20 & -0.15 \\ 0.50 & -0.50 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.31 & 0.23 \\ -0.12 & 0.17 \end{bmatrix}, \]
\[ C_1 = \begin{bmatrix} 0.28 & 0.02 \\ 0 & 0.50 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.51 & 0.24 \\ 0 & -0.44 \end{bmatrix}, \]
\[ D_1 = \begin{bmatrix} 0.20 & 0 \\ 0 & 0.15 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.30 & 0 \\ 0 & -0.10 \end{bmatrix}, \]
\[ E_i = F_i = 0, \quad (i = 1, 2), \]

(60)

The transition rate matrix is considered as follows:

\[ \Omega = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}. \]

(61)
Furthermore, as a result of $E_i = F_i = 0$, only the nonlinear function $f_1(x_k(t))$ is effective and given as

$$f_1(x_k(t)) = \begin{bmatrix} 0.5x_{k1}(t) - \tanh(0.2x_{k1}(t)) + 0.2x_{k2}(t) & 0.95x_{k2}(t) - \tanh(0.75x_{k2}(t)) \end{bmatrix}^T.$$  

(62)

Then, it is easy to verify that

$$W_{1}^{(1)} = \begin{bmatrix} 0.3 & 0.2 \\ 0 & 0.3 \end{bmatrix}, \quad W_{2}^{(1)} = \begin{bmatrix} 0.5 & 0.2 \\ 0 & 0.95 \end{bmatrix}. \quad (63)$$

The interval mode-dependent time-varying neutral delays and discrete delays are, respectively, assumed to be

$$\tau_1(t) = 0.5 \left(1 + \sin^3(t)\right), \quad \tau_2(t) = 0.5 \left(1 + \cos^3(t)\right),$$
$$d_1(t) = 0.1 + |\sin t|, \quad d_2(t) = 0.1 + |\cos t|. \quad (64)$$

They are governed by the Markov process $\{r(t), t \geq 0\}$ and shown in Figures 1 and 2. It can be readily obtained that

$$\tau_{11} = 0, \quad \tau_{21} = 1; \quad \tau_{12} = 0, \quad \tau_{22} = 1; \quad \nu_1 = \nu_2 = \frac{\sqrt{3}}{3};$$
$$d_{11} = 0.1, \quad d_{21} = 1.1; \quad d_{12} = 0.1, \quad d_{22} = 1.1. \quad (65)$$
\(H_\infty\) cluster synchronization of this NCDN based on the above criterion is tested. Choose \(\tau_{m1} = 0.2\), \(\tau_{m2} = 0.3\), \(d_{m1} = 0.4\), \(d_{m2} = 0.5\), and the initial conditions

\[
x_1(s) = \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}, \quad x_2(s) = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \\
\]

\[
x_3(s) = \begin{bmatrix} 0.3 \\ -0.3 \end{bmatrix}, \quad x_4(s) = \begin{bmatrix} 0.3 \\ -0.2 \end{bmatrix},
\]

\(s \in [-\zeta, 0]\).

Let the disturbance attenuation level \(\delta = 0.5\), and let the initial positive definite matrix \(Y = 3I_6\). With Theorem 12, by using the Matlab LMI Toolbox, a group of matrices as a feasible solution can be obtained in the following (for simplicity, we only list the matrices for \(P_i\) and \(Q_i\), \(i \in S, j = 1, 2, \ldots, 6\)):

\[
P_1 = \begin{bmatrix}
1.7802 & 0.0659 & -0.0047 & 0.0028 \\
* & 1.0304 & 0.0012 & -0.0015 \\
* & * & 1.8546 & 0.0326 \\
* & * & * & 1.1325
\end{bmatrix},
\]

\[
P_2 = \begin{bmatrix}
1.6372 & 0.1644 & -0.0042 & 0.0040 \\
* & 1.4526 & 0.0033 & -0.0025 \\
* & * & 1.3748 & 0.0727 \\
* & * & * & 1.2369
\end{bmatrix},
\]

\[
Q_1 = \begin{bmatrix}
2.3589 & 0.0467 & -0.0019 & 0.0016 \\
* & 3.1324 & 0.0016 & -0.0015 \\
* & * & 2.1046 & 0.4326 \\
* & * & * & 3.1433
\end{bmatrix},
\]

\[
Q_2 = \begin{bmatrix}
3.0811 & 0.0259 & -0.0034 & 0.0027 \\
* & 3.3245 & 0.0029 & -0.0038 \\
* & * & 3.6435 & 0.0037 \\
* & * & * & 3.1046
\end{bmatrix},
\]

\[
Q_3 = \begin{bmatrix}
2.3042 & 0.1654 & -0.0003 & 0.0002 \\
* & 1.6345 & 0.0018 & -0.0014 \\
* & * & 1.8673 & 0.0756 \\
* & * & * & 1.0564
\end{bmatrix},
\]

\[
Q_4 = \begin{bmatrix}
2.3632 & 0.0735 & -0.0011 & 0.0007 \\
* & 2.0411 & 0.0134 & -0.0001 \\
* & * & 1.7745 & 0.0542 \\
* & * & * & 1.0643
\end{bmatrix},
\]

\[
Q_5 = \begin{bmatrix}
3.1822 & -0.0453 & -0.0003 & -0.0005 \\
* & 3.3314 & 0 & 0 \\
* & * & 3.2446 & -0.0443 \\
* & * & * & 3.0418
\end{bmatrix},
\]

\[
Q_6 = \begin{bmatrix}
2.6433 & -0.0059 & -0.0050 & 0.0042 \\
* & 2.3074 & 0.0014 & -0.0005 \\
* & * & 2.0435 & 0.0926 \\
* & * & * & 1.8663
\end{bmatrix},
\]

(67)

It can be concluded that this neutral complex dynamical network (NCDN) has achieved \(H_\infty\) cluster synchronization, which illustrates the effectiveness of Theorem 12.

\textbf{Example 2.} Particularly, consider \(\tau_j(t) \equiv 0\) in Example 1 and other elements are identical with Example 1. With Corollary 15, by utilizing Matlab LMI Toolbox, the LMIs (54) can be solved. Then a group of matrices as a feasible solution can be obtained as follows (for simplicity, we only list the matrices for \(P_i\) and \(Q_i\), \(i \in S, j = 4, 5, 6\)):

\[
P_4 = \begin{bmatrix}
1.5433 & 0.0049 & -0.0006 & 0.0003 \\
* & 1.0241 & 0.0018 & -0.0017 \\
* & * & 1.0327 & 0.0034 \\
* & * & * & 1.0065
\end{bmatrix},
\]

\[
P_5 = \begin{bmatrix}
1.5638 & 0.1536 & -0.0032 & 0.0028 \\
* & 1.3674 & 0.0026 & -0.0011 \\
* & * & 1.2655 & 0.0424 \\
* & * & * & 1.0258
\end{bmatrix},
\]

\[
P_6 = \begin{bmatrix}
2.1844 & 0.0632 & -0.0009 & 0.0006 \\
* & 2.0087 & 0.0136 & -0.0001 \\
* & * & 1.7549 & 0.0466 \\
* & * & * & 1.0557
\end{bmatrix},
\]

\[
Q_4 = \begin{bmatrix}
3.1756 & -0.0346 & -0.0003 & -0.0004 \\
* & 3.3267 & 0 & 0 \\
* & * & 3.2338 & -0.0365 \\
* & * & * & 3.0344
\end{bmatrix},
\]

\[
Q_5 = \begin{bmatrix}
2.6368 & -0.0047 & -0.0028 & 0.0035 \\
* & 2.2866 & 0.0006 & -0.0003 \\
* & * & 2.0337 & 0.0677 \\
* & * & * & 1.8359
\end{bmatrix},
\]

(68)

It also can be proved that the complex dynamical network (CDN) has achieved \(H_\infty\) cluster synchronization, which verifies the effectiveness of Corollary 15.

\textbf{5. Conclusions}

In this paper, \(H_\infty\) cluster synchronization of neutral complex dynamical networks with Markovian switching is considered for the first time. By interval mode-dependent delays dividing, a new augmented Lyapunov functional containing some triple-integral terms is constructed to reduce conservativeness. Then the delay-range-dependent \(H_\infty\) cluster synchronization criteria are obtained by the Lyapunov stability theory, integral matrix inequalities, and convex combination. Finally, numerical examples are given to illustrate the feasibility and effectiveness of the proposed result.

\textbf{Conflict of Interests}

The author declares that there is no conflict of interests regarding the publication of this paper.

\textbf{Acknowledgments}

The author would like to thank the associate editor and the anonymous reviewers for their constructive comments and suggestions to improve the quality of the paper. This work was...
supported in part by the Fundamental Research Funds of the Central Universities.

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