Research Article

Radius Constants for Analytic Functions with Fixed Second Coefficient

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Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in the unit disk with the second coefficient a_2 satisfying $|a_2| = 2b$, $0 \leq b \leq 1$. Sharp radius of Janowski starlikeness is obtained for functions f whose n th coefficient satisfies $|a_n| \leq cn + d$ ($c, d \geq 0$) or $|a_n| \leq c/n$ ($c > 0$ and $n \geq 3$). Other radius constants are also obtained for these functions, and connections with earlier results are made.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f defined in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, normalized by $f(0) = 0 = f'(0) - 1$, and let $\mathcal{S}$ denote its subclass consisting of univalent functions. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$, de Branges [1] obtained the sharp coefficient bound that $|a_n| \leq n$ ($n \geq 2$). However, the inequality $|a_n| \leq n$, $n \geq 2$, is not sufficient for f to be univalent; for example, $f(z) = z + 2z^2$ is clearly not a member of $\mathcal{S}$.

Several subclasses of $\mathcal{S}$ possess a similar coefficient bound. For instance, the n th coefficients of starlike functions, convex functions in the direction of imaginary axis, and close-to-convex functions satisfy $|a_n| \leq n$ ($n \geq 2$) [2–4]. Other examples include functions which are convex, starlike of order $1/2$, and starlike with respect to symmetric points. The n th coefficients of these functions satisfy $|a_n| \leq 1$ ($n \geq 2$) [5–7]. The n th coefficient of close-to-convex functions with argument β satisfies $|a_n| \leq 1 + (n-1)\cos\beta$ [8], and the coefficients of uniformly starlike functions are bounded by $2/n$ [9], while $|a_n| \leq 1/n$ [10] for uniformly convex functions. Simple examples show that these bounds are not sufficient to characterize the geometric properties of the classes of functions.

In the sequel, we will assume that $f \in \mathcal{A}$ has the Taylor expansion of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Gavrilo [11] showed that the radius of univalence for functions $f \in \mathcal{A}$ satisfying $|a_n| \leq n$ ($n \geq 2$) is the real root $r_0 \approx 0.164$ of the equation $2(1-r)^3 - (1+r) = 0$, and the result is sharp for $f(z) = 2z - z/(1-z)^2$. Gavrilo also proved that the radius of univalence for functions $f \in \mathcal{A}$ satisfying the coefficient bound $|a_n| \leq M$ ($n \geq 2$) is $1 - \sqrt{M/(1+M)}$. The condition $|a_n| \leq M$ clearly holds for functions $f \in \mathcal{A}$ satisfying $|f(z)| \leq M$, and for these functions, Landau [12] proved that the radius of univalence is $M - \sqrt{M^2 - 1}$. In fact, Yamashita [13] showed that the radius of univalence obtained by Gavrilo [11] is also the radius of starlikeness for functions $f \in \mathcal{A}$ satisfying $|a_n| \leq n$ or $|a_n| \leq M$. Additionally, Yamashita [13] determined that the radius of convexity for functions $f \in \mathcal{A}$ satisfying $|a_n| \leq n$ is the real root $r_0 \approx 0.090$ of the equation $2(1-r)^4 - (1+4r+r^2) = 0$, while the radius of convexity for functions $f \in \mathcal{A}$ satisfying $|a_n| \leq M$ is the real root of

$$(M+1)(1-r)^3 - M(1+r) = 0. \quad (1)$$

Recently, Kalaj et al. [14] obtained the radii of univalence, starlikeness, and convexity for harmonic mappings satisfying certain coefficient inequalities.

For two analytic functions f and g , the function f is subordinate to g , denoted by $f < g$, if there is an analytic self-map w of $\mathbb{D}$ with $w(0) = 0$ satisfying $f(z) = g(w(z))$. If g is univalent, then $f < g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$.

For $\beta \in \mathbb{R} \setminus \{1\}$, $\alpha \geq 0$, the class $\mathcal{L}(\alpha, \beta)$ consists of functions $f \in \mathcal{A}$ satisfying

$$\alpha \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} < \frac{1 + (1 - 2\beta)z}{1 - z}. \quad (2)$$

Denote by $\mathcal{L}_0(\alpha, \beta)$ its subclass consisting of functions $f \in \mathcal{A}$ satisfying

$$\left| \alpha \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} - 1 \right| \leq |1 - \beta| \quad (\beta \in \mathbb{R} \setminus \{1\}, \alpha \geq 0). \quad (3)$$

These classes were investigated in [15–24].

For $\beta < 1$, the class $\mathcal{L}(0, \beta)$ is the class of starlike functions of order β , while, for the case $\beta > 1$, the class was studied in [25–28].

The class $\mathcal{ST}[A, B]$ of Janowski starlike functions [29] consists of $f \in \mathcal{A}$ satisfying the subordination

$$\frac{zf'(z)}{f(z)} < \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1). \quad (4)$$

Certain well-known subclasses of starlike functions are special cases of $\mathcal{ST}[A, B]$ for appropriate choices of the parameters A and B . For example, for $0 \leq \beta < 1$, $\mathcal{ST}(\beta) := \mathcal{ST}[1 - 2\beta, -1]$ is the familiar class of starlike functions of order β . Denote by $\mathcal{ST}_\beta$ the class $\mathcal{ST}_\beta := \mathcal{L}_0(0, \beta) = \mathcal{ST}[1 - \beta, 0]$. Janowski [29] obtained the sharp radius of convexity for $\mathcal{ST}[A, B]$.

This paper studies the class $\mathcal{A}_b$ consisting of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, ($|a_2| = 2b$, $0 \leq b \leq 1$), in the disk $\mathbb{D}$. The subclass of univalent functions in $\mathcal{A}_b$ have been studied in [30–33]. In [33], Ravichandran obtained sharp radii of starlikeness and convexity of order α for functions $f \in \mathcal{A}_b$ satisfying $|a_n| \leq n$ or $|a_n| \leq M$, $n \geq 3$. The author also obtained the radius of uniform convexity and parabolic starlikeness for functions $f \in \mathcal{A}_b$ satisfying $|a_n| \leq n$, $n \geq 3$.

This paper finds radius constants for functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}_b$ satisfying either $|a_n| \leq cn + d$ ($c, d \geq 0$) or $|a_n| \leq c/n$ ($c > 0, n \geq 3$). In the next section, sharp $\mathcal{L}(\alpha, \beta)$ -radius and $\mathcal{ST}[A, B]$ -radius are derived for these classes. Several known radius constants are shown to be special cases of the results obtained.

2. Radius Constants

A sufficient condition for functions $f \in \mathcal{A}$ to belong to the class $\mathcal{L}(\alpha, \beta)$ is given in the following lemma.

Lemma 1 (see [24, 34]). Let $\beta \in \mathbb{R} \setminus \{1\}$ and $\alpha \geq 0$. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ satisfies the inequality

$$\sum_{n=2}^{\infty} (\alpha n^2 + (1 - \alpha)n - \beta) |a_n| \leq |1 - \beta|, \quad (5)$$

then $f \in \mathcal{L}(\alpha, \beta)$.

Making use of this lemma, the sharp $\mathcal{L}(\alpha, \beta)$ -radius is obtained for $f \in \mathcal{A}_b$ satisfying the coefficient inequality $|a_n| \leq cn + d$.

Theorem 2. Let $\beta \in \mathbb{R} \setminus \{1\}$, $6\alpha + 3 - \beta \geq 0$, and $\alpha \geq 0$. The $\mathcal{L}(\alpha, \beta)$ -radius for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}_b$ satisfying the coefficient inequality $|a_n| \leq cn + d$, $c, d \geq 0$, $n \geq 3$, is the real root in $(0, 1)$ of the equation

$$\begin{aligned} & ((c + d)(1 - \beta) + |1 - \beta| \\ & + (2\alpha + 2 - \beta)(2(c - b) + d)r)(1 - r)^4 \\ & = c\alpha(1 + 4r + r^2) + ((1 - \alpha)c + \alpha d)(1 - r^2) \\ & + ((1 - \alpha)d - \beta c)(1 - r)^2 - \beta d(1 - r)^3. \end{aligned} \quad (6)$$

For $\beta < 1$, this number is also the $\mathcal{L}_0(\alpha, \beta)$ -radius of $f \in \mathcal{A}_b$. The results are sharp.

Proof. The number r_0 is the $\mathcal{L}(\alpha, \beta)$ -radius for $f \in \mathcal{A}_b$ if and only if $f(r_0 z)/r_0 \in \mathcal{L}(\alpha, \beta)$. Therefore, by Lemma 1, it is sufficient to verify the inequality

$$\sum_{n=2}^{\infty} (\alpha n^2 + (1 - \alpha)n - \beta) |a_n| r_0^{n-1} \leq |1 - \beta|, \quad (7)$$

where r_0 is the real root in $(0, 1)$ of (6). Using the known expansions

$$\sum_{n=3}^{\infty} r_0^{n-1} = \frac{1}{1 - r_0} - 1 - r_0, \quad (8)$$

$$\sum_{n=3}^{\infty} n r_0^{n-1} = \frac{1}{(1 - r_0)^2} - 1 - 2r_0, \quad (9)$$

$$\sum_{n=3}^{\infty} n^2 r_0^{n-1} = \frac{1 + r_0}{(1 - r_0)^3} - 1 - 4r_0, \quad (10)$$

$$\sum_{n=3}^{\infty} n^3 r_0^{n-1} = \frac{1 + 4r_0 + r_0^2}{(1 - r_0)^4} - 1 - 8r_0 \quad (11)$$

leads to

$$\begin{aligned} & \sum_{n=2}^{\infty} (\alpha n^2 + (1 - \alpha)n - \beta) |a_n| r_0^{n-1} \\ & \leq 2(2\alpha + 2 - \beta) b r_0 \\ & + \sum_{n=3}^{\infty} (\alpha n^2 + (1 - \alpha)n - \beta) (cn + d) r_0^{n-1} \\ & = 2(2\alpha + 2 - \beta) b r_0 + c\alpha \left(\frac{1 + 4r_0 + r_0^2}{(1 - r_0)^4} - 1 - 8r_0 \right) \\ & + ((1 - \alpha)c + \alpha d) \left(\frac{1 + r_0}{(1 - r_0)^3} - 1 - 4r_0 \right) \\ & + ((1 - \alpha)d - \beta c) \left(\frac{1}{(1 - r_0)^2} - 1 - 2r_0 \right) \\ & - \beta d \left(\frac{1}{1 - r_0} - 1 - r_0 \right) \end{aligned}$$

$$\begin{aligned}
&= (c+d)(\beta-1) - (2\alpha+2-\beta)(2(c-b)+d)r_0 \\
&\quad + (c\alpha(1+4r_0+r_0^2) + ((1-\alpha)c+\alpha d)(1-r_0^2) \\
&\quad + ((1-\alpha)d-\beta c)(1-r_0)^2 \\
&\quad - \beta d(1-r_0)^3) \times (1-r_0)^{-4} \\
&= |1-\beta|.
\end{aligned} \tag{12}$$

For $\beta < 1$, consider the function

$$\begin{aligned}
f_0(z) &= z - 2bz^2 - \sum_{n=3}^{\infty} (cn+d)z^n \\
&= (c+1)z + 2(c-b)z^2 - \frac{cz}{(1-z)^2} - \frac{dz^3}{1-z}.
\end{aligned} \tag{13}$$

At the root $z = r_0$ in $(0, 1)$ of (6), f_0 satisfies

$$\operatorname{Re} \left(\alpha \frac{z^2 f_0''(z)}{f_0(z)} + \frac{zf_0'(z)}{f_0(z)} \right) = 1 - \frac{N(r_0)}{D(r_0)} = \beta, \tag{14}$$

where

$$\begin{aligned}
N(r_0) &= -2(c-b)(2\alpha+1)r_0 + \frac{2cr_0(2\alpha+1)}{(1-r_0)^3} \\
&\quad + \frac{6c\alpha r_0^2}{(1-r_0)^4} + \frac{2dr_0^2(3\alpha+1)}{1-r_0} \\
&\quad + \frac{dr_0^3(6\alpha+1)}{(1-r_0)^2} + \frac{2dr_0^4\alpha}{(1-r_0)^3},
\end{aligned} \tag{15}$$

$$D(r_0) = c+1+2(c-b)r_0 - \frac{c}{(1-r_0)^2} - \frac{dr_0^2}{1-r_0}.$$

This shows that r_0 is the sharp $\mathcal{L}(\alpha, \beta)$ -radius for $f \in \mathcal{A}_b$. For $\beta < 1$, (14) shows that the rational expression $N(r_0)/D(r_0)$ is positive, and therefore the equality

$$\left| \alpha \frac{z^2 f_0''(z)}{f_0(z)} + \frac{zf_0'(z)}{f_0(z)} - 1 \right| = 1 - \beta \tag{16}$$

holds. Thus, r_0 is the sharp $\mathcal{L}_0(\alpha, \beta)$ -radius for $f \in \mathcal{A}_b$ when $\beta < 1$.

For $\beta > 1$, the function

$$\begin{aligned}
f_0(z) &= z + 2bz^2 + \sum_{n=3}^{\infty} (cn+d)z^n \\
&= (1-c)z + 2(b-c)z^2 + \frac{cz}{(1-z)^2} + \frac{dz^3}{1-z}
\end{aligned} \tag{17}$$

demonstrates sharpness of the result. The derivation is similar to the case $\beta < 1$ and is omitted. $\square$

Theorem 3. Let $\beta \in \mathbb{R} \setminus \{1\}$ and $\alpha \geq 0$. The $\mathcal{L}(\alpha, \beta)$ -radius of $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}_b$ satisfying the coefficient inequality $|a_n| \leq c/n$ for $n \geq 3$ and $c > 0$ is the real root in $(0, 1)$ of the equation

$$\begin{aligned}
&\left[c(1-\beta) + |1-\beta| + (2\alpha+2-\beta)r \left(\frac{c}{2} - 2b \right) \right] (1-r)^2 \\
&= c\alpha + (1-\alpha)c(1-r) + \beta c(1-r)^2 \frac{\log(1-r)}{r}.
\end{aligned} \tag{18}$$

For $\beta < 1$, this number is also the $\mathcal{L}_0(\alpha, \beta)$ -radius of $f \in \mathcal{A}_b$. The results are sharp.

Proof. By Lemma 1, r_0 is the $\mathcal{L}(\alpha, \beta)$ -radius of functions $f \in \mathcal{A}_b$ when inequality (7) holds for the real root r_0 of (18) in $(0, 1)$. Using (8) and (9) together with

$$\sum_{n=3}^{\infty} \frac{r_0^{n-1}}{n} = -\frac{\log(1-r_0)}{r_0} - 1 - \frac{r_0}{2} \tag{19}$$

leads to

$$\begin{aligned}
&\sum_{n=2}^{\infty} (\alpha n^2 + (1-\alpha)n - \beta) |a_n| r_0^{n-1} \\
&\leq 2(2\alpha+2-\beta)br_0 \\
&\quad + \sum_{n=3}^{\infty} (\alpha n^2 + (1-\alpha)n - \beta) \left(\frac{c}{n} \right) r_0^{n-1} \\
&= 2(2\alpha+2-\beta)br_0 + c\alpha \left(\frac{1}{(1-r_0)^2} - 1 - 2r_0 \right) \\
&\quad + (1-\alpha)c \left(\frac{1}{1-r_0} - 1 - r_0 \right) \\
&\quad - \beta c \left(-\frac{\log(1-r_0)}{r_0} - 1 - \frac{r_0}{2} \right) \\
&= c(\beta-1) + (2\alpha+2-\beta)r_0 \left(2b - \frac{c}{2} \right) \\
&\quad + \frac{c\alpha r_0 + (1-\alpha)c(1-r_0)r_0 + \beta c(1-r_0)^2 \log(1-r_0)}{(1-r_0)^2 r_0} \\
&= |1-\beta|.
\end{aligned} \tag{20}$$

To verify sharpness for $\beta < 1$, consider the function

$$\begin{aligned}
f_0(z) &= z - 2bz^2 - \sum_{n=3}^{\infty} \frac{c}{n} z^n \\
&= (1+c)z + \left(\frac{c}{2} - 2b \right) z^2 + c \log(1-z).
\end{aligned} \tag{21}$$

At the root $z = r_0$ in $(0, 1)$ of (18), f_0 satisfies

$$\begin{aligned} & \operatorname{Re} \left(\alpha \frac{z^2 f_0''(z)}{f_0(z)} + \frac{z f_0'(z)}{f_0(z)} \right) \\ &= 1 - \left(- \left(\frac{c}{2} - 2b \right) r_0 (2\alpha + 1) + \frac{c r_0 \alpha}{(1 - r_0)^2} \right. \\ & \quad \left. + \frac{c}{1 - r_0} + \frac{c \log(1 - r_0)}{r_0} \right) \\ & \quad \times \left((1 + c) + \left(\frac{c}{2} - 2b \right) r_0 + \frac{c \log(1 - r_0)}{r_0} \right)^{-1} = \beta. \end{aligned} \quad (22)$$

Thus, r_0 is the sharp $\mathcal{L}(\alpha, \beta)$ -radius for $f \in \mathcal{A}_b$. For $\beta < 1$, the rational expression in (22) is positive, and therefore

$$\left| \alpha \frac{z^2 f_0''(z)}{f_0(z)} + \frac{z f_0'(z)}{f_0(z)} - 1 \right| = 1 - \beta, \quad (23)$$

which shows that r_0 is the sharp $\mathcal{L}_0(\alpha, \beta)$ -radius for $f \in \mathcal{A}_b$. For $\beta > 1$, sharpness of the result is demonstrated by the function f_0 given by

$$\begin{aligned} f_0(z) &= z + 2bz^2 + \sum_{n=3}^{\infty} \frac{c}{n} z^n \\ &= (1 - c)z + \left(2b - \frac{c}{2} \right) z^2 - c \log(1 - z). \end{aligned} \quad (24)$$

□

Remark 4. The results obtained above yield the following special cases.

- (1) For $\alpha = 0$, $\beta = 0$, $c = 1$, $d = 0$, and $0 \leq b \leq 1$, Theorem 2 yields the radius of starlikeness obtained by Yamashita [13].
- (2) For $\alpha = 0$, $c = 1$, and $d = 0$, Theorem 2 reduces to Theorem 2.1 in [33, page 3]. When $\alpha = 0$, $c = 0$, and $d = M$, Theorem 2 leads to Theorem 2.5 in [33, page 5].
- (3) For $\alpha = 0$, Theorem 3 yields the radius of starlikeness of order β for $f \in \mathcal{A}_b$ obtained by Ravichandran [33, Theorem 2.8].

The following result of Goel and Sohi [35] will be required in our investigation of the class of Janowski starlike functions.

Lemma 5 (see [35]). Let $-1 \leq B < A \leq 1$. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ satisfies the inequality

$$\sum_{n=2}^{\infty} ((1 - B)n - (1 - A)) |a_n| \leq A - B, \quad (25)$$

then $f \in \mathcal{ST}[A, B]$.

The next result finds the sharp $\mathcal{ST}[A, B]$ -radius for $f \in \mathcal{A}_b$ satisfying the coefficient inequality $|a_n| \leq cn + d$.

Theorem 6. Let $-1 \leq B < A \leq 1$. The $\mathcal{ST}[A, B]$ -radius for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}_b$ satisfying the coefficient inequality $|a_n| \leq cn + d$, $n \geq 3$ and $c, d \geq 0$, is the real root in $(0, 1)$ of the equation

$$\begin{aligned} & [(A - B)(c + d + 1) \\ & - (2b - 2c - d)(2(1 - B) - (1 - A))r](1 - r)^3 \\ &= c(1 - B)(1 + r) + (d(1 - B) - c(1 - A))(1 - r) \\ & - (1 - A)d(1 - r)^2. \end{aligned} \quad (26)$$

This radius is sharp.

Proof. It is evident that r_0 is the $\mathcal{ST}[A, B]$ -radius of $f \in \mathcal{A}_b$ if and only if $f(r_0 z)/r_0 \in \mathcal{ST}[A, B]$. Hence, by Lemma 5, it suffices to show that

$$\begin{aligned} & \sum_{n=2}^{\infty} ((1 - B)n - (1 - A)) |a_n| r_0^{n-1} \leq A - B \\ & (-1 \leq B < A \leq 1), \end{aligned} \quad (27)$$

where r_0 is the root in $(0, 1)$ of (26). From (8), (9), and (10), it follows that

$$\begin{aligned} & \sum_{n=2}^{\infty} ((1 - B)n - (1 - A)) |a_n| r_0^{n-1} \\ & \leq 2(2(1 - B) - (1 - A)) b r_0 \\ & + \sum_{n=3}^{\infty} ((1 - B)n - (1 - A)) (cn + d) r_0^{n-1} \\ &= 2(2(1 - B) - (1 - A)) b r_0 \\ & + c(1 - B) \left(\frac{1 + r_0}{(1 - r_0)^3} - 1 - 4r_0 \right) \\ & + (d(1 - B) - c(1 - A)) \left(\frac{1}{(1 - r_0)^2} - 1 - 2r_0 \right) \\ & - (1 - A)d \left(\frac{1}{1 - r_0} - 1 - r_0 \right) \\ &= (B - A)(c + d) + (2b - 2c - d) \\ & \times (2(1 - B) - (1 - A)) r_0 \\ & + (c(1 - B)(1 + r_0) \\ & + (d(1 - B) - c(1 - A))(1 - r_0) \\ & - (1 - A)d(1 - r_0)^2) \times (1 - r_0)^{-3} \\ &= A - B. \end{aligned} \quad (28)$$

The function f_0 given by (13) shows that the result is sharp. Indeed, at the point $z = r_0$ where r_0 is the root in $(0, 1)$ of (26), the function f_0 satisfies

$$\begin{aligned} & \left| \frac{zf'_0(z)}{f_0(z)} - 1 \right| \\ &= \left(-2(c-b)r_0 + \frac{2dr_0^2}{1-r_0} + \frac{dr_0^3}{(1-r_0)^2} + \frac{2cr_0}{(1-r_0)^3} \right) \\ & \quad \times \left(c+1+2(c-b)r_0 - \frac{c}{(1-r_0)^2} - \frac{dr_0^2}{1-r_0} \right)^{-1}, \\ & \left| A - B \frac{zf'_0(z)}{f_0(z)} \right| \\ &= \frac{(c+1)(A-B) + 2(c-b)r_0(A-2B)}{c+1+2(c-b)r_0 - c/(1-r_0)^2 - dr_0^2/(1-r_0)} \\ & \quad - \left(\frac{c(A-B)}{(1-r_0)^2} + \frac{2cr_0B}{(1-r_0)^3} \right. \\ & \quad \left. - \frac{dr_0^2(A-3B)}{1-r_0} + \frac{dr_0^3B}{(1-r_0)^2} \right) \\ & \quad \times \left(c+1+2(c-b)r_0 - \frac{c}{(1-r_0)^2} - \frac{dr_0^2}{1-r_0} \right)^{-1}. \end{aligned} \quad (29)$$

Then, (26) yields

$$\left| \frac{zf'_0(z)}{f_0(z)} - 1 \right| = \left| A - B \frac{zf'_0(z)}{f_0(z)} \right| \quad (-1 \leq B < A \leq 1, z = r_0), \quad (30)$$

or equivalently $f_0 \in \mathcal{ST}[A, B]$. $\square$

Theorem 7. Let $-1 \leq B < A \leq 1$. The $\mathcal{ST}[A, B]$ -radius for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}_b$ satisfying the coefficient inequality $|a_n| \leq c/n$, $n \geq 3$ and $c > 0$, is the real root in $(0, 1)$ of the equation

$$\begin{aligned} & \left((c+1)(A-B) - (2(1-B) - (1-A))r \left(2b - \frac{c}{2} \right) \right) \\ & \quad \times (1-r) \\ &= c(1-B) + c(1-A)(1-r) \frac{\log(1-r)}{r}. \end{aligned} \quad (31)$$

This radius is sharp.

Proof. By Lemma 5, condition (27) assures that r_0 is the $\mathcal{ST}[A, B]$ -radius of $f \in \mathcal{A}_b$ where r_0 is the real root of (31). Therefore, using (8) and (19) for $f \in \mathcal{A}_b$ yields

$$\begin{aligned} & \sum_{n=2}^{\infty} ((1-B)n - (1-A)) |a_n| r_0^{n-1} \\ & \leq 2(2(1-B) - (1-A)) br_0 \\ & \quad + \sum_{n=3}^{\infty} ((1-B)n - (1-A)) \left(\frac{c}{n} \right) r_0^{n-1} \\ &= 2(2(1-B) - (1-A)) br_0 \\ & \quad + c(1-B) \left(\frac{1}{1-r_0} - 1 - r_0 \right) \\ & \quad - c(1-A) \left(-\frac{\log(1-r_0)}{r_0} - 1 - \frac{r_0}{2} \right) \\ &= c(B-A) + (2(1-B) - (1-A))r_0 \left(2b - \frac{c}{2} \right) \\ & \quad + \frac{c(1-B)r_0 + c(1-A)(1-r_0)\log(1-r_0)}{(1-r_0)r_0} \\ &= A-B. \end{aligned} \quad (32)$$

The result is sharp for the function f_0 given by (21). Indeed, f_0 satisfies

$$\begin{aligned} & \left| \frac{zf'_0(z)}{f_0(z)} - 1 \right| \\ &= \frac{-(c/2 - 2b)r_0 + c/(1-r_0) + (c \log(1-r_0))/r_0}{(1+c) + (c/2 - 2b)r_0 + (c \log(1-r_0))/r_0}, \\ & \left| A - B \frac{zf'_0(z)}{f_0(z)} \right| \\ &= \left((1+c)(A-B) + (A-2B) \left(\frac{c}{2} - 2b \right) r_0 \right. \\ & \quad \left. + \frac{cB}{1-r_0} + \frac{cA \log(1-r_0)}{r_0} \right) \\ & \quad \times \left((1+c) + \left(\frac{c}{2} - 2b \right) r_0 + \frac{c \log(1-r_0)}{r_0} \right)^{-1}, \end{aligned} \quad (33)$$

at the root $z = r_0$ in $(0, 1)$ of (31). Evidently, the function f_0 satisfies (30), and hence the result is sharp. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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