Research Article

Feedback Control in a General Almost Periodic Discrete System of Plankton Allelopathy

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We study the properties of almost periodic solutions for a general discrete system of plankton allelopathy with feedback controls and establish a theorem on the uniformly asymptotic stability of almost periodic solutions.

1. Introduction

Allelopathy is first used by Hans (see [1]) in 1937, which is a biological phenomenon that is characteristic of any process involving secondary metabolites produced by some plants, algae, bacteria, and fungi which influences the growth and development of biological systems. There are many investigations which have been conducted to study the toxic effect on plankton allelopathy (see [2–9]). In this paper, we investigate the following system with feedback controls:

$$\begin{aligned} x_i(n+1) &= x_i(n) \exp \left[r_i(n) - a_{ii}(n)x_i(n) - a_{ij}(n)x_j(n) \right. \\ &\quad \left. - b_i(n)x_i(n)x_j(n) - c_i(n)u_i(n) \right], \\ \Delta u_i(n) &= -d_i(n)u_i(n) + e_i(n)x_i(n), \quad i, j = 1, 2; i \neq j. \end{aligned} \quad (1)$$

Here $x_i(n)$ represent the densities of species x_i at the n th generation, $r_i(n)$ are the intrinsic growth rates of species x_i at the n th generation, $a_{ii}(n)$ measure the intraspecific effects of the generation of species x_i on their own population, and $a_{ij}(n)$ stand for the interspecific effects of the n th generation of species x_i on species x_j ; $b_i(n)$ are the rates of toxic inhibition of species x_i by species x_j and vice versa at the n th generation, Δ is the first-order forward difference operator $\Delta u_i(n) = u_i(n+1) - u_i(n)$, and $u_i(n)$ are the feedback control variables ($i, j = 1, 2; i \neq j$). Mean-

while, $\{r_i(n)\}$, $\{a_{ij}(n)\}$, $\{b_i(n)\}$, $\{c_i(n)\}$, $\{d_i(n)\}$, and $\{e_i(n)\}$ are bounded nonnegative sequences such that

$$\begin{aligned} 0 &< r_i^l \leq r_i(n) \leq r_i^u, \\ 0 &< a_{ij}^l \leq a_{ij}(n) \leq a_{ij}^u, \\ 0 &< b_i^l \leq b_i(n) \leq b_i^u, \\ 0 &< c_i^l \leq c_i(n) \leq c_i^u, \\ 0 &< e_i^l \leq e_i(n) \leq e_i^u, \\ 0 &< d_i^l \leq d_i(n) \leq d_i^u < 1, \end{aligned} \quad (2)$$

for $i, j = 1, 2$. Here, we use the notations $f^u = \sup_{n \in \mathbb{Z}^+} f(n)$ and $f^l = \inf_{n \in \mathbb{Z}^+} f(n)$ for any bounded sequence $\{f(n)\}$. For the simplicity and convenience of exposition, throughout this paper we let $\mathbb{Z}$, $\mathbb{Z}^+$, and $\mathbb{R}^+$ denote the sets of all integers, and nonnegative integers, nonnegative real numbers, respectively.

From the point of view of biology, we only consider $\{x_1(n), x_2(n), u_1(n), u_2(n)\}$ are positive; that is,

$$x_i(n) > 0, \quad u_i(n) > 0, \quad i = 1, 2. \quad (3)$$

For convenience, we give the following definition of persistence of system (1).

Definition 1. System (1) is said to be persistent if there exist positive constants x_i^* , u_i^* , x_i^* , and u_i^* which are independent

of the solutions of system (1), such that any positive solution $\{x_1(n), x_2(n), u_1(n), u_2(n)\}$ of system (1) satisfies

$$\begin{aligned} x_i^* &\leq \liminf_{n \rightarrow +\infty} x_i(n) \leq \limsup_{n \rightarrow +\infty} x_i(n) \leq x_i^*, \\ u_i^* &\leq \liminf_{n \rightarrow +\infty} u_i(n) \leq \limsup_{n \rightarrow +\infty} u_i(n) \leq u_i^*, \end{aligned} \quad (4)$$

for $i = 1, 2$.

The rest of this paper is organized as follows. We first give some preliminaries in Section 2. Next, we study the persistent property of (1) in Section 3 and the almost periodic property in Section 4. To the best of our knowledge, no work has been done for the discrete system (1) with feedback controls.

2. Preliminaries

In this section, we first introduce some definitions. For our purpose, we introduce the following notions with their properties. For convenience, $n, m, l, s, h, i, j, k, \tau, \delta$, and r in the sequel always denote integers, and the relevant intervals and inequalities are discrete ones. This section, devoted to investigate the persistent property of system (1). To do so, we need to make some preparations.

We first introduce some definitions.

Definition 2 (see [10]). A sequence $x : \mathbb{Z} \rightarrow \mathbb{R}^k$ is called an almost periodic sequence if the ε -translation set of x

$$E\{\varepsilon, x\} := \{\tau \in \mathbb{Z} : |x(n + \tau) - x(n)| < \varepsilon \forall n \in \mathbb{Z}\} \quad (5)$$

is a relatively dense set in $\mathbb{Z}$ for all $\varepsilon > 0$; that is, for any given $\varepsilon > 0$, there exists an integer $l(\varepsilon) > 0$ such that each discrete interval of length $l(\varepsilon)$ contains a $\tau = \tau(\varepsilon) \in E\{\varepsilon, x\}$ such that

$$|x(n + \tau) - x(n)| < \varepsilon \quad (6)$$

for all $n \in \mathbb{Z}$. τ is called the ε -translation number of $x(n)$.

Definition 3 (see [10]). Let $f : \mathbb{Z} \times \mathbb{D} \rightarrow \mathbb{R}^k$, where $\mathbb{D}$ is an open set in $\mathbb{R}^k$. $f(n, x)$ is said to be almost periodic in n uniformly for $x \in \mathbb{D}$ or uniformly almost periodic for short if, for any $\varepsilon > 0$ and any compact set S in $\mathbb{D}$, there exists a positive integer $l(\varepsilon, S)$ such that any interval of length $l(\varepsilon, S)$ contains a τ for which

$$|f(n + \tau, x) - f(n, x)| < \varepsilon \quad (7)$$

for all $n \in \mathbb{Z}$ and all $x \in S$. τ is called the ε -translation number of $f(n, x)$.

Next, we will introduce the following Lemmas.

Lemma 4 (see [10]). $\{x(n)\}$ is an almost periodic sequence if and only if for any sequence $\{h_k\} \subset \{h_k\}$ such that $x(n + h_k)$ converges uniformly on $n \in \mathbb{Z}$ as $k \rightarrow \infty$. Furthermore, the limit sequence is also an almost periodic sequence.

Lemma 5 (see [11]). Assume that $\{x(n)\}$ satisfies $x(n) > 0$ and

$$x(n + 1) \leq x(n) \exp[a(n) - b(n)x(n)] \quad (8)$$

for $n \in \mathbb{Z}^+$, where $a(n)$ and $b(n)$ are nonnegative sequences bounded above and below by positive constants. Then

$$\limsup_{n \rightarrow +\infty} x(n) \leq \frac{1}{b^l} \exp(a^u - 1). \quad (9)$$

Lemma 6 (see [11]). Assume that $\{x(n)\}$ satisfies

$$x(n + 1) \geq x(n) \exp[a(n) - b(n)x(n)], \quad n \geq n_0, \quad (10)$$

and $\limsup_{n \rightarrow +\infty} x(n) \leq x^*$, $b^u x^* > a^l$, $x(n_0) > 0$, where $a(n)$ and $b(n)$ are nonnegative sequences bounded above and below by positive constants and $n_0 \in \mathbb{Z}^+$. Then

$$\liminf_{n \rightarrow +\infty} x(n) \geq \frac{a^l}{b^u} \exp(a^l - b^u x^*). \quad (11)$$

3. Persistence

In this section, we will establish sufficient conditions for the persistence of system (1).

Proposition 7. Assume that (2) and (3) hold; any solution $\{x_1(n), x_2(n), u_1(n), u_2(n)\}$ of system (1) satisfies

$$\limsup_{n \rightarrow +\infty} x_i(n) \leq x_i^*, \quad \limsup_{n \rightarrow +\infty} u_i(n) \leq u_i^*, \quad i = 1, 2, \quad (12)$$

where

$$x_i^* \stackrel{\text{def}}{=} \frac{\exp(r_i^u - 1)}{a_{ii}^l}, \quad u_i^* \stackrel{\text{def}}{=} \frac{f_i^u \exp(r_i^u - 1)}{a_{ii}^l e_i^l}. \quad (13)$$

Proof. Let $\{x_1(n), x_2(n), u_1(n), u_2(n)\}$ be any positive solution of system (1) with initial condition (3). By assumption (2), it follows from the first and second equations of system (1) that

$$x_i(n + 1) \leq x_i(n) \exp[r_i(n) - a_{ii}(n)x_i(n)], \quad i = 1, 2. \quad (14)$$

Applying Lemma 5, one obtains

$$\limsup_{n \rightarrow +\infty} x_i(n) \leq \frac{\exp(r_i^u - 1)}{a_{ii}^l} = x_i^*, \quad i = 1, 2. \quad (15)$$

Thus, for any $\varepsilon > 0$ small enough, it follows from (15) that there exists a large enough integer n_1 such that

$$x_i(n) \leq x_i^* + \varepsilon \quad \text{for } n \geq n_1, \quad i = 1, 2. \quad (16)$$

Then the third and fourth equations of system (1) lead to

$$\begin{aligned} u_i(n) &= \prod_{p=0}^{n-1} (1 - d_i(p)) \left[u_i(0) + \sum_{p=0}^{n-1} \frac{e_i(p)x_i(p)}{\prod_{q=0}^{n-1} (1 - d_i(q))} \right] \\ &\leq (1 - d_i^l)^n (u_i(0) + v_i) \\ &\quad + e_i^u (x_i^* + \varepsilon) \sum_{p=n_0}^{n-1} \prod_{q=p+1}^{n-1} (1 - e_i(q)) \\ &\leq (1 - d_i^l)^n (u_i(0) + v_i) + e_i^u (x_i^* + \varepsilon) \sum_{p=n_0}^{n-1} (1 - e_i^l)^{n-p-1}, \end{aligned} \quad (17)$$

where $v_i = \sum_{p=0}^{n-1} (e_i(p)x_i(p)/\prod_{q=0}^{n-1} 1 - d_i(q))$, $i = 1, 2$. Since $0 < d_1^l < 1$, we can find a $g \in \mathbb{Z}^+$ such that $1 - d_1^l = e^{-g}$; by Stolz's theorem, we have

$$\begin{aligned} \sum_{p=n_0}^{n-1} (1 - e_i^l)^{n-p-1} &= \frac{\sum_{p=n_0}^{n-1} e^{g(p+1)}}{e^{gn}} \rightarrow \frac{1}{1 - e^{-g}} \\ &= \frac{1}{d_1^l}, \quad (n \rightarrow \infty). \end{aligned} \quad (18)$$

Thus,

$$\limsup_{n \rightarrow +\infty} u_i(n) \leq \frac{(x_i^* + \varepsilon) f_i^u}{e_i^l}. \quad (19)$$

Letting $\varepsilon \rightarrow 0$ and substituting (16) in the above inequality lead to

$$\limsup_{n \rightarrow +\infty} u_i(n) \leq \frac{f_i^u \exp(r_i^u - 1)}{a_{ii}^l e_i^l} = u_i^*, \quad i = 1, 2. \quad (20)$$

□

Proposition 8. Assume that (2) and (3); furthermore

$$\begin{aligned} r_i^l &\geq a_{ij}^u x_j^* + c_i^u u_i^*, \\ a_{ii}^u + c_i^u u_i^* + (b_i^u + a_{ij}^u) x_j^* &\geq r_i^l, \end{aligned} \quad (21)$$

for $i, j = 1, 2$, $i \neq j$, are satisfied, where u_i^* and x_i^* are the same as those in Proposition 7. Then

$$\liminf_{n \rightarrow +\infty} x_i(n) \geq x_i^*, \quad \liminf_{n \rightarrow +\infty} u_i(n) \geq u_i^*, \quad (22)$$

where

$$\begin{aligned} u_i^* &\stackrel{\text{def}}{=} \frac{x_i^* f_i^l}{e_i^u}, \\ x_i^* &\stackrel{\text{def}}{=} \frac{r_i^l - a_{ij}^u x_j^* - c_i^u x_i^*}{a_{ii}^u + b_i^u x_j^*} \end{aligned} \quad (23)$$

$$\times \exp \left\{ (r_i^l - a_{ij}^u x_j^* - c_i^u u_i^*) - (a_{ii}^u + b_i^u x_j^*) x_i^* \right\}.$$

Proof. For $\varepsilon > 0$ and $n > n_2$, according to Proposition 7, there exists a $n_2 \in \mathbb{Z}^+$ such that

$$x_i(n) \leq x_i^* + \varepsilon, \quad u_i(n) \leq u_i^* + \varepsilon. \quad (24)$$

By Lemma 6, associating (21) with (1), one has

$$\begin{aligned} \liminf_{n \rightarrow +\infty} x_i(n) &\geq \frac{r_i^l - a_{ij}^u (x_j^* + \varepsilon) - c_i^u (x_i^* + \varepsilon)}{a_{ii}^u + b_i^u (x_j^* + \varepsilon)} \\ &\times \exp \left\{ (r_i^l - a_{ij}^u (x_j^* + \varepsilon) - c_i^u (u_i^* + \varepsilon)) \right. \\ &\quad \left. - [a_{ii}^u + b_i^u (x_j^* + \varepsilon)] x_i^* \right\}. \end{aligned} \quad (25)$$

Letting $\varepsilon \rightarrow 0$ in the above inequality leads to

$$\begin{aligned} \liminf_{n \rightarrow +\infty} x_i(n) &\geq \frac{r_i^l - a_{ij}^u x_j^* - c_i^u x_i^*}{a_{ii}^u + b_i^u x_j^*} \\ &\times \exp \left\{ (r_i^l - a_{ij}^u x_j^* - c_i^u u_i^*) \right. \\ &\quad \left. - (a_{ii}^u + b_i^u x_j^*) x_i^* \right\} = x_i^*. \end{aligned} \quad (26)$$

Also from (26), for each $\varepsilon > 0$, there exists a large enough integer n_0 such that

$$x_i(n) \geq x_i^* - \varepsilon \quad \text{for } n \geq n_0, \quad i = 1, 2. \quad (27)$$

By the third and fourth equations of system (1), we can get that

$$\begin{aligned} u_i(n) &= \prod_{p=0}^{n-1} (1 - d_i(p)) \left[u_i(0) + \sum_{i=0}^{n-1} \frac{e_i(p) x_i(p)}{\prod_{q=0}^p 1 - d_i(q)} \right] \\ &\geq (1 - d_i^u)^n (u_i(0) + v_i) + e_i^u (x_i^* - \varepsilon) \\ &\times \sum_{p=n_0}^{n-1} \prod_{q=p+1}^{n-1} (1 - e_i(j)) \\ &\geq (1 - d_i^u)^n (u_i(0) + v_i) \\ &\quad + e_i^u (x_i^* - \varepsilon) \sum_{p=n_0}^{n-1} (1 - e_i^u)^{n-p-1}, \end{aligned} \quad (28)$$

where $v_i = \sum_{p=0}^{n-1} (e_i(p)x_i(p)/\prod_{q=0}^{n-1} 1 - d_i(q))$, $i = 1, 2$. As $0 < d_i^u < 1$, similar to the analysis in the proof of Proposition 7, we have

$$\liminf_{n \rightarrow +\infty} u_i(n) \geq \frac{x_i^* f_i^l}{e_i^u} = u_i^*, \quad i = 1, 2. \quad (29)$$

□

Now, we are in a position to state Theorem 9 whose proof is a direct consequence of Propositions 7 and 8.

Theorem 9. If the inequalities in (2), (3), and (21) hold, then system (1) is persistent.

4. Almost Periodic Solution

Based on Theorem 9, in Section 3, we discuss the almost periodic property for system (1).

Consider the following almost periodic difference system:

$$x(n+1) = f(n, x(n)), \quad n \in \mathbb{Z}^+, \quad (30)$$

where $f: \mathbb{Z} \times S_B \rightarrow \mathbb{R}^k$, $S_B = \{x \in \mathbb{R}^k : \|x\| < B\}$, and $f(n, x)$ is almost periodic in n uniformly for $x \in S_B$ and is continuous in x . The product system of system (30) is as follows:

$$x(n+1) = f(n, x(n)), \quad y(n+1) = f(n, y(n)). \quad (31)$$

Lemma 10 (see [2]). Suppose that there exists a Lyapunov function $V(n, x, y)$ defined for $n \in \mathbb{N}$, $\|x\| < B$, $\|y\| < B$ satisfying that

- (i) $a(\|x - y\|) \leq V(n, x, y) \leq b(\|x - y\|)$, where $a, b \in K$, with $K = \{a \in C(\mathbb{R}^+, \mathbb{R}^+) : a(0) = 0\}$, and a is increasing;
- (ii) $|V(n, x_1, y_1) - V(n, x_2, y_2)| \leq L(\|x_1 - x_2\| + \|y_1 - y_2\|)$, where $L > 0$ is a constant;
- (iii) $\Delta V_{(10)}(n, x, y) \leq -aV(n, x, y)$, where $0 < a < 1$ is a constant and $\Delta V_{(10)}(n, x, y) = V(n+1, f(n, x), f(n, y)) - V(n, x, y)$.

Moreover, if there exists a solution $\varphi(n)$ of system (30) such that $\|\varphi(n)\| \leq B^* < B$ for $n \in \mathbb{Z}^+$, then there exists a unique uniformly asymptotically stable almost periodic solution $p(n)$ of system (30) which is bounded by B^* . In particular, if $f(n, x)$ is periodic of period ω , then there exists a unique uniformly asymptotically stable periodic solution of (30) of period ω .

According to Lemma 10, we first prove that there exists a adaptive Lyapunov functional for system (1) and then construct an adaptive Lyapunov functional for system (1). We denote by Ω the set of all solutions $X(n) = \{x_1(n), x_2(n), u_1(n), u_2(n)\}$ of system (1) satisfying $x_i^* \leq x_i(n) \leq x_i^*, u_i^* \leq u_i(n) \leq u_i^*$.

Lemma 11. Assume that (2), (3), and the conditions of Theorem 9 hold; then $\Omega \neq \emptyset$.

Proof. It is now possible to show by an inductive argument that the system (1) leads to

$$\begin{aligned} x_i(n) &= x_i(0) \exp \sum_{l=0}^{n-1} [r_i(l) - a_{ii}(l)x_i(l) - a_{ij}(l)x_j(l) \\ &\quad - b_i(l)x_i(l)x_j(l) - c_i(l)u_i(l)], \\ u_i(n) &= u_i(0) - \sum_{l=0}^{n-1} \{d_i(l)u_i(l) - e_i(l)x_i(l)\}, \end{aligned} \quad (32)$$

for $i, j = 1, 2, i \neq j$. From Theorem 9, for any solution $X(n) = \{x_1(n), x_2(n), u_1(n), u_2(n)\}$ of system (1) with initial condition (2) satisfies Definition 1. Hence, for any $\varepsilon > 0$, there exists a n_0 ; if n_0 is sufficiently large, we have

$$\begin{aligned} x_{i^*} - \varepsilon \leq x_i(n) \leq x_i^* + \varepsilon, \quad u_{i^*} - \varepsilon \leq u_i(n) \leq u_i^* + \varepsilon, \\ i = 1, 2. \end{aligned} \quad (33)$$

Let $\{\tau_\alpha\}$ be any integer valued sequence such that $\tau_\alpha \rightarrow \infty$ as $\alpha \rightarrow \infty$; we claim that there exists a subsequence of $\{\tau_\alpha\}$, and we still denote it by $\{\tau_\alpha\}$, such that $x_i(n + \tau_\alpha) \rightarrow x_i^*(n)$ uniformly in n on any finite subset B of $\mathbb{Z}$ as $\alpha \rightarrow \infty$, where

$$B = \{a_1, a_2, \dots, a_m\}, \quad a_h \in \mathbb{Z}, \quad h = (1, 2, \dots, m), \quad (34)$$

and m is a finite number.

In fact, for any finite subset $B \subset \mathbb{Z}$, when α is large enough, $\tau_\alpha + a_\alpha > n_0, h = 1, 2, \dots, m$. So

$$\begin{aligned} x_{i^*} - \varepsilon \leq x_i(n + \tau_\alpha) \leq x_i^* + \varepsilon, \\ u_{i^*} - \varepsilon \leq u_i(n + \tau_\alpha) \leq u_i^* + \varepsilon. \end{aligned} \quad (35)$$

That is, $\{x_i(k + \tau_\alpha), u_i(k + \tau_\alpha)\}$ are uniformly bounded for large enough α .

Now, for $a_1 \in B$, we can choose a subsequence $\tau_\alpha^{(1)}$ of $\{\tau_\alpha\}$ such that $\{x_i(a_1 + \tau_\alpha^{(1)}), u_i(a_1 + \tau_\alpha^{(1)})\}$ uniformly converges on $\mathbb{Z}^+$ for α large enough.

Similarly, for $a_2 \in B$, we can choose a subsequence $\tau_\alpha^{(2)}$ of $\{\tau_\alpha\}$ such that $\{x_i(a_2 + \tau_\alpha^{(2)}), u_i(a_2 + \tau_\alpha^{(2)})\}$ uniformly converges on $\mathbb{Z}^+$ for α large enough.

Repeating this procedure, for $a_m \in B$, we can choose a subsequence $\{\tau_\alpha^{(m-1)}\}$ of $\{\tau_\alpha^{(m)}\}$ such that $\{x_i(a_m + \tau_\alpha^{(m)}), u_i(a_m + \tau_\alpha^{(m)})\}$ uniformly converges on $\mathbb{Z}^+$ for α large enough.

Now pick the sequence $\{\tau_\alpha^{(m)}\}$ which is a subsequence of $\{\tau_\alpha\}$; we still denote it by $\{\tau_\alpha\}$; then for all $k \in B$, we have

$$\{x_i(n + \tau_\alpha) \rightarrow x_i^*, u_i(n + \tau_\alpha) \rightarrow u_i^*\} \quad (36)$$

uniformly in $k \in B$ as $\alpha \rightarrow \infty$.

By the arbitrary of B , the conclusion is valid. $\square$

Since $\{r_i(n)\}, \{a_{ij}(n)\}, \{b_i(n)\}, \{c_i(n)\}, \{d_i(n)\}$, and $\{e_i(n)\}$ are almost periodic sequence, for the above sequence $\tau_\alpha, \tau_\alpha \rightarrow \infty$ as $\alpha \rightarrow \infty$, there exists a subsequence which we still denote it by $\{\tau_\alpha\}$ (if necessary, we take subsequence), such that

$$\begin{aligned} r_i(n + \tau_\alpha) &\rightarrow r_i(n), \quad a_{ij}(n + \tau_\alpha) \rightarrow a_{ij}(n), \\ b_i(n + \tau_\alpha) &\rightarrow b_i(n), \quad c_i(n + \tau_\alpha) \rightarrow c_i(n), \\ d_i(n + \tau_\alpha) &\rightarrow d_i(n), \quad e_i(n + \tau_\alpha) \rightarrow e_i(n), \end{aligned} \quad (37)$$

as $\alpha \rightarrow \infty$ uniformly on $\mathbb{Z}^+$.

For any $\delta \in \mathbb{Z}$, we can assume that $\tau_\alpha + \delta \geq n_0$ for δ large enough. Letting $n \geq 0$ and $n \in \mathbb{Z}$, by an inductive argument of (1) from $\tau_\alpha + \delta$ to $n + \tau_\alpha + \delta$, leads to

$$\begin{aligned} &x_i(n + \tau_\alpha + \delta) \\ &= x_i(\tau_\alpha + \delta) \\ &\quad \times \exp \sum_{l=\tau_\alpha+\delta}^{n+\tau_\alpha+\delta-1} [r_i(l) - a_{ii}(l)x_i(l) - a_{ij}(l)x_j(l) \\ &\quad - b_i(l)x_i(l)x_j(l) - c_i(l)u_i(l)], \\ &u_i(n + \tau_\alpha + \delta) \\ &= u_i(\tau_\alpha + \delta) - \sum_{l=\tau_\alpha}^{n+\tau_\alpha+\delta-1} \{d_i(l)u_i(l) - e_i(l)x_i(l)\}. \end{aligned} \quad (38)$$

Then, for $i, j = 1, 2, i \neq j$, we have

$$\begin{aligned} &x_i(n + \tau_\alpha + \delta) \\ &= x_i(\tau_\alpha + \delta) \exp \sum_{l=\delta}^{n+\delta-1} [r_i(l + \tau_\alpha) - a_{ii}(l + \tau_\alpha)x_i(l + \tau_\alpha) \\ &\quad - a_{ij}(l + \tau_\alpha)x_j(l + \tau_\alpha) \\ &\quad - b_i(l + \tau_\alpha)x_i(l + \tau_\alpha)x_j(l + \tau_\alpha) \\ &\quad - c_i(l + \tau_\alpha)u_i(l + \tau_\alpha)], \\ &u_i(n + \tau_\alpha + \delta) \\ &= u_i(\tau_\alpha + \delta) - \sum_{l=\delta}^{n+\delta-1} \{d_i(l + \tau_\alpha)u_i(l + \tau_\alpha) \\ &\quad - e_i(l + \tau_\alpha)x_i(l + \tau_\alpha)\}. \end{aligned} \quad (39)$$

Let $\alpha \rightarrow \infty$; for any $k \geq 0$,

$$\begin{aligned} x_i^*(n+\delta) &= x_i(\delta) \exp \sum_{l=\delta}^{n+\delta-1} [r_i(l) - a_{ii}(l)x_i(l) - a_{ij}(l)x_j(l) \\ &\quad - b_i(l)x_i(l)x_j(l) - c_i(l)u_i(l)], \\ u_i^*(n+\delta) &= u_i(\delta) - \sum_{l=\delta}^{n+\delta-1} \{d_i(l)u_i(l) - e_i(l)x_i(l)\}. \end{aligned} \quad (40)$$

By the arbitrariness of δ , $X^*(n) = \{x_1^*(n), x_2^*(n), u_1^*(n), u_2^*(n)\}$ is a solution of system (1) on $\mathbb{Z}^+$. It is clear that $0 \leq x_{i^*} \leq x_i(n) \leq x_i^*$; $0 \leq u_{i^*} \leq u_i(n) \leq u_i^*$, $n \in \mathbb{Z}^+$.

So $\Omega \neq \Phi$. Lemma 10 is valid.

Before stating Theorem 12, for the sake of convenience, we set

$$\begin{aligned} r_{ij} &= 2(a_{ii}^l + b_i^l x_{i^*}^*) - [(a_{ii}^u + b_i^u x_j^*)^2 + e_i^{2u}] x_i^{*2} \\ &\quad - [a_{ji}^u + b_j^u x_j^*]^2 x_i^{*2} + (a_{ij}^u + b_i^u x_i^*) + (a_{ji}^u + b_j^u x_j^*) \\ &\quad - (a_{ii}^u + b_i^u x_j^*)(a_{ij}^u + b_i^u x_i^*) x_i^* x_j^* \\ &\quad - (a_{jj}^u + b_j^u x_i^*)(a_{ji}^u + b_j^u x_j^*) x_i^* x_j^*, \\ &\quad + c_i^l (a_{ii}^l + b_i^l x_{j^*}^*) x_i^* - c_i^u - (1 - d_i^l) e_i^u x_i^* \\ &\quad - c_j^u (a_{ji}^u + b_j^u x_i^*) x_i^*, \\ r_{ij}^* &= -c_i^u - (1 - d_i^l) e_i^u x_i^* + c_i^l (a_{ii}^l + b_i^l x_{j^*}^*) x_i^* \\ &\quad - c_i^u (a_{ij}^u + b_j^u x_i^*) x_j^* - d_i^{2u} + 2d_i^l - c_i^u. \end{aligned} \quad (41)$$

Theorem 12. Suppose that the conditions of Lemma 10 are satisfied; moreover, $0 < \beta < 1$, where

$$\beta = \min \{r_{ij}, r_{ij}^*\}, \quad \text{for } i, j = 1, 2, i \neq j. \quad (42)$$

Then there exists a unique uniformly asymptotically stable almost periodic solution $X(n) = \{x_1(n), x_2(n), u_1(n), u_2(n)\}$ of system (1) which is bounded by Ω for all $n \in \mathbb{Z}^+$.

Proof. Let $x_i(n) = \exp p_i(n)$; from system (1), we have

$$\begin{aligned} p_i(n+1) &= p_i(n) + r_i(n) - a_{ii}(n) \exp p_i(n) \\ &\quad - a_{ij}(n) \exp p_j(n) - b_i(n) \exp p_i(n) \exp p_j(n) \\ &\quad - c_i(n) u_i(n), \\ \Delta u_i(n) &= -d_i(n) u_i(n) + e_i(n) \exp p_i(n), \end{aligned} \quad (43)$$

where $i, j = 1, 2, i \neq j$. From Lemma 10, it shows that system (43) has a bounded solution $Y(n) = \{p_1(n), p_2(n), u_1(n), u_2(n)\}$ satisfying

$$\ln x_{i^*} \leq p_i(n) < \ln x_i^*, \quad u_{i^*} \leq u_i(n) \leq u_i^*, \quad i = 1, 2, n \in \mathbb{Z}^+. \quad (44)$$

Thus,

$$\begin{aligned} |p_i(n)| &\leq A_i, \quad |u_i(n)| \leq B_i, \\ A_i &= \max \{|\ln x_{i^*}|, |\ln x_i^*|\}, \quad B_i = \max \{u_{i^*}, u_i^*\}, \\ &\quad i = 1, 2. \end{aligned} \quad (45)$$

For $(X, U) \in \mathbb{R}^{2+2}$, we define a norm

$$\|(X, U)\| = \sum_{i=1}^2 |x_i| + \sum_{i=1}^2 |u_i|. \quad (46)$$

For the product form of system (43)

$$\begin{aligned} p_i(n+1) &= p_i(n) + r_i(n) - a_{ii}(n) \exp p_i(n) \\ &\quad - a_{ij}(n) \exp p_j(n) - b_i(n) \exp p_i(n) \exp p_j(n) \\ &\quad - c_i(n) u_i(n), \\ \Delta u_i(n) &= -d_i(n) u_i(n) + e_i(n) \exp p_i(n), \\ q_i(n+1) &= q_i(n) + r_i(n) - a_{ii}(n) \exp q_i(n) \\ &\quad - a_{ij}(n) \exp q_j(n) - b_i(n) \exp q_i(n) \exp q_j(n) \\ &\quad - c_i(n) v_i(n), \\ \Delta v_i(n) &= -d_i(n) v_i(n) + e_i(n) \exp q_i(n). \end{aligned} \quad (47)$$

Let $Z = \{p_1(n), p_2(n), u_1(n), u_2(n)\}$ and $W = \{q_1(n), q_2(n), v_1(n), v_2(n)\}$ be any two solutions of system (30) defined on $\mathbb{Z}^+ \times S^* \times S^*$; then $\|Z\| \leq B$ and $\|W\| \leq B$, where $B = \sum_{i=1}^2 \{A_i + B_i\}$, and

$$\begin{aligned} S^* &= \{Y(n) \mid \ln x_{i^*} \leq p_i(n) \leq \ln x_i^*, \\ &\quad u_{i^*} \leq u_i(n) \leq u_i^*, i = 1, 2\}. \end{aligned} \quad (48)$$

For convenience, we denote

$$\begin{aligned} M_i(n) &= p_i(n) - q_i(n), \quad N_i(n) = u_i(n) - v_i(n), \\ \widetilde{M}_i(n) &= \widetilde{p}_i(n) - \widetilde{q}_i(n), \quad \widetilde{N}_i(n) = \widetilde{u}_i(n) - \widetilde{v}_i(n). \end{aligned} \quad (49)$$

Let

$$V(n, Z, W) = \sum_{i=1}^2 M_i^2(n) + N_i^2(n) \quad (50)$$

be a function on $\mathbb{Z}^+ \times S^* \times S^*$.

Then

$$\begin{aligned} \|Z - W\| &= \sum_{i=1}^2 \{|M_i(n)| + |N_i(n)|\}, \\ \|Z - W\|_* &= \left\{ \sum_{i=1}^2 \{M_i^2(n) + N_i^2(n)\} \right\}^{1/2} \end{aligned} \quad (51)$$

are equivalent. It is easy to see that

$$(C_1 \|Z - W\|)^2 \leq V(n, Z, W) \leq (C_2 \|Z - W\|)^2. \quad (52)$$

Let $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $a(x) = C_1^2 x^2$, $b \in C(\mathbb{R}^+, \mathbb{R}^+)$, and $b(x) = C_2^2 x^2$; thus condition (i) in Lemma 10 is satisfied.

In addition,

$$\begin{aligned} &|V(n, Z, W) - V(n, \widetilde{Z}, \widetilde{W})| \\ &= \left| \sum_{i=1}^2 (M_i^2 + N_i^2) - \sum_{i=1}^2 (\widetilde{M}_i^2 + \widetilde{N}_i^2) \right| \\ &\leq \sum_{i=1}^2 \{|M_i(n) + \widetilde{M}_i(n)| \cdot |M_i(n) - \widetilde{M}_i(n)|\} \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^2 \left\{ \left| N_i(n) + \tilde{N}_i(n) \right| \cdot \left| N_i(n) - \tilde{N}_i(n) \right| \right\} \\
& \leq L \left\{ \sum_{i=1}^2 \left(\left| M_i(n) \right| + \left| \tilde{M}_i(n) \right| \right) + \sum_{i=1}^2 \left(\left| N_i(n) \right| + \left| \tilde{N}_i(n) \right| \right) \right\} \\
& = L \left\{ \|Z - \tilde{Z}\| + \|W - \tilde{W}\| \right\},
\end{aligned} \tag{53}$$

where $L = 4 \max\{A_i, B_i\}$, $i = 1, 2$. And Lemma 10 (ii) is held.

Finally, calculating the ΔV along the solutions of (47) leads to

$$\begin{aligned}
\Delta V_{(12)}(n) &= \sum_{i=1}^2 \left\{ M_{i+1}^2(n) + N_{i+1}^2(n) \right\} - \sum_{i=1}^2 \left\{ M_i^2(n) + N_i^2(n) \right\} \\
&= \sum_{i=1}^2 \left[\left(a_{ii}(n) + b_i(n) e^{p_j(n)} \right)^2 + e_i^2(n) \right] \left(e^{p_i(n)} - e^{q_i(n)} \right)^2 \\
&\quad + \left[a_{ij}(n) + b_i(n) e^{q_i(n)} \right]^2 \left(e^{p_j(n)} - e^{q_j(n)} \right)^2 \\
&\quad + 2 \left[a_{ii}(n) + b_i(n) e^{p_j(n)} \right] \\
&\quad \cdot \left[a_{ij}(n) + b_i(n) e^{q_i(n)} \right] \left(e^{p_i(n)} - e^{q_i(n)} \right) \\
&\quad \times \left(e^{p_j(n)} - e^{q_j(n)} \right) - 2 M_i(n) \\
&\quad \cdot \left[a_{ii}(n) + b_i(n) e^{p_j(n)} \right] \left(e^{p_i(n)} - e^{q_i(n)} \right) \\
&\quad - 2 M_i(n) \left[a_{ij}(n) + b_i(n) e^{q_j(n)} \right] \\
&\quad \cdot \left(e^{p_j(n)} - e^{q_j(n)} \right) + 2 c_i M_i(n) N_i(n) \\
&\quad + 2 \left[(1 - d_i(n)) e_i(n) + c_i(n) \right. \\
&\quad \cdot \left. \left(a_{ii}(n) + b_i(n) e^{p_j(n)} \right) \right] N_i(n) \left(e^{p_i(n)} - e^{q_i(n)} \right) \\
&\quad + 2 c_i \left[a_{ij}(n) + b_i(n) e^{q_i(n)} \right] \\
&\quad \cdot N_i(n) \left(e^{p_j(n)} - e^{q_j(n)} \right) + \left[d_i^2(n) - 2 d_i(n) + c_i(n) \right]^2 N_i^2(n).
\end{aligned} \tag{54}$$

By the mean value theorem we get $e^{p_i(n)} - e^{q_i(n)} = \xi(n) M_i(n)$, where $\xi(n)$ lies between $e^{p_i(n)}$ and $e^{q_i(n)}$.

From (49), (54), one has

$$\begin{aligned}
\Delta V_{(12)}(n) &= \sum_{i=1}^2 \left\{ \left[\left(a_{ii}(n) + b_i(n) e^{p_j(n)} \right)^2 + e_i^2(n) \right] \right. \\
&\quad \times \xi_i^2 \left[(p_i(n) - q_i(n))^2 \right. \\
&\quad + \left. \left[a_{ij}(n) + b_i(n) e^{q_i(n)} \right]^2 \xi_j^2 \left[p_j(n) - q_j(n) \right]^2 \right. \\
&\quad + 2 \left[a_{ii}(n) + b_i(n) e^{p_j(n)} \right] \left[a_{ij}(n) + b_i(n) e^{q_i(n)} \right] \\
&\quad \cdot \xi_i \xi_j \left[p_i(n) - q_i(n) \right] \left[p_j(n) - q_j(n) \right] \\
&\quad - 2 \left[p_i(n) - q_i(n) \right] \left[a_{ii}(n) + b_i(n) e^{p_j(n)} \right] \\
&\quad \times \xi_i \left[p_i(n) - q_i(n) \right] \\
&\quad + 2 \left[p_i(n) - q_i(n) \right] \left[a_{ij}(n) + b_i(n) e^{q_j(n)} \right] \\
&\quad \times \xi_j \left[p_j(n) - q_j(n) \right]
\end{aligned}$$

$$\begin{aligned}
& + 2 c_i \left[p_i(n) - q_i(n) \right] \left[u_i(n) - v_i(n) \right] \\
& + 2 \left[(1 - d_i(n)) e_i(n) + c_i(n) \left(a_{ii}(n) + b_i(n) e^{p_j(n)} \right) \right] \\
& \cdot \left(u_i(n) - v_i(n) \right) \xi_i \left[p_i(n) - q_i(n) \right] \\
& + 2 c_i \left[a_{ij}(n) + b_i(n) e^{q_i(n)} \right] \left(u_i(n) - v_i(n) \right) \\
& \times \xi_j \left[p_j(n) - q_j(n) \right] \\
& + \left[d_i^2(n) - 2 d_i(n) + c_i(n) \right]^2 \left(u_i(n) - v_i(n) \right)^2 \} \\
& = \sum_{i=1}^2 \left\{ \left[\left(a_{ii}(n) + b_i(n) e^{p_j(n)} \right)^2 + e_i^2(n) \right] \xi_i^2 \right. \\
& \quad - 2 \left[a_{ii}(n) + b_i(n) e^{p_j(n)} \right] \xi_i \} \\
& \quad \cdot M_i^2(n) + \left[a_{ij}(n) + b_i(n) e^{q_i(n)} \right]^2 \xi_j^2 M_j^2(n) \\
& \quad + \left(2 \left[a_{ii}(n) + b_i(n) e^{p_j(n)} \right] \left[a_{ij}(n) + b_i(n) e^{q_i(n)} \right] \xi_i \xi_j \right. \\
& \quad \quad \left. - 2 \left[a_{ij}(n) + b_i(n) \cdot e^{q_j(n)} \right] \xi_j \right) M_i(n) M_j(n) \\
& \quad + \left[d_i^2(n) - 2 d_i(n) + c_i(n) \right]^2 N_i^2(n) \\
& \quad + 2 \left[c_i + \left[(1 - d_i(n)) e_i(n) \right. \right. \\
& \quad \quad \left. \left. + c_i(n) \left(a_{ii}(n) + b_i(n) e^{p_j(n)} \right) \right] \right] \xi_i \} \\
& \quad \times M_i(n) N_i(n) + 2 c_i \left[a_{ij}(n) + b_i(n) e^{q_i(n)} \right] \\
& \quad \times N_i(n) \xi_j \left[p_j(n) - q_j(n) \right] \}.
\end{aligned} \tag{55}$$

We get for $j = 1, 2$ that

$$\Delta V_{(12)}(n) \leq \sum_{i=1}^2 \left\{ V_{1ij} + V_{2ij} + V_{3ij} + V_{4ij} + V_{5ij} + V_{6ij} \right\}, \tag{56}$$

where

$$\begin{aligned}
V_{1ij} &= \left\{ \left[\left(a_{ii}(n) + b_i(n) e^{p_j(n)} \right)^2 + e_i^2(n) \right] \xi_i^2 \right. \\
&\quad \left. - 2 \left[a_{ii}(n) + b_i(n) e^{p_j(n)} \right] \xi_i \right\} M_i^2(n) \\
&\leq \left\{ \left(a_{ii}^u + b_i^u x_j^* \right)^2 + e_i^{2u} \right\} x_i^{*2} - 2 \left(a_{ii}^l + b_i^l x_i^* \right) x_i^* \} M_i^2(n); \\
V_{2ij} &= \left\{ \left[a_{ij}(n) + b_i(n) e^{q_i(n)} \right]^2 \xi_j^2 \right\} M_j^2(n), \\
&\leq \left\{ \left[a_{ij}^u + b_i^u x_i^* \right]^2 x_j^{*2} \right\} M_j^2(n); \\
V_{3ij} &= \left\{ 2 \left[a_{ii}(n) + b_i(n) e^{p_j(n)} \right] \left[a_{ij}(n) + b_i(n) e^{q_i(n)} \right] \xi_i \xi_j \right. \\
&\quad \left. - 2 \left[a_{ij}(n) + b_i(n) e^{q_j(n)} \right] \xi_j \right\} M_i(n) M_j(n) \\
&\leq \left\{ \left(a_{ii}^u + b_i^u x_j^* \right) \left(a_{ij}^u + b_i^u x_i^* \right) x_i^* x_j^* - \left(a_{ij}^u + b_i^u x_i^* \right) \right\} \\
&\quad \times \left[M_i^2(n) + M_j^2(n) \right]; \\
V_{4ij} &= \left\{ 2 c_i + 2 \left[(1 - d_i(n)) e_i(n) \right. \right. \\
&\quad \left. \left. + c_i(n) \left(a_{ii}(n) + b_i(n) e^{p_j(n)} \right) \right] \right\} \xi_i \} \\
&\quad \times M_i(n) N_i(n) \\
&\leq \left\{ c_i^u + \left[(1 - d_i^l) e_i^u + c_i^l \left(a_{ii}^l + b_i^l x_j^* \right) \right] x_i^* \right\} \\
&\quad \times \left[M_i^2(n) + N_i^2(n) \right];
\end{aligned}$$

$$\begin{aligned}
V_{5ij} &= \{2c_i [a_{ij}(n) + b_j(n) e^{q_i(n)}] \xi_j\} M_j(n) N_i(n) \\
&\leq \{c_i^u (a_{ij}^u + b_j^u x_i^*) x_j^*\} [M_j^2(n) + N_i^2(n)]; \\
V_{6ij} &= \{[d_i^2(n) - 2d_i(n) + c_i(n)]^2\} N_i^2(n) \\
&\leq \{d_i^{2u} - 2d_i^l + c_i^u\} N_i^2(n).
\end{aligned} \tag{57}$$

Hence,

$$\begin{aligned}
&\Delta V_{(12)}(n) \\
&\leq \sum_{i=1}^2 \{ [(a_{ii}^u + b_i^u x_j^*)^2 + e_i^{2u}] x_i^{*2} - 2(a_{ii}^l + b_i^l x_{i^*}^*) x_i^* \} \\
&\quad \cdot [(p_i(n) - q_i(n))^2 + [(a_{ij}^u + b_i^u x_j^*)^2 x_j^{*2}] M_j^2(n) \\
&\quad + [(a_{ii}^u + b_i^u x_j^*)(a_{ij}^u + b_i^u x_i^*) x_i^* x_j^* - (a_{ij}^u + b_i^u x_i^*)] \\
&\quad \times M_i^2(n) M_j^2(n) \\
&\quad + [c_i^u + [(1 - d_i^l) e_i^u - c_i^l (a_{ii}^l + b_i^l x_{i^*}^*)] x_i^*] \\
&\quad \times M_i^2(n) N_i^2(n) \\
&\quad + [c_i^u (a_{ij}^u + b_i^u x_i^*) x_j^*] [M_j^2(n) + N_i^2(n)] \\
&\quad + [d_i^{2u} - 2d_i^l + c_i^u] N_i^2(n) \} \\
&= - \sum_{i=1}^2 \{ [2(a_{ii}^l + b_i^l x_{i^*}^*) - [(a_{ii}^u + b_i^u x_j^*)^2 + e_i^{2u}] x_i^{*2} \\
&\quad - [a_{ji}^u + b_j^u x_j^*]^2 x_i^{*2} + (a_{ij}^u + b_i^u x_i^*) + (a_{ji}^u + b_j^u x_j^*) \\
&\quad - (a_{ii}^u + b_i^u x_j^*)(a_{ij}^u + b_i^u x_i^*) x_i^* x_j^* \\
&\quad - (a_{jj}^u + b_j^u x_i^*)(a_{ji}^u + b_j^u x_j^*) x_i^* x_j^* \\
&\quad + c_i^l (a_{ii}^l + b_i^l x_{i^*}^*) x_i^* - c_i^u - (1 - d_i^l) e_i^u x_i^* \\
&\quad - c_j^u (a_{ji}^u + b_i^u x_i^*) x_i^*] \cdot M_i^2(n) \\
&\quad + [-c_i^u - (1 - d_i^l) e_i^u x_i^* + c_i^l (a_{ii}^l + b_i^l x_{i^*}^*) x_i^* \\
&\quad - c_i^u (a_{ij}^u + b_i^u x_i^*) x_j^* - d_i^{2u} + 2d_i^l - c_i^u] N_i^2(n) \} \\
&\leq - \sum_{i=1}^2 [r_{ij} M_i^2(n) + r_{ij}^* N_i^2(n)] \\
&\leq -\beta \sum_{i=1}^2 [M_i^2(n) + N_i^2(n)] \\
&\leq -\beta V(n),
\end{aligned} \tag{58}$$

where $\beta = \min\{r_{ij}, r_{ij}^*\}$, $i, j = 1, 2, i \neq j$. That is, there exists a positive constant $0 < \beta < 1$ such that

$$\Delta V_{(12)}(n) \leq -\beta V(n). \tag{59}$$

From $0 < \beta < 1$, condition (iii) of Lemma 10 is satisfied. So, from Lemma 10, there exists a uniqueness uniformly asymptotically stable almost periodic solution $X(n) = \{x_1(n), x_2(n), u_1(n), u_2(n)\}$ of system (43) which is bounded by S^* for all $n \in \mathbb{Z}^+$, which means that there exists a

uniqueness uniformly asymptotically stable almost periodic solution $X(n) = \{x_1(n), x_2(n), u_1(n), u_2(n)\}$ of system (1) which is bounded by Ω for all $n \in \mathbb{Z}^+$. This completes the proof. $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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