Research Article

Coefficient Inequalities for a Subclass of p -Valent Analytic Functions

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The aim of this paper is to study the problem of coefficient bounds for a newly defined subclass of p -valent analytic functions. Many known results appear as special consequences of our work.

1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (p \in \mathbb{N}), \quad (1)$$

which are analytic and multivalent in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. Also let $\mathcal{S}_p^*$ and $\mathcal{K}_p$ denote the well-known classes of p -valent starlike functions and p -valent convex functions, respectively.

For $f(z) \in \mathcal{A}(p)$ given by (1) and $g(z) \in \mathcal{A}(p)$ given by

$$g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n, \quad (p \in \mathbb{N}), \quad (2)$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n, \quad (z \in \mathbb{U}). \quad (3)$$

Motivated by Ruscheweyh operator [1], Goel and Sohi [2] introduced a differential operator $D^{\delta+p-1}$ for p -valent analytic functions given by

$$D^{\delta+p-1} f(z) = \frac{z}{(1-z)^{\delta+p}} * f(z) = z^p + \sum_{n=p+1}^{\infty} \varphi_n(\delta) a_n z^n, \quad (4)$$

with $\delta > -p$,

$$\varphi_n(\delta) = \frac{(\delta+p)_{n-p}}{(n-p)!}, \quad (5)$$

and $(x)_n$ is a Pochhammer symbol given by

$$(x)_n = \begin{cases} 1, & n = 0, \\ x(x+1)(x+2)\cdots(x+n-1), & n \in \mathbb{N}. \end{cases} \quad (6)$$

It is obvious that when δ is any integer greater than $-p$,

$$D^{\delta+p-1} f(z) = \frac{z^p (z^{\delta-1} f(z))^{\delta+p-1}}{(\delta+p-1)!}. \quad (7)$$

The following identity can be easily established:

$$(\delta+p) D^{\delta+p} f(z) = \delta D^{\delta+p-1} f(z) + z (D^{\delta+p-1} f(z))'. \quad (8)$$

Using the generalized Ruscheweyh operator, we define a subclass of $\mathcal{V}\mathcal{D}_p^\lambda(\delta, b, \alpha, \beta)$ of p -valent analytic functions as follows.

Definition 1. An analytic p -valent function $f(z)$ of the form (1) belongs to the class $\mathcal{V}\mathcal{D}_p^\lambda(\delta, b, \alpha, \beta)$, if and only if

$$\operatorname{Re} \left\{ e^{i\lambda} \left(1 - \frac{2}{b} + \frac{2}{b} \frac{D^{\delta+p} f(z)}{D^{\delta+p-1} f(z)} \right) \right\} > \alpha \left| \frac{2}{b} \left(\frac{D^{\delta+p} f(z)}{D^{\delta+p-1} f(z)} - 1 \right) \right| + \beta \cos \lambda, \quad (9)$$

where $\alpha \geq 0$, $b \in \mathbb{C} \setminus \{0\}$, $\delta > -p$, λ is real with $|\lambda| < (\pi/2)$, and $0 \leq \beta < 1$.

By giving specific values to α , β , λ , p , b , and δ in $\mathcal{V}\mathcal{D}_p^\lambda(\delta, b, \alpha, \beta)$, we obtain many important subclasses studied by various authors in earlier papers; see for details [3–6]; we list some of them as follows:

- (i) $\mathcal{V}\mathcal{D}_1^\lambda(0, 2, 0, 0) \equiv \mathcal{S}_\lambda^*$ and $\mathcal{V}\mathcal{D}_1^\lambda(1, 1, 0, 0) \equiv \mathcal{K}_\lambda$, studied by Spacek [7] and Robertson [8], respectively; for the advancement work see [9–11];
- (ii) $\mathcal{V}\mathcal{D}_1^0(0, 2, \alpha, \beta) \equiv \mathcal{SD}(\alpha, \beta)$ and $\mathcal{V}\mathcal{D}_1^0(1, 1, \alpha, \beta) \equiv \mathcal{K}(\alpha, \beta)$, studied by both Owa et al. and Shams et al. [12, 13];
- (iii) $\mathcal{V}\mathcal{D}_1^\lambda(0, 2, 1, 0) \equiv \mathcal{USP}(\lambda)$ and $\mathcal{V}\mathcal{D}_1^\lambda(1, 1, 1, 0) \equiv \mathcal{UCSP}(\lambda)$, introduced by Ravichandran et al. [14];
- (iv) $\mathcal{V}\mathcal{D}_1^0(\delta, b, \alpha, \beta) \equiv \mathcal{VD}(\delta, b, \alpha, \beta)$, considered by Latha [15];
- (v) $\mathcal{V}\mathcal{D}_1^0(0, 2, 0, \beta) \equiv \mathcal{S}^*(\beta)$ and $\mathcal{V}\mathcal{D}_1^0(1, 1, 0, \beta) \equiv \mathcal{K}(\beta)$, the well-known classes of starlike and convex functions of order β .

From the above special cases we note that this class provides a continuous passage from the class of starlike functions to the class of convex functions.

We will assume throughout our discussion, unless otherwise stated, that $\alpha \geq 0$, $0 \leq \beta < 1$, $\delta > -1$, λ is real with $|\lambda| < (\pi/2)$, and $b \in \mathbb{C} \setminus \{0\}$.

2. Main Results

Theorem 2. Let $f(z) \in \mathcal{V}\mathcal{D}_p^\lambda(\delta, b, \alpha, \beta)$ with $0 \leq \alpha \leq \beta$. Then $f(z) \in \mathcal{V}\mathcal{D}_p^\lambda(\delta, b, 0, \zeta)$, where $\zeta = (\beta - \alpha)/(1 - \alpha)$.

Proof. Let $f(z) \in \mathcal{V}\mathcal{D}_p^\lambda(\delta, b, \alpha, \beta)$. Then we obtain

$$\begin{aligned} \operatorname{Re} e^{i\lambda} \left\{ 1 - \frac{2}{b} + \frac{2}{b} \frac{D^{\delta+p} f(z)}{D^{\delta+p-1} f(z)} \right\} &> \alpha \left| \frac{2}{b} \left(\frac{D^{\delta+p} f(z)}{D^{\delta+p-1} f(z)} - 1 \right) \right| + \beta \cos \lambda \\ &> \alpha \operatorname{Re} e^{i\lambda} \left(1 - \frac{2}{b} + \frac{2}{b} \frac{D^{\delta+p} f(z)}{D^{\delta+p-1} f(z)} \right) \\ &\quad - \alpha \operatorname{Re} e^{i\lambda} + \beta \cos \lambda, \end{aligned} \quad (10)$$

and this implies

$$\operatorname{Re} \left[e^{i\lambda} \left(1 - \frac{2}{b} + \frac{2}{b} \frac{D^{\delta+p} f(z)}{D^{\delta+p-1} f(z)} \right) \right] > \zeta \cos \lambda. \quad (11)$$

Also if $0 \leq \alpha \leq \beta$, then we can easily obtain

$$0 \leq \zeta < 1, \quad (12)$$

and this completes the proof. $\square$

Theorem 3. If $f(z) \in \mathcal{V}\mathcal{D}_p^\lambda(\delta, b, \alpha, \beta)$, then

$$|a_2| \leq \frac{|b| |\eta|}{|1 - \alpha|}, \quad (13)$$

and

$$\begin{aligned} |a_{n+p-1}| &\leq \frac{(\delta + p) |b| |\eta|}{(n-1) |1 - \alpha| \varphi_{n+p-1}(\delta)} \\ &\quad \times \prod_{j=1}^{n-2} \left(1 + \frac{(\delta + p) |b| |\eta|}{j |1 - \alpha|} \right), \quad n \geq 3, \end{aligned} \quad (14)$$

where $\varphi_{n+p-1}(\delta)$ is given by (5) and

$$\eta = (1 - \beta) \cos \lambda + i(1 - \alpha) \sin \lambda. \quad (15)$$

Proof. Let $f(z) \in \mathcal{V}\mathcal{D}_p^\lambda(\delta, b, \alpha, \beta)$. Then by Theorem 2, we have

$$\begin{aligned} \operatorname{Re} \left[e^{i\lambda} \left(1 - \frac{2}{b} + \frac{2}{b} \frac{D^{\delta+p} f(z)}{D^{\delta+p-1} f(z)} \right) \right] &> \left(\frac{\beta - \alpha}{1 - \alpha} \right) \cos \lambda, \quad (z \in E). \end{aligned} \quad (16)$$

Let us define $p(z)$ by

$$\begin{aligned} e^{i\lambda} \left(1 - \frac{2}{b} + \frac{2}{b} \frac{D^{\delta+p} f(z)}{D^{\delta+p-1} f(z)} \right) &= \left[\left(\frac{1 - \beta}{1 - \alpha} \right) p(z) + \left(\frac{\beta - \alpha}{1 - \alpha} \right) \right] \cos \lambda + i \sin \lambda. \end{aligned} \quad (17)$$

Then $p(z)$ is analytic in E with $p(0) = 1$ and $\operatorname{Re} p(z) > 0$, $z \in E$. Let

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad z \in E. \quad (18)$$

Then (17) becomes

$$\begin{aligned} 1 - \frac{2}{b} + \frac{2}{b} \frac{D^{\delta+p} f(z)}{D^{\delta+p-1} f(z)} \\ = 1 + \frac{(1-\beta) \cos \lambda + i(1-\alpha) \sin \lambda}{e^{i\lambda}(1-\alpha)} \sum_{n=1}^{\infty} p_n z^n. \end{aligned} \quad (19)$$

That is,

$$\begin{aligned} 2e^{i\lambda}(1-\alpha) [D^{\delta+p} f(z) - D^{\delta+p-1} f(z)] \\ = b\eta D^{\delta+p-1} f(z) \sum_{n=1}^{\infty} p_n z^n, \end{aligned} \quad (20)$$

where η is given by (15). Using (8) in (20), we obtain

$$\begin{aligned} 2e^{i\lambda}(1-\alpha) [z(D^{\delta+p-1} f(z))' - pD^{\delta+p-1} f(z)] \\ = b(\delta+p)\eta D^{\delta+p-1} f(z) \sum_{n=1}^{\infty} p_n z^n, \end{aligned} \quad (21)$$

or, equivalently,

$$\begin{aligned} 2e^{i\lambda}(1-\alpha) \left[\sum_{k=p+1}^{\infty} (k-p) \varphi_k(\delta) a_k z^k \right] \\ = b(\delta+p)\eta \left[z^p + \sum_{k=p+1}^{\infty} \varphi_k(\delta) a_k z^k \right] \left(\sum_{n=1}^{\infty} p_n z^n \right). \end{aligned} \quad (22)$$

Comparing the coefficients of z^{n+p-1} on both sides,

$$\begin{aligned} 2e^{i\lambda}(1-\alpha)(n-1)\varphi_{n+p-1}(\delta)a_{n+p-1} \\ = b(\delta+p)\eta \{p_1 a_{n+p-2} \varphi_{n+p-2}(\delta) + \dots + p_{n-1}\}. \end{aligned} \quad (23)$$

Taking absolute on both sides and then applying the coefficient estimates $|p_n| \leq 2$ for Caratheodory functions [3], we have

$$\begin{aligned} |a_{n+p-1}| \leq \frac{|b|(\delta+p)|\eta|}{(n-1)|1-\alpha|\varphi_{n+p-1}(\delta)} \\ \times \{1 + \varphi_{p+1}(\delta)|a_{p+1}| + \dots + \varphi_{n+p-2}(\delta)|a_{n+p-2}|\}. \end{aligned} \quad (24)$$

We apply mathematical induction on (24). So for $n = 2$,

$$|a_{p+1}| \leq \frac{|b||\eta|}{|1-\alpha|}, \quad (25)$$

which shows that (13) is true. For $n = 3$,

$$|a_{p+2}| \leq \frac{|b||\eta|(\delta+p)}{|1-\alpha|(2)\varphi_{p+2}(\delta)} \{1 + \varphi_{p+1}(\delta)|a_{p+1}|\}, \quad (26)$$

and using (13), we have

$$|a_{p+2}| \leq \frac{|b||\eta|(\delta+p)}{(1-\alpha)(2)\varphi_{p+2}(\delta)} \left\{ 1 + \frac{|b||\eta|(\delta+p)}{(1-\alpha)} \right\}. \quad (27)$$

Therefore, (14) holds for $n = 3$.

Assume that (14) is true for $n = k$; that is,

$$\begin{aligned} |a_{k+p-1}| \\ \leq \frac{|b||\eta|(\delta+p)}{|1-\alpha|(k-1)\varphi_{k+p-1}(\delta)} \prod_{j=1}^{k-2} \left(1 + \frac{|b||\eta|(\delta+p)}{|1-\alpha|j} \right). \end{aligned} \quad (28)$$

Consider

$$\begin{aligned} |a_{k+p}| \leq \frac{|b||\eta|(\delta+p)}{|1-\alpha|(k)\varphi_{k+p}(\delta)} \\ \times \left\{ \left(1 + \frac{|b||\eta|(\delta+p)}{|1-\alpha|} \right) + \frac{|b||\eta|(\delta+p)}{|1-\alpha|(2)} \right. \\ \times \left(1 + \frac{|b||\eta|(\delta+p)}{|1-\alpha|} \right) \\ + \dots + \frac{|b||\eta|(\delta+p)}{|1-\alpha|(k-1)} \\ \times \prod_{j=1}^{k-2} \left(1 + \frac{|b||\eta|(\delta+p)}{|1-\alpha|j} \right) \Big\} \\ = \frac{|b||\eta|(\delta+p)}{|1-\alpha|(k)\varphi_{k+p}(\delta)} \prod_{j=1}^{k-1} \left(1 + \frac{|b||\eta|(\delta+p)}{|1-\alpha|j} \right). \end{aligned} \quad (29)$$

Therefore, the result is true for $n = k + 1$, and hence by using mathematical induction, (14) holds true for all $n \geq 3$.

If we put $\lambda = 0$, $p = 1$, $b = 2$, and $\delta = 0$ in Theorem 3, we obtain the result proved in [12]. $\square$

Corollary 4. If $f(z) \in \mathcal{SD}(\alpha, \beta)$, then

$$\begin{aligned} |a_2| \leq \frac{2(1-\beta)}{|1-\alpha|}, \\ |a_n| \leq \frac{2(1-\beta)}{(n-1)|1-\alpha|} \prod_{j=1}^{n-2} \left(1 + \frac{2(1-\beta)}{j|1-\alpha|} \right), \quad (n \geq 3). \end{aligned} \quad (30)$$

If one takes $\alpha = 0$ in Corollary 4, one obtains the following inequality:

$$|a_n| \leq \frac{1}{(n-1)!} \prod_{j=2}^n (j-2\beta), \quad (n \geq 2), \quad (31)$$

which was proved by Robertson [16].

By setting $\lambda = 0$, $p = 1$, $b = 1$, and $\delta = 1$ in Theorem 3, one obtains the result proved in [12].

Corollary 5. If $f(z) \in \mathcal{K}\mathcal{D}(\alpha, \beta)$, then

$$|a_2| \leq \frac{(1-\beta)}{|1-\alpha|},$$

$$|a_n| \leq \frac{2(1-\beta)}{n(n-1)|1-\alpha|} \prod_{j=1}^{n-2} \left(1 + \frac{2(1-\beta)}{j|1-\alpha|}\right), \quad (n \geq 3). \quad (32)$$

Letting $\alpha = 0$ in Corollary 5, one gets the following inequality proved by Robertson [16]:

$$|a_n| \leq \frac{1}{n!} \prod_{j=2}^n (j-2\beta), \quad (n \geq 2). \quad (33)$$

Theorem 6. If $f(z) \in \mathcal{A}(p)$ and satisfies

$$2(\alpha+1)(\delta+p)|b|\sin\frac{\lambda}{2} + \beta(\delta+p)|b|\cos\lambda$$

$$+ \sum_{k=p+1}^{\infty} \varphi_k(\delta)|a_k|$$

$$\times \left[(\alpha+1) \left\{ 2(\delta+p)|b|\sin\frac{\lambda}{2} + 2(k-p) \right\} \right. \quad (34)$$

$$\left. - \beta(\delta+p)|b|\cos\lambda + (\delta+p)|b| \right]$$

$$< (\delta+p)|b|,$$

where $\varphi_k(\delta)$ is given by (5), then $f(z) \in \mathcal{V}\mathcal{D}_p^\lambda(\delta, b, \alpha, \beta)$.

Proof. Suppose (34) holds. Also let us suppose

$$Q(z) = e^{i\lambda} \left(1 - \frac{2}{b} + \frac{2}{b} \frac{D^{\delta+p} f(z)}{D^{\delta+p-1} f(z)} \right). \quad (35)$$

Then

$$|Q(z) - 1|$$

$$= \left| \frac{(be^{i\lambda} - 2e^{i\lambda} - b) D^{\delta+p-1} f(z) + 2e^{i\lambda} D^{\delta+p} f(z)}{b D^{\delta+p-1} f(z)} \right|. \quad (36)$$

Using (8) and then simplifications gives

$$|Q(z) - 1|$$

$$= \left| \left(\{b\delta(e^{i\lambda} - 1) + bp(e^{i\lambda} - 1) - 2pe^{i\lambda}\} \right. \right.$$

$$\left. \times D^{\delta+p-1} f(z) + 2e^{i\lambda} z (D^{\delta+p-1} f(z))' \right)$$

$$\left. \times (b(\delta+p) D^{\delta+p-1} f(z))^{-1} \right|$$

$$\leq \left(2(\delta+p)|b|\sin\frac{\lambda}{2} + \left| \sum_{k=p+1}^{\infty} \varphi_k(\delta) a_k z^k \right| \right. \quad (37)$$

$$\left. \times \left[2(\delta+p)|b|\sin\frac{\lambda}{2} + (k-p) \right] \right)$$

$$\times \left((\delta+p)|b| \left(1 - \sum_{k=p+1}^{\infty} \varphi_k(\delta) |a_k| \right) \right)^{-1}.$$

Now consider

$$\alpha |Q(z) - 1| - \operatorname{Re} |Q(z) - 1| + \beta \cos \lambda$$

$$\leq (\alpha+1) |Q(z) - 1| + \beta \cos \lambda$$

$$\leq \left((\alpha+1) \left\{ 2(\delta+p)|b|\sin\frac{\lambda}{2} + \left| \sum_{k=p+1}^{\infty} \varphi_k(\delta) a_k z^k \right| \right. \right.$$

$$\left. \times \left[2(\delta+p)|b|\sin\frac{\lambda}{2} + (k-p) \right] \right\} \right)$$

$$\times \left((\delta+p)|b| \left(1 - \sum_{k=p+1}^{\infty} \varphi_k(\delta) |a_k| \right) \right)^{-1}$$

$$+ \beta \cos \lambda. \quad (38)$$

The last expression is bounded by 1 if

$$2(\alpha+1)(\delta+p)|b|\sin\frac{\lambda}{2} + \beta(\delta+p)|b|\cos\lambda$$

$$+ \sum_{k=p+1}^{\infty} \varphi_k(\delta)|a_k|$$

$$\times \left[(\alpha+1) \left\{ 2(\delta+p)|b|\sin\frac{\lambda}{2} + 2(k-p) \right\} \right.$$

$$\left. - \beta(\delta+p)|b|\cos\lambda + (\delta+p)|b| \right] < (\delta+p)|b|, \quad (39)$$

and this completes the proof. $\square$

For $\lambda = 0$, $\delta = -p + 1$, $b = 2$, and $\alpha = 0$ in Theorem 6, we obtain the following.

Corollary 7. If $f(z) \in \mathcal{A}(p)$ and satisfies

$$\sum_{k=p+1}^{\infty} (k+1-p-\beta) |a_k| < (1-\beta), \quad (40)$$

then $f(z) \in \mathcal{S}_p^*(\beta)$, the class of p -valent starlike functions of order β .

For $\lambda = 0$, $\delta = -p+2$, $b = 1$, and $\alpha = 0$ in Theorem 6, one has the following.

Corollary 8. If $f(z) \in \mathcal{A}(p)$ and satisfies

$$\sum_{k=p+1}^{\infty} (k-p+1)(k+1-p-\beta) |a_k| < (1-\beta), \quad (41)$$

then $f(z) \in \mathcal{K}_p(\beta)$, the class of p -valent convex functions of order β .

Further for $p = 1$ in both the last two corollaries, one obtains the results for the classes $\mathcal{S}^*(\beta)$ and $\mathcal{K}(\beta)$ which was proved by Merkes et al. [17] and Silverman [18], respectively.

Conflict of Interests

The authors declare that they have no conflict of interests. Please consider this paper for further process.

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