Research Article

The Structure of EAP-Groups and Self-Autopermutable Subgroups

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A subgroup H of a given group G is said to be *autopermutable*, if $HH^\alpha = H^\alpha H$ for all $\alpha \in \text{Aut}(G)$. We also call H a *self-autopermutable subgroup* of G , when $HH^\alpha = H^\alpha H$ implies that $H^\alpha = H$. Moreover, G is said to be *EAP-group*, if every subgroup of G is *autopermutable*. One notes that if α runs over the inner automorphisms of the group, one obtains the notions of conjugate-permutability, self-conjugate-permutability, and *ECP-groups*, which were studied by Foguel in 1997, Li and Meng in 2007, and Xu and Zhang in 2005, respectively. In the present paper, we determine the structure of a finite EAP-group when its centre is of index 4 in G . We also show that self-autopermutability and characteristic properties are equivalent for nilpotent groups.

1. Introduction

Let H be a subgroup of a given group G . Then we call H to be *autopermutable*, if $HH^\alpha = H^\alpha H$ for all $\alpha \in \text{Aut}(G)$. The subgroup H is said to be *self-autopermutable*, if $HH^\alpha = H^\alpha H$ implies that $H^\alpha = H$. Moreover, we call the group G to be an *EAP-group* if every subgroup of G is *autopermutable*. Clearly, if α runs over the inner automorphisms of the group, we obtain the notions of conjugate-permutability [1], self-conjugate-permutability [2], and *ECP-groups* [3], respectively. One notes that the subgroup $H = \langle b \rangle$ of the Dihedral group $D_8 = \langle a, b : a^4 = b^2 = 1, a^b = a^{-1} \rangle$ is conjugate-permutable, which is not autopermutable. To see this, consider the automorphism α which sends a and b into a and ab , respectively. Clearly HH^α is not a subgroup of G , which means that $HH^\alpha \neq H^\alpha H$. It is easily seen that similar examples can be obtained by taking a direct product of D_8 with any other group. Also, every noncharacteristic normal subgroup of a given group is an example for a self-conjugate-permutable subgroup which is not self-autopermutable. Moreover, D_8 is an *ECP-group*, which is not an *EAP-group*.

In the present paper, we determine the structure of a finite EAP-group, when its centre is of index 4. We also prove that self-autopermutability and characteristic properties are equivalent in nilpotent groups.

2. Finite EAP-Groups

In this section, we determine the structure of finite EAP-groups, when their centres are of index 4. In fact we prove the following theorem.

Theorem 1. *Let G be a finite group with the centre of index 4. Then G is an EAP-group if and only if the Sylow 2-subgroup of G is one of the following forms:*

- (i) Q_8 ;
- (ii) $\langle a, b \mid a^{2^{n+1}} = b^2 = 1, a^b = a^{2^n+1} \rangle, n \geq 3$;
- (iii) $\mathbb{Z}_2 \times Q_8$;
- (iv) $\langle a, b, c, d \mid a^2 = b^2 = c^4 = d^4 = 1, b \in Z(G), [a, c] = [a, d] = c^3 d, [c, d] = 1 \rangle$;

- (v) $\langle a, b, c, d, e \mid a^2 = b^2 = c^2 = d^2 = e^4 = 1, a, b, c \in Z(G), [d, e] = b \rangle$;
 (vi) $\langle a, b, c, d, e \mid a^2 = b^2 = c^2 = d^4 = e^4 = 1, a, b, c \in Z(G), [d, e] = a \rangle$.

We remind that a nonabelian group is said to be Hamiltonian, if all of its subgroups are normal. The following result gives our claim, when G is a 2-group with cyclic centre of index 4.

Theorem 2. *Let G be a finite 2-group with cyclic centre of index 4. Then G is an EAP-group if and only if $G \cong Q_8$ or $\langle a, b \mid a^{2^{n+1}} = b^2 = 1, a^b = a^{2^{n+1}} \rangle$, for all $n \geq 3$.*

Proof. Consider the group G to be Q_8 . Since Q_8 is Hamiltonian group, the result follows easily. Now assume $G = \langle a, b \mid a^{2^{n+1}} = b^2 = 1, a^b = a^{2^{n+1}} \rangle$, $n \geq 3$. One can easily check that G contains exactly three proper subgroups of orders 2^i , for $1 \leq i \leq n+1$. We also observe that the subgroups of orders 2 are autopermutable and as the subgroups of orders 2^{n+1} are normal, they are also autopermutable. Now, one can check that there are exactly two cyclic and one noncyclic subgroups of orders 2^i , $2 \leq i \leq n$, so that one of the cyclic subgroups is central and hence all the subgroups of G satisfy the required property.

Conversely, assume that G is an EAP-group, $Z(G) = \langle x \mid x^{2^n} = 1 \rangle$, $G/Z(G) = \{Z(G), aZ(G), bZ(G), abZ(G)\}$, where $a^2, b^2 \in Z(G)$ and so $|a|, |b| \leq 2^{n+1}$. In case $n = 1$, then the group G is either D_8 or Q_8 . As explained before, D_8 cannot be an EAP-group and hence $G \cong Q_8$. Now suppose $n > 1$ and the elements a and b are both of order 2. Then every element $y \in G$ has the following form (as G is nilpotent of class 2):

$$y = a^i b^j x^k, \quad \text{where } 0 \leq i, j < 2, \quad 0 \leq k < 2^n. \quad (1)$$

Clearly, the map α given by $\alpha(y) = b^i a^j x^k$ is an automorphism of G , which sends a into b . Thus $HH^\alpha \neq H^\alpha H$ for the subgroup $H = \langle b \rangle$, which contradicts the assumption. Now, if $|a|, |b| < 2^{n+1}$ we may replace a and b by the elements ax^i and bx^j , both of which are of order 2. This reduces to the previous case. Therefore we must have a or b of order 2^{n+1} . Then G has a cyclic subgroup of order 2^{n+1} and so G is of order 2^{n+2} ($n > 1$) with the centre of index 4. Hence, by [4, 5.3.4], the group G has the following presentation:

$$G = \langle a, b \mid a^{2^{n+1}} = b^2 = 1, a^b = a^{2^{n+1}} \rangle, \quad n \geq 3. \quad (2)$$

This is an EAP-group and so the proof is completed. $\square$

The following result considers the case when G is a 2-group with noncyclic centre of index 4.

Theorem 3. *Let G be a finite 2-group with noncyclic centre of index 4. Then G is an EAP-group if and only if G is one of the following forms:*

- (i) $G = \mathbb{Z}_2 \times Q_8$;
 (ii) $G = \langle a, b, c, d \mid a^2 = b^2 = c^4 = d^4 = 1, b \in Z(G), [a, c] = [a, d] = c^3 d, [c, d] = 1 \rangle$;

- (iii) $G = \langle a, b, c, d, e \mid a^2 = b^2 = c^2 = d^2 = e^4 = 1, a, b, c \in Z(G), [d, e] = b \rangle$;
 (iv) $G = \langle a, b, c, d, e \mid a^2 = b^2 = c^2 = d^4 = e^4 = 1, a, b, c \in Z(G), [d, e] = a \rangle$.

Proof. The sufficient condition is obvious. We only need to prove the necessity condition. Let G be an EAP-group and $G/Z(G) = \{Z(G), aZ(G), bZ(G), abZ(G)\}$, where $a^2, b^2 \in Z(G)$. Assume that $Z(G)$ is not an elementary abelian 2-group. Since $Z(G)$ is the direct product of its cyclic subgroups, by the same argument as in Theorem 2, there are no EAP-groups in this case. Now, assume that $Z(G)$ is an elementary abelian 2-group. Clearly G must be a group of order either 16 or 32. The structure of such groups is given as follows in [5]. If $|G| = 16$, then

- (i) $G = \mathbb{Z}_2 \times D_8$;
 (ii) $G = \langle a, b \mid a^4 = b^4 = 1, bab^{-1} = a^3 \rangle \cong \mathbb{Z}_4 \rtimes \mathbb{Z}_4$;
 (iii) $G = \mathbb{Z}_2 \times Q_8$;
 (iv) $G = \langle a, b, c, d \mid a^2 = b^2 = c^4 = d^4 = 1, b \in Z(G), [a, c] = [a, d] = c^3 d, [c, d] = 1 \rangle$.

As D_8 is not an EAP-group, hence the group of form (i) cannot be an EAP-group. For the group of form (ii) we can consider $H = \langle b \rangle$ and $\alpha \in \text{Aut}(G)$ which sends a and b into a and ab , respectively. Clearly, $HH^\alpha \neq H^\alpha H$ and hence G cannot be an EAP-group. Thus when $|a| = |b|$, then G is of the form given in either (iii) or (iv).

Assume $|G| = 32$. Then such groups in the list of small groups with elementary abelian centres of index 4 are only of the following forms:

- (i) $G = \langle a, b, c \mid a^4 = b^4 = c^2 = 1, [a, b] = c, [a, c] = [b, c] = 1 \rangle$;
 (ii) $G = \langle a, b, c, d, e \mid a^2 = b^2 = c^2 = d^4 = e^4 = 1, a, b, c \in Z(G), [d, e] = b \rangle$;
 (iii) $G = \langle a, b, c, d, e \mid a^2 = b^2 = c^2 = d^2 = e^2 = 1, c, d \in Z(G), [a, e] = [b, e] = ab, [a, b] = 1 \rangle$;
 (iv) $G = \langle a, b, c, d, e \mid a^2 = b^2 = c^2 = d^2 = e^4 = 1, a, b, c \in Z(G), [d, e] = b \rangle$;
 (v) $G = \langle a, b, c, d, e \mid a^2 = b^2 = c^2 = d^4 = e^4 = 1, a, b, c \in Z(G), [d, e] = a \rangle$.

For the group of form (i) we may consider the cyclic subgroup $H = \langle b \rangle$ and $\alpha \in \text{Aut}(G)$, which sends a, b , and c into a, ab , and c , respectively. In case the group G is of form (ii), we consider $H = \langle e \rangle$ and $\alpha \in \text{Aut}(G)$ which sends a, b, c, d , and e into a, ab, ac, be , and ed , respectively. Also if the group is considered to be of form (iii), one may consider $H = \langle e \rangle$ and $\alpha \in \text{Aut}(G)$ which sends a, b, c, d , and e into e, b, c, d , and a , respectively. Now, one can easily check that in these cases $HH^\alpha \neq H^\alpha H$ and so G cannot be an EAP-group. Hence, when $|G| = 32$, then G is of either form (iv) or form (v). The proof is complete. $\square$

Proof of Theorem 1. The necessity condition is obvious and Theorems 2 and 3 establish the result, when G is a 2-group. If G is not a 2-group, then we may write $G = S_1 \times S_2 \cdots \times S_k$,

in such a way that S_1 is a Sylow 2-subgroup and S_i is an abelian Sylow p_i -subgroup, where p_i is an odd prime number, for $2 \leq i \leq k$. Clearly, $\text{Aut}(G) \cong \text{Aut}(S_1) \times \text{Aut}(S_2) \times \cdots \times \text{Aut}(S_k)$ and for any subgroup H of G , $H \cong H_1 \times H_2 \times \cdots \times H_k$, where $H_i \leq S_i$ for $1 \leq i \leq k$. Thus H is an autopermutable subgroup of G if H_1 is an autopermutable subgroup of S_1 . This completes the proof. $\square$

3. Self-Autopermutable Subgroups in Nilpotent Groups

We call a subgroup H of a given group G to be *weakly characteristic*, when $H^\alpha \leq N_G(H)$ implies that $H^\alpha = H$ for all $\alpha \in \text{Aut}(G)$. Also, given the subgroups H and K , then H satisfies the *subcharacteriser condition*, if $H \trianglelefteq K$ implies that $N_{\text{Aut}(G)}(K) \leq N_{\text{Aut}(G)}(H)$, where $N_{\text{Aut}(G)}(K) = \{\alpha \in \text{Aut}(G); K^\alpha = K\} \leq \text{Aut}(G)$. Clearly, if one considers the inner automorphisms of the group then *weakly normal* and *normaliser condition* properties are obtained.

The following result of [6] shows that self-conjugate-permutability, weakly normal property, and subnormaliser condition are equivalent for p -subgroups of a given group.

Theorem 4 (see [6], Proposition 3.3). *Let H be a p -subgroup of a group G . Then the following properties are equivalent:*

- (i) H is a self-conjugate-permutable subgroup;
- (ii) H is a weakly normal subgroup;
- (iii) H satisfies the subnormaliser condition.

In this section, it is shown that self-autopermutable subgroups in nilpotent groups are always characteristic.

Proposition 5. *Let H be a subgroup of a group G .*

- (i) *If H is self-autopermutable, then H is weakly characteristic in G .*
- (ii) *If H is weakly characteristic, then H satisfies the subcharacteriser condition in G .*

Proof. (i) If $H^\alpha \leq N_G(H)$, as $H \trianglelefteq N_G(H)$, we have $HH^\alpha = H^\alpha H$. Applying the condition that H is self-autopermutable subgroup of the group G , we get $H^\alpha = H$. By definition, H is weakly characteristic.

(ii) Let $K \leq G$, such that $H \trianglelefteq K$. We have $H^\alpha \leq K^\alpha = K \leq N_G(H)$ for every $\alpha \in N_{\text{Aut}(G)}(K)$. Since H is weakly characteristic in G , we have $H^\alpha = H$. Thus $\alpha \in N_{\text{Aut}(G)}(H)$ and the result is obtained. $\square$

The following theorem is one of the main results in this section.

Theorem 6. *Let H be a subgroup of a nilpotent finite group G . If H satisfies the subcharacteriser condition then H is characteristic in G .*

Proof. Write $G \cong P_1 \times P_2 \times \cdots \times P_t$, where P_i is a Sylow p_i -subgroup of G , for $1 \leq i \leq t$. We may also write $H \cong H_1 \times H_2 \times \cdots \times H_t$, with $H_i \leq P_i$, $1 \leq i \leq t$. Since H satisfies

the subcharacteriser condition in G , one can easily see that H_i satisfies the subcharacteriser condition in P_i . Therefore $H_i \trianglelefteq P_i$ implies that H_i is characteristic in P_i , which proves the result. $\square$

Finally, we show that self-autopermutability, weakly characteristic, and subcharacteriser conditions are equivalent, for every subgroup of a nilpotent group.

Corollary 7. *Let H be a subgroup of a finite nilpotent group G . Then*

- (i) H is a self-autopermutable;
- (ii) H is a weakly characteristic;
- (iii) H satisfies the subcharacteriser condition in G .

Proof. The result follows by Proposition 5 and Theorem 6. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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