Research Article

On F-Algebras $M^p (1 < p < \infty)$ of Holomorphic Functions

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We consider the classes $M^p (1 < p < \infty)$ of holomorphic functions on the open unit disk $D$ in the complex plane. These classes are in fact generalizations of the class $M^1$ introduced by Kim (1986). The space $M^p$ equipped with the topology given by the metric

$$\rho_p(f, g) = \|f - g\|_p = \left(\int_0^{2\pi} \log^+(1 + M(f - g)(\theta))(d\theta/2\pi)\right)^{1/p},$$

with $f, g \in M^p$ and $Mf(\theta) = \sup_{0 < r < 1} |f(re^{i\theta})|$, becomes an $F$-space. By a result of Stoll (1977), the Privalov space $N^p (1 < p < \infty)$ with the topology given by the Stoll metric $d_p$ is an $F$-algebra. By using these two facts, we prove that the spaces $M^p$ and $N^p$ coincide and have the same topological structure. Consequently, we describe a general form of continuous linear functionals on $M^p$ (with respect to the metric $\rho_p$). Furthermore, we give a characterization of bounded subsets of the spaces $M^p$. Moreover, we give the examples of bounded subsets of $M^p$ that are not relatively compact.

1. Introduction and Preliminaries

Let $D$ denote the open unit disk in the complex plane and let $\mathbb{T}$ denote the boundary of $D$. Let $L^q(\mathbb{T}) (0 < q \leq \infty)$ be the familiar Lebesgue spaces on the unit circle $\mathbb{T}$.

Following Kim ([1, 2]), the class $M$ consists of all holomorphic functions $f$ on $D$ for which

$$\int_0^{2\pi} \log^+ Mf(\theta) \frac{d\theta}{2\pi} < \infty,$$

where $\log^+|a| = \max\{|\log a|, 0\}$ and

$$Mf(\theta) = \sup_{0 < r < 1} |f(re^{i\theta})|$$

is the maximal radial function of $f$. The Privalov class $N^p (1 < p < \infty)$ consists of all holomorphic functions $f$ on $D$ for which

$$\sup_{0 < r < 1} \int_0^{2\pi} \left(\log^+ |f(re^{i\theta})|\right)^p \frac{d\theta}{2\pi} < +\infty.$$

These classes were firstly considered by Privalov in [3, page 93], where $N^p$ is denoted as $A_q$.

Notice that for $p = 1$, the condition (3) defines the Nevanlinna class $N$ of holomorphic functions in $D$. Recall that the Smirnov class $N^+$ is the set of all functions $f$ holomorphic on $D$ such that

$$\lim_{r \to 1} \int_0^{2\pi} \log^+ \left|f(re^{i\theta})\right| \frac{d\theta}{2\pi} = \int_0^{2\pi} \log^+ \left|f^*(e^{i\theta})\right| \frac{d\theta}{2\pi} < +\infty,$$

where $f^*$ is the boundary function of $f$ on $\mathbb{T}$; that is,

$$f^*(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta})$$

is the radial limit of $f$ which exists for almost every $e^{i\theta}$. We denote by $H^q (0 < q \leq \infty)$ the classical Hardy space on $D$. It is known (see [4, 5]) that

$$N^+ \subseteq N^p (r > p), \quad \bigcup_{q > 0} H^q \subseteq \bigcap_{p > 1} N^p,$$

$$\bigcup_{p > 1} N^p \subseteq M \subseteq N^+ \subseteq N,$$

where the above containment relations are proper.

The study of the spaces $N^p (1 < p < \infty)$ was continued in 1977 by Stoll [6] (with the notation $(\log^+ H)^a$ in [6]). Further,
the topological and functional properties of these spaces were studied in [4, 5, 7–14]; typically, the notation varied and these spaces are called the Privalov spaces in \[12–15\].

It is well known [16, page 26] that a function \( f \in N^p \) if and only if \( f = IF \), where \( I \) is an inner function on \( \mathbb{D} \) and \( F \) is an outer function given by

\[
F(z) = \exp \left( \int_0^{2\pi} e^{it} \log |F(e^{it})| \frac{dt}{2\pi} \right),
\]

(7)

where \( \log |F(z)| \in L^1(\mathbb{T}) \).

Privalov [3, page 98] showed that \( f \in N^p \) if and only if \( f = IF \), where \( I \) is an inner function on \( \mathbb{D} \) and \( F \) is an outer function as given above with \( \log |F(z)| \in L^1(\mathbb{T}) \).

Stoll [6, Theorem 4.2] showed that the space \( N^p \) (with the notation \( \log^+ H^a \) in [6]) with the topology given by the metric \( d_p \) defined by

\[
d_p(f, g) = \left( \int_0^{2\pi} \left( \log (1 + |f^*(e^{it}) - g^*(e^{it})|) \frac{dt}{2\pi} \right)^{1/p} \right),
\]

(8)

becomes an \( F \)-algebra. Recall that the function \( d_1 = d \) defined on the Smirnov class \( N^1 \) by (8) with \( p = 1 \) induces the metric topology on \( N^1 \). Yanagihara [17] showed that, under this topology, \( N^1 \) is an \( F \)-space.

Furthermore, in connection with the spaces \( N^p (1 < p < \infty) \), Stoll [6] (also see [7] and [12, Section 3]) also studied the spaces \( F^q (0 < q < \infty) \) (with the notation \( F_{1/q} \) in [6]), consisting of those functions \( f \) holomorphic on \( \mathbb{D} \) for which

\[
\lim_{r \to 1} (1 - r)^{1/2q} \log^+ M_{\infty}(r, f) = 0,
\]

(9)

where

\[
M_{\infty}(r, f) = \max_{|z| < r} |f(z)|.
\]

(10)

Stoll [6, Theorem 3.2] proved that the space \( F^q \) with the topology given by the family of seminorms \( \|\cdot\|_{q, c} \) defined for \( f \in F^q \) as

\[
\|f\|_{q, c} = \sum_{n=0}^{\infty} |\widehat{f}(n)| e^{-cn^{1/(q-1)}} < \infty,
\]

(11)

for each \( c > 0 \), where \( \widehat{f}(n) \) is the \( n \)th Taylor coefficient of \( f \), becomes a countably normed Fréchet algebra. By a result of Eoff [7, Theorem 4.2], \( F^p \) is the Fréchet envelope of \( N^p \), and hence \( F^p \) and \( N^p \) have the same topological duals.

Here, as always in the sequel, we will need some of Stoll’s results concerning the spaces \( F^p \) only with \( 1 < q < \infty \), and hence we will assume that \( q = p > 1 \) is any fixed number.

The study of the class \( M \) has been extensively investigated by Kim in [1, 2], Gavrilov and Zaharyan [18], and Nawrocky [19]. Kim [2, Theorems 3.1 and 6.1] showed that the space \( M \) with the topology given by the metric \( \rho \) defined by

\[
\rho(f, g) = \int_0^{2\pi} \log (1 + M(f - g)(\theta)) \frac{d\theta}{2\pi}, \quad f, g \in M
\]

(12)

becomes an \( F \)-algebra. Furthermore, Kim [2, Theorems 5.2 and 5.3] gave an incomplete characterization of multipliers of \( M \) into \( F^p \). Consequently, the topological dual of \( M \) is not exactly determined in [2], but, as an application, it was proved in [2, Theorem 5.4] (also cf. [19, Corollary 4]) that \( M \) is not locally convex space. Furthermore, the space \( M \) is not locally bounded ([2, Theorem 4.5] and [19, Corollary 5]).

Although the class \( M \) is essentially smaller than the class \( N^p \), Nawrocky [19] showed that the class \( M \) and the Smirnov class \( N^p \) have the same corresponding locally convex structure which was already established by Yanagihara for the Smirnov class in [17, 20]. More precisely, it was proved in [19, Theorem 1] that the Fréchet envelope of the class \( M \) can be identified with the space \( F^p \) of holomorphic functions on the open unit disk \( \mathbb{D} \) such that

\[
\|f\|_c := \sum_{n=0}^{\infty} |\widehat{f}(n)| e^{-cn^{1/2q}} < \infty,
\]

(13)

for each \( c > 0 \), where \( \widehat{f}(n) \) is the \( n \)th Taylor coefficient of \( f \). Notice that \( F^p \) coincides with the space \( F^3 \) defined above. It was shown in [17, 21] that \( E^p \) is actually the containing Fréchet space for \( N^p \). Moreover, Nawrocky [19, Theorem 1] characterized the set of all continuous linear functionals on \( M \) which by a result of Yanagihara [17] coincides with those on the Smirnov class \( N^p \).

Motivated by the mentioned investigations of the classes \( M \) and \( N^p \), and the fact that the classes \( N^p (1 < p < \infty) \) are generalizations of the Smirnov class \( N^1 \), in Section 2, we consider the classes \( M^p (1 < p < \infty) \) as generalizations of the class \( M \). Accordingly, the \( M^p (1 < p < \infty) \) consists of all holomorphic functions \( f \) on \( \mathbb{D} \) for which

\[
\int_0^{2\pi} \log^+ \left( \int_0^{2\pi} M(f(\theta)) \frac{d\theta}{2\pi} \right)^{1/p} < \infty.
\]

(14)

Obviously,

\[
\bigcup_{p>1} M^p \subset M.
\]

(15)

Following [2], by analogy with the space \( M \), the space \( M^p \) is equipped with the topology induced by the metric \( \rho_p \) defined as

\[
\rho_p(f, g) = \|f - g\|_p = \left( \int_0^{2\pi} \log^+ \left( 1 + M(f - g)(\theta) \right) \frac{d\theta}{2\pi} \right)^{1/p},
\]

(16)

with \( f, g \in M^p \).

In Section 2, we give the integral limit criterion for a function \( f \) holomorphic on the disk \( \mathbb{D} \) to belong to the class \( M^p \) (Lemma 3). Furthermore, we prove that the space \( M^p \) is closed under integration (Theorem 4).

In Section 3 we study and compare the uniform convergence on compact subsets of \( \mathbb{D} \) and the convergences induced by the metrics \( \rho_p \) and \( d_p \) in the space \( M^p \), respectively. It is proved (Theorem 11) that \( M^p = N^p \) for each \( p > 1 \).
It is proved in Section 4 that the space of all polynomials on \( C \) is a dense subset of \( M^p \) (Theorem 13). Hence, \( M^p \) is a separable metric space. We show that the space \( M^p \) with the topology given by the metric \( \rho_p \) becomes an \( F \)-space (Theorem 15). As an application, we prove that the metric spaces \((M^p, \rho_p)\) and \((N^p, d_p)\) have the same topological structure (Theorem 16). Consequently, we obtain a characterization of continuous linear functionals on \( M^p \) (Theorem 17). Notice that Theorem 17 with \( p = 1 \) characterizes the set of all continuous linear functionals on the space \( M \), which is in fact the Nawrocky result [19, Theorem 1] mentioned above.

In Section 5 we obtain a characterization of bounded subsets of the spaces \( M^p(=N^p) \) (Theorem 19). It is also given another necessary condition for a subset of \( C \)

Recall that, for a fixed \( M \), \( 1 < p < \infty \)

Combining the inequalities \( \log(|a| + 1) \leq \log|a| + \log 2 \) and \( \frac{1}{1+p} (\log^+ |b| + |c|)^p \leq 2^p - 1 \left( \frac{1}{1+p} (\log^+ |a|)^p + (\log 2)^p \right) \) \((a, b, c \in C)\). The last inequality implies the fact that the condition (17) is equivalent to

\[
\|f\|_p := \left( \int_0^{2\pi} \left( \log^+ (1 + Mf(\theta)) \right)^p \frac{d\theta}{2\pi} \right)^{\frac{1}{p}} < \infty. \tag{18}
\]

**Theorem 2.** The function \( \rho_p \) defined on \( M^p \) as

\[
\rho_p(f, g) = \|f - g\|_p = \left( \int_0^{2\pi} \log^+ (1 + M(f - g)(\theta)) \frac{d\theta}{2\pi} \right)^{\frac{1}{p}}, \tag{23}
\]

\( f, g \in M^p \)

is a translation invariant metric on \( M^p \). Further, the space \( M^p \) is a complete metric space with respect to the metric \( \rho_p \).

**Proof.** If we suppose that \( \rho_p(f, g) = 0 \), for some \( f, g \in M^p \), then by (23) it follows that \( M(f - g)(\theta) = 0 \) for almost every \( \theta \in [0, 2\pi] \). Hence, \( f^*(e^{i\theta}) = g^*(e^{i\theta}) \) for almost every \( e^{i\theta} \in \mathbb{T} \), and, by Riesz uniqueness theorem, we infer that \( f(z) = g(z) \) for all \( z \in D \). As, by (19), the triangle inequality is satisfied, it follows that \( \rho_p \) is a metric on \( M^p \). Finally, by the obvious inequality

\[
\rho_p(f + h, g + h) = \rho_p(f, g), \quad f, g, h \in M^p, \tag{24}
\]

we see that \( \rho_p \) is a translation invariant metric. This concludes the proof.

For simplicity, here as always in the sequel, we shall write \( M^p \) instead of the metric space \((M^p, \rho_p)\). For a function \( f \) holomorphic in \( D \) and for any fixed \( 0 \leq \rho < 1 \), denote by \( f_\rho \) the function defined on \( D \) as \( f_\rho(z) = f(\rho z), z \in D \). Furthermore, for a given holomorphic function \( f \) on \( D \), let

\[
Mf_\rho(\theta) = \sup_{0 \leq r \leq \rho} \left| f(\rho e^{i\theta}) \right|, \quad 0 \leq \rho < 1. \tag{25}
\]

**Lemma 3.** A function \( f \) holomorphic on the unit disk \( D \) belongs to the class \( M^p \) if and only if it satisfies

\[
\lim_{\rho \to 1} \int_0^{2\pi} \left( \log^+ Mf_\rho(\theta) \right)^p \frac{d\theta}{2\pi} < \infty. \tag{26}
\]

**Proof.** The condition (26) implies that \( f \in M^p \). Conversely, assume that \( f \in M^p \). Then

\[
Mf_\rho(\theta) \to Mf(\theta) \quad \text{as} \quad \rho \to 1
\]

for almost every \( \theta \in [0, 2\pi] \).

Since, by the assumption, \( f \in M^p \); that is, \( \int_0^{2\pi} \left( \log^+ Mf_\rho(\theta) \right)^p (d\theta/2\pi) < \infty \), using (27) and applying the Lebesgue dominated convergence theorem, we obtain

\[
\lim_{\rho \to 1} \int_0^{2\pi} \left( \log^+ Mf_\rho(\theta) \right)^p \frac{d\theta}{2\pi} = \int_0^{2\pi} \left( \log^+ Mf(\theta) \right)^p \frac{d\theta}{2\pi}, \tag{28}
\]

which completes the proof.
Theorem 4. The space $M^p$ is closed under integration.

Proof. For a given function $f \in M^p$, define

$$ F(z) = \int_0^z f(z)\,dz = \int_0^r f(\rho e^{i\theta})\,e^{i\theta}\,d\theta. $$

(29)

It follows that $|F(re^{i\theta})| \leq Mf(\theta)$, and thus $Mf(\theta) \leq Mf(\theta)$ for almost every $\theta \in [0,2\pi]$. Therefore $F \in M^p$, as desired. \qed

3. Convergences in the Space $M^P$

Theorem 5. For each function $f \in M^p$, $f_\rho \to f$ in $M^p$ as $\rho \to 1$.

Proof. Assume that $f \in M^p$. Since $f \in N^p$, by Fatou’s theorem, the radial limit $f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$ exists for almost every $\theta \in [0,2\pi]$. Hence, for each a fixed $\theta$, the function $r \mapsto f(re^{i\theta})$ is a continuous on $[0,1]$, and thus it is uniformly continuous on $[0,1]$. Therefore, for such a $\theta$, we have

$$ M\left(f - f_\rho\right)(\theta) \to 0 \text{ as } \rho \to 1 - . \quad (30) $$

By the inequality

$$ \log \left(1 + M\left(f - f_\rho\right)(\theta)\right) $$

$$ \leq \log \left(1 + Mf(\theta)\right) + \log \left(1 + Mf_\rho(\theta)\right) $$

$$ \leq 2 \log \left(1 + Mf(\theta)\right), $$

in view of the fact that (18) is satisfied for $f \in M^p$, we obtain

$$ \log^p \left(1 + M\left(f - f_\rho\right)(\theta)\right) \leq 2^p \log^p \left(1 + Mf(\theta)\right) \in L^1(\mathbb{T}). $$

(32)

From this and (30), by the Lebesgue dominated convergence theorem, we obtain

$$ \int_0^{2\pi} \left(\log \left(1 + M\left(f - f_\rho\right)(\theta)\right)\right)^p \frac{d\theta}{2\pi} \to 0, $$

$$ \text{as } \rho \to 1 - . \quad (33) $$

That is, $f_\rho \to f$ in $M^p$ as $\rho \to 1$.

So by the triangle inequality, for each $n \geq k$, we have

$$ \rho_p\left(f_n, (f_n)_{\rho}\right) \leq \rho_p\left(f_n, f_k\right) + \rho_p\left(f_k, (f_k)_{\rho}\right) $$

$$ + \rho_p\left((f_k)_{\rho}, (f_n)_{\rho}\right) $$

$$ \leq 2 \rho_p\left(f_n, f_k\right) + \rho_p\left(f_k, (f_k)_{\rho}\right) $$

$$ < 2 \varepsilon \frac{3}{3} + \rho_p\left(f_k, (f_k)_{\rho}\right). $$

(35)

By Theorem 5, there exists $0 < \rho_0 < 1$ sufficiently near to 1, for which

$$ \rho_p\left(f_l, (f_l)_{\rho}\right) < \frac{\varepsilon}{3} \text{ for each } \rho_0 < \rho < 1, $$

(36)

for each $l = 1, \ldots, k$.

Hence, by (35), we immediately obtain

$$ \rho_p\left(f_n, (f_n)_{\rho}\right) < \varepsilon \text{ for each } \rho_0 < \rho < 1, \text{ for each } n \in \mathbb{N}. $$

(37)

This completes proof of Lemma 6. \qed

Lemma 7. For any $p > 1$, $M^p \subseteq N^p$ and

$$ d_p\left(f, g\right) \leq \rho_p\left(f, g\right) \text{ for each } f, g \in M^p, $$

(38)

where $d_p$ is the metric of $N^p$ defined by (8).

Proof. The inclusion $M^p \subseteq N^p$ is obvious, and (38) follows by the definition of the metrics $d_p$ and $\rho_p$. \qed

Lemma 8. The convergence with respect to the metric $d_p$ of the space $N^p$ is stronger than the metric of uniform convergence on compact subsets of the disk $\mathbb{D}$.

Proof. The assertion immediately follows from the inequality on [5, page 898], which implies that, for any function $f \in N^p$ and $0 \leq r < 1$, we have

$$ \max_{|z|=r} |f(z)| \leq \exp\left(\left(\frac{1+r}{1-r}\right)^{1/p} d_p(f,0)\right). $$

(39)

This completes proof of Lemma 8. \qed

Lemma 9. If $\{f_n\}$ is a Cauchy sequence in the space $M^p$, then $\{f_n\}$ converges uniformly on compact subsets of $\mathbb{D}$ to some holomorphic function $f$ on $\mathbb{D}$.

Proof. From the inequality (38) of Lemma 7, it follows that $\{f_n\}$ is a Cauchy sequence in $N^p$. Therefore, there exists $f \in N^p$ such that $f_n \to f$ in $N^p$, and so, by Lemma 8, $f_n \to f$ uniformly on compact subsets of $\mathbb{D}$.

The following result is a maximal theorem of Hardy and Littlewood.
Lemma 10 (see [16, page11]). Let $1 < p \leq +\infty$ and let $\varphi$ be a function in the Lebesgue space $L^p(T)$. Let
\[
u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - t) + r^2} \varphi(t) \, dt, \quad 0 \leq r < 1
\] (40)
be the Poisson integral of the function $\varphi$. Define
\[U(\theta) = \sup_{0 < r < 1} |\nu(r, \theta)|, \quad \theta \in [0, 2\pi].
\] (41)
Then $U \in L^p(T)$ and there is a constant $A_p$, depending only on $p$ such that
\[\|U\|_{L^p} \leq A_p \|\varphi\|_{L^p}, \quad (42)
\]
where $\| \cdot \|_{L^p}$ is the usual norm of the space $L^p(T)$.

We are now ready to state the following result.

Theorem 11. $M^p = N^p$ for each $p > 1$; that is, the spaces $M^p$ and $N^p$ coincide.

Proof. By Lemma 7, $M^p \subseteq N^p$ for each $p > 1$. For the proof of the converse of this inclusion, assume that $f \in N^p$. We will show that $f \in M^p$. As noticed in Section 1, $f$ can be factorized as
\[f(z) = I(z)F(z), \quad z \in D,
\] (43)
where $I(z)$ is the inner function and $F(z)$ is an outer function for the class $N^p$; that is,
\[F(z) = \omega \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - t) + r^2} \log |f^*(e^{it})| \, dt \right), \quad (44)
\]
where $\omega$ is a constant of unit modulus. Furthermore, $\log |f^*| \in L^p(T)$. As $|I(z)| \leq 1$, for each $z \in D$, the previous factorization and the fact that $F \in M^p$ immediately imply that $f \in M^p$. Since
\[\frac{e^{it} + z}{e^{it} - z} = \frac{1 - r^2}{1 - 2r \cos(\theta - t) + r^2}, \quad z = re^{it},
\] (45)
from (44), we immediately obtain
\[
\log |F(\rho e^{it})| = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - t) + r^2} \log |f^*(e^{it})| \, dt, \quad 0 \leq r < 1,
\] (46)
whence it follows that, for $0 \leq r < 1$,
\[
\log^+ |F(\rho e^{it})| = \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - t) + r^2} \log^+ |f^*(e^{it})| \, dt \right)^+ \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - t) + r^2} \log^+ |f^*| \, dt.
\] (47)

The above inequality yields
\[
\log^+ MF(\theta) \leq \sup_{0 < r < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - t) + r^2} \log^+ |f^*| \, dt \right).
\] (48)

From the above inequality and the fact that $\log^+ |f^*| \in L^p(T)$, we conclude by Lemma 10 that $\log^+ MF(\theta) \in L^p(T)$, which means that $F \in M^p$ and therefore $f \in M^p$. Thus $N^p \subseteq M^p$, and therefore $M^p = N^p$. This completes the proof.

Corollary 12. Let $f \in M^p$. Then
\[
\int_0^{2\pi} (\log^+ MF(\theta))^p \, d\theta \leq C_p \int_0^{2\pi} \left( \log^+ |f^*(e^{it})| \right)^p \, d\theta,
\] (49)
where $C_p$ is a nonnegative constant depending only on $p$.

Proof. Let $F$ be the outer factor in the canonical factorization of $f \in M^p$. From the proof of Theorem 11, we see that for the functions $U(\theta) = \log^+ MF(\theta)$ and $\varphi(\theta) = \log^+ |f^*(e^{it})|$ the inequality (42) can be applied from Lemma 10. The obtained inequality is in fact (49) with $F$ instead of $f$. Since $|f(\theta)| \leq |F(\theta)|$, for each $\theta \in D$, it follows that $MF(\theta) \leq MF(\theta)$ at almost every $\theta \in [0, 2\pi]$; thus (49) is obviously satisfied.

4. $M^p$ as an $F$-Algebra

Theorem 13. The space of all polynomials over $\mathbb{C}$ is a dense subset of $M^p$. Hence, $M^p$ is a separable metric space.

Proof. Suppose that $f \in M^p$. Since, for a fixed $0 \leq \rho < 1$, $f_\rho$ is a holomorphic function on the closed unit disk $\overline{D} : |z| \leq 1$, by Runge’s theorem, $f_\rho$ can be uniformly approximated by polynomials on $\overline{D}$. This together with the fact that, by Theorem 5, $f_\rho \rightarrow f$ in $M^p$ as $\rho \rightarrow 1$, yields that the space of all polynomials over $\mathbb{C}$ is a dense subset of $M^p$. Therefore, the set of all polynomials whose coefficients have rational real parts and rational imaginary parts becomes a countable dense subset of $M^p$. This concludes the proof.

Theorem 14. $M^p$ is a complete metric space.

Proof. Let $\{f_n\}$ be a Cauchy sequence in $M^p$. Then since $N^p$ is complete, there is a $f \in N^p$ such that $f_n \rightarrow f$ in $N^p$. Since, by Theorem 11, $M^p = N^p$, it follows that $f \in M^p$, and thus it remains to show that $f_n \rightarrow f$ in $M^p$. By Theorem 5 and Lemma 6, there exist $0 < r < 1$ and $n_1 \in \mathbb{N}$ such that
\[
\rho_p(f_n, f) < \frac{\varepsilon}{3}, \quad \rho_p(f_n, f_{n_1}) < \frac{\varepsilon}{3} \quad \text{for each } n \geq n_1.
\] (50)
Since, by Lemma 9, a sequence \( \{f_n\} \) converges uniformly on each closed disk \( |z| \leq \rho < 1 \) to some function \( f \), it follows that there exists \( n_2 \in \mathbb{N} \) such that
\[
|\rho_\rho (f_n, f) < \frac{\varepsilon}{3} \quad \text{for each } n \geq n_2. \tag{51}
\]
Taking \( n_0 = \max\{n_1, n_2\} \), by (50) and (51), the triangle inequality implies that
\[
|\rho_\rho (f_n, f) < \varepsilon \quad \forall n \geq n_0. \tag{52}
\]
This shows that \( f_n \to f \) in \( M^p \), which completes the proof.

**Theorem 15.** \( M^p \) with the topology given by the metric \( \rho_p \) defined by (23) becomes an F-space.

**Proof.** By [22, page 51], it suffices to show the following properties:

1. \( \rho_p \) is an additive-invariant metric,
2. for any fixed \( f \in M^p \), \( c \mapsto cf \) is a continuous map from \( C \) into \( M^p \),
3. for any fixed \( c \in C \), \( f \mapsto cf \) is a continuous map from \( M^p \) into \( M^p \), and
4. \( M^p \) is a complete metric space.

The assertion (i) follows from Theorem 2.

By the Lebesgue dominated convergence theorem, we have
\[
|\rho_p (cf, 0) = \left( \frac{1}{2\pi} \int_0^{2\pi} \log^+ (1 + |c|^p Mf (\theta)) \frac{d\theta}{2\pi} \right)^{1/p} \to 0 \tag{53}
\]
as \( c \to 0 \).

Let \( k \in \mathbb{N} \) such that \( |c| \leq k \). Then the triangle inequality yields
\[
|\rho_p (cf, 0) \leq \rho_p (kf, 0) \leq k \rho_p (f, 0), \tag{54}
\]
whence we see that \( f \mapsto cf \) is a continuous map from \( M^p \) into \( M^p \).

The assertion (iv) is in fact the assertion of Theorem 14. This concludes the proof.

We are now ready to prove that the (metric) spaces \( (M^p, \rho_p) \) and \( (N^p, d_p) \) have the same topological structure.

**Theorem 16.** For each \( p > 1 \), the classes \( M^p \) and \( N^p \) coincide, and the metric spaces \( (M^p, \rho_p) \) and \( (N^p, d_p) \) have the same topological structure.

**Proof.** Consider the identity map \( j : M^p \to N^p \). Then, by the inequality (38) of Lemma 7, \( j \) is continuous. Since, by Theorem 11, \( M^p = N^p \), \( j \) maps \( M^p \) onto \( N^p \). Since \( M^p \) and \( N^p \) are both F-spaces, it follows, by the open mapping theorem [23, Corollary 2.12 (b)], that the inverse map \( j^{-1} \) of \( j \) is continuous. Hence, \( j \) is a homeomorphism, and so the metrics \( d_p \) and \( \rho_p \) induce the same topology on \( N^p \) and \( M^p \), respectively.

As an application of Theorem 16 and using the characterization of topological dual of the space \( F^p \) (which is by [7, Theorem 4.2] the Fréchet envelope of \( N^p \)) given by Stoll [6, Theorem 3.3] (cf. also [12, Theorem 3.5] and [13, Theorem 2]), we immediately get the following result.

**Theorem 17.** If \( y \) is a continuous linear functional on \( M^p \), then there exists a sequence \( \{y_n\} \) of complex numbers with \( y_n = O(\exp(-cn^{1/(p+1)}) \), for some \( c > 0 \), such that
\[
\gamma (f) = \sum_{n=0}^\infty d_n y_n, \tag{55}
\]
where \( f(z) = \sum_{n=0}^\infty a_n z^n \in M^p \), with convergence being absolute. Conversely, if \( \{y_n\} \) is a sequence of complex numbers for which
\[
\gamma_n = O \left( \exp \left( -cn^{1/(p+1)} \right) \right), \tag{56}
\]
then (55) defines a continuous linear functional on \( M^p \).

**Corollary 18.** \( M^p \) is an F-algebra.

**Proof.** By Theorem 15, \( M^p \) becomes an F-space. As \( N^p \) is an F-algebra, by Theorem 16, the multiplication is also continuous on \( M^p \). Hence, \( M^p \) is an F-algebra.

5. Bounded Subsets of \( M^p \)

It is proved in Section 4 (Theorem 16) that the spaces \( M^p \) and \( N^p \) coincide and have the same topological structure. Since \( N^p \) and \( M^p \) are not Banach spaces, it is of interest to obtain a characterization of bounded subsets of these spaces in terms of both metrics \( d_p \) and \( \rho_p \).

Recall that, for a function \( f \in N^p \), its boundary function \( f^* \) is defined as the radial limit \( f^*(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta}) \) which exists for almost every \( e^{i\theta} \in \mathbb{T} \).

The following result gives a characterization of bounded subsets of \( N^p(= M^p) \). Recall that the assertion (i)=\( \iff \) (iii) is analogous to Theorem 1 in [21] that describes bounded subsets of \( N^* \).

**Theorem 19.** For given set \( L \subset M^p \), the following conditions are equivalent:

1. \( L \) is a bounded subset of \( M^p \);
2. for all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that
\[
\int_E (\log^+ Mf (\theta))^p \frac{d\theta}{2\pi} < \varepsilon \quad \forall f \in L, \tag{57}
\]
for every measurable set \( E \subset \mathbb{T} \) with the Lebesgue measure \( |E| < \delta \);
3. for all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that
\[
\int_E (\log^+ |f^*(e^{i\theta})|)^p \frac{d\theta}{2\pi} < \varepsilon \quad \forall f \in L, \tag{58}
\]
for each measurable set \( E \subset \mathbb{T} \) with the Lebesgue measure \( |E| < \delta \).
Proof. (ii)⇒(iii). It follows that from the obvious inequality
\[ |f^*(e^{i\theta})| \leq M_f(\theta), f \in M^p, \text{for almost every } \theta \in [0, 2\pi]. \]
(iii)⇒(i). Let
\[ V = \{ g \in N^p : d_p(g, 0) < \eta \} \]
be an arbitrary neighborhood of zero in \( N^p \). Choose sufficiently small \( \epsilon > 0 \) such that
\[ \log^p (1 + \epsilon) + 2p^{-1} \log^p 2\delta + 2p^{-1} \epsilon < \eta^p. \]
Now it follows that there exists \( \delta, 0 < \delta < \epsilon, \) such that (iii) holds. Choose an \( n \in \mathbb{N} \) for which \( 1/n < \delta \). Set
\[ E_k = \{ e^{i\theta} : \theta \in \left[ \frac{2(k-1)\pi}{n}, \frac{2k\pi}{n} \right], \ k = 1, 2, \ldots, n. \]
Then \( |E_k| = 1/n < \delta, \) and thus by (iii) we have
\[ \int_{2\pi}^{0} (\log^p |f^*(e^{i\theta})|)^p \frac{d\theta}{2\pi} = \sum_{k=1}^{n} \int_{E_k} |e^{i\theta}| < ne \ \forall f \in L. \] (62)

By (62) and Chebyshev’s inequality, we conclude that for every function \( f \in L^p \) there exists a measurable set \( E_f \subset \mathbb{T} \) depending on \( f \) such that
\[ |T \setminus E_f| < \delta, \quad (\log^p |f^*(e^{i\theta})|)^p \leq \frac{ne}{\delta} \text{ on } E_f. \] (63)

From (63), we obtain
\[ |f^*(e^{i\theta})| \leq \exp \left( \frac{ne}{\delta} \right)^{1/p} = K(\delta) = K \text{ on } E_f. \] (64)

Choose \( \alpha \) such that \( 0 < \alpha < \epsilon/\delta. \) Then using the inequality
\[ \log^p (1 + |\alpha|) \leq 2p^{-1} \left( (\log^p |\alpha|)^p + \log^p 2 \right), \] (65)
and (iii), for every \( f \in L, \) we obtain
\[ (d_p(\alpha f, 0))^p \]
\[ = \int_{2\pi}^{0} \log^p \left( 1 + |\alpha f^*(e^{i\theta})| \right) \frac{d\theta}{2\pi} \]
\[ = \int_{E_f} + \int_{T \setminus E_f} \]
\[ \leq \int_{E_f} \log^p (1 + \epsilon) \frac{d\theta}{2\pi} \]
\[ + 2p^{-1} \left( \int_{T \setminus E_f} \log^p 2 \frac{d\theta}{2\pi} + \int_{T \setminus E_f} \left( \log^p |f^*(e^{i\theta})| \right)^p \frac{d\theta}{2\pi} \right) \]
\[ \leq \log^p (1 + \epsilon) + 2p^{-1} \log^p 2 \delta + 2p^{-1} \epsilon \]
\[ < \eta^p. \] (66)

Therefore, \( d_p(\alpha f, 0) < \eta, \) from which it follows that \( \alpha L \subset V. \) Hence, \( L \) is a bounded subset of \( N^p. \)

(i)⇒(ii). Assume that \( L \) is a bounded subset of \( M^p. \) Then for any given \( \eta > 0 \) there is a \( \alpha_0 = \alpha_0(\eta), 0 < \alpha_0 < 1, \) such that
\[ \left( \rho_p(\alpha f, 0) \right)^p = \int_{0}^{2\pi} \log^p (1 + |\alpha|Mf(\theta)) \frac{d\theta}{2\pi} < \eta^p \] (67)
for each \( f \in L \) and \( |\alpha| \leq \alpha_0. \) It follows that
\[ \int_{0}^{2\pi} (\log^p |\alpha|Mf(\theta))^p \frac{d\theta}{2\pi} < \eta^p \] (68)
for each \( f \in L, |\alpha| \leq \alpha_0. \) Since
\[ \log^p Mf(\theta) \leq \log^p \alpha_0 Mf(\theta) + \log \frac{1}{\alpha_0}, \] (69)
we obtain
\[ (\log^p Mf(\theta))^p \leq 2p^{-1} \left( (\log^p \alpha_0 Mf(\theta))^p + \left( \log \frac{1}{\alpha_0} \right)^p \right). \] (70)

For given \( \epsilon > 0, \) choose \( \eta > 0 \) satisfying
\[ \eta < \frac{\epsilon^{1/p}}{2}, \] (71)
and \( \alpha_0 = \alpha_0(\eta) \) satisfying (67) and so also satisfying (68). Next, take \( \delta > 0 \) such that
\[ \delta \log^p \frac{1}{\alpha_0} < \frac{\epsilon}{2p}. \] (72)

Then for each set \( E \subset \mathbb{T} \) with \( |E| < \delta, \) by (68)–(72), for every \( f \in L, \) we obtain
\[ \int_{E} (\log^p Mf(\theta))^p \frac{d\theta}{2\pi} \]
\[ \leq 2p^{-1} \left( \int_{E} (\log^p \alpha_0 Mf(\theta))^p \frac{d\theta}{2\pi} + \int_{E} \log^p \frac{1}{\alpha_0} \frac{d\theta}{2\pi} \right) \]
\[ \leq 2p^{-1} \eta^p + 2p^{-1} |E| \log^p \frac{1}{\alpha_0} \]
\[ \leq \epsilon. \]
(73)

Therefore, the condition (ii) of the theorem is satisfied, which concludes the proof. \( \square \)

Remark 20. Note that the condition (ii) from Theorem 19 in fact means that the family \( \{ (\log^p Mf(\theta))^p : f \in L \} \) is uniformly integrable on \( \mathbb{T}. \) The same assertion is also valid for the condition (iii). On the other hand, from the proof of Theorem 19, we see that (ii) implies that the family \( \{ (\log^p Mf(\theta))^p : f \in L \} \) forms a bounded subset of the space \( L^1(\mathbb{T}); \) that is, there holds
\[ \limsup_{f \in L} \int_{0}^{2\pi} (\log^p Mf(\theta))^p \frac{d\theta}{2\pi} < +\infty. \] (74)
Similarly, it follows from (iii) that the family \( \{ (\log^+ |f^* (e^\theta)|)^p : f \in L \} \) is bounded in \( L^1(\mathbb{T}) \).

**Corollary 21.** If \( L \) is a subset of \( M^p \) for which the family
\[
\left\{ (\log^+ |f^* (e^\theta)|)^p : f \in L \right\}
\]
is uniformly integrable, then the family
\[
\left\{ (\log^+ |f (re^\theta)|)^p : f \in L, 0 \leq r < 1 \right\}
\]
is also uniformly integrable.

**Proof.** The condition of Corollary 21 and (iii)\( \Rightarrow \) (ii) of Theorem 19 immediately yield that the family \( \{ (\log^+ Mf(\theta))^p : f \in L \} \) is uniformly integrable on the circle \( \mathbb{T} \). This fact and the obvious inequality \( |f(re^\theta)| \leq Mf(\theta), f \in M^p, 0 \leq r < 1 \), for almost every \( \theta \in [0, 2\pi] \), imply that the family \( \{ (\log^+ |f(re^\theta)|)^p : f \in L, 0 \leq r < 1 \} \) is uniformly integrable. \( \square \)

The following result gives a necessary condition for a subset of \( M^p(= N^p) \) to be bounded.

**Theorem 22.** Let \( L \) be a subset of \( M^p \). If \( L \) is bounded in \( M^p \), then
\[
M_\infty (r, f) \leq K \exp \left( \frac{\omega(r)}{(1-r)^{3/p}} \right) \quad \text{for each } f \in L,
\]
where \( M_\infty (r, f) = \max_{0 \leq \theta < 2\pi} |f(re^\theta)|, K \) is a positive constant, and \( \omega(r), 0 \leq r < 1 \), is a positive continuous function that does not depend on \( f \in L \) and for which \( \omega(r) \downarrow 0 \) as \( r \to 1 \).

**Proof.** By the inequality (5.4) from the proof of Theorem 5.2 in [4], for all \( f \in N^p \), we have
\[
\left( \log^+ |f (re^\theta)| \right)^p \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{1-r^2}{1-2r \cos(\theta-t)+r^2} (\log^+ |f^* (e^\theta)|)^p \, dt.
\]
(78)
As, by the assumption, \( L \) is a bounded subset of \( N^p \), by Theorem 19 (iii), for all \( \epsilon > 0 \) there exists \( \delta = \delta(\epsilon) > 0 \), such that
\[
\int_0^{2\pi} (\log^+ |f^* (e^\theta)|)^p \, d\theta < \frac{\epsilon}{2} \quad \forall f \in L
\]
(79)
and for every measurable set \( E \subset \mathbb{T} \) with the Lebesgue measure \( |E| < \delta \).

Further, from the proof of (iii)\( \Rightarrow \) (i) of Theorem 19, we see that for each \( f \in N^p \) there is a measurable set \( E_f \subset \mathbb{T} \) depending on \( f \) for which
\[
|\mathbb{T} \setminus E_f| < \delta, \quad \left( \log^+ |f^* (e^\theta)| \right)^p \leq \frac{\epsilon}{p} \delta
\]
(80)
for almost every \( e^\theta \in E_f \). From (78)–(80), we obtain
\[
\left( \log^+ |f (re^\theta)| \right)^p = \int_{E_f} + \int_{E_f} \leq \frac{\epsilon}{p} + \frac{1}{1-r^2} \frac{\epsilon}{\delta}.
\]
(81)
whence it follows that
\[
(1-r) \left( \log^+ M_\infty (r, f) \right)^p \leq \frac{(1-r) \epsilon}{\delta} + \frac{\epsilon}{2}.
\]
(82)
Choose a sequence \( \{ \epsilon_k \} \) of positive numbers such that \( \epsilon_k \downarrow 0 \).

For each \( k \in \mathbb{N} \), let \( r_k > 0 \) be a number such that
\[
\frac{(1-r_k) \epsilon_k}{\delta_k} + \frac{\epsilon_k}{2} < \epsilon_k,
\]
(83)
where \( \epsilon_k = \delta(\epsilon_k) \) and
\[
r_k-1 < r_k < 1, \quad r_k \uparrow 1 \quad \text{as } k \to \infty.
\]
(84)
Put
\[
\omega_1 (r) = \epsilon_k \quad \text{for } r_k \leq r < r_{k+1}, \quad k = 1, 2, \ldots.
\]
(85)
From (82), (83), and (85) we obtain
\[
\left( \log^+ M_\infty (r, f) \right)^p \leq \frac{\omega_1 (r)}{1-r} \quad \forall 0 \leq r < 1.
\]
(86)
Since
\[
\omega_1 (r) \to 0 \quad \text{as } r \to 1,
\]
(87)
we conclude that there exists a continuous function \( \omega_2 (r) \) satisfying
\[
\omega_1 (r) \leq \omega_2 (r), \quad \omega_2 (r) \downarrow 0 \quad \text{as } r \to 1.
\]
(88)
Therefore,
\[
\left( \log^+ M_\infty (r, f) \right)^p \leq \frac{\omega_2 (r)}{1-r} \quad \text{for each } 0 \leq r < 1,
\]
(89)
whence by setting
\[
\omega_2 (r) = \left( \omega_2 (r) \right)^{1/p} \quad \text{for each } 0 \leq r < 1,
\]
(90)
we obtain
\[
M_\infty (r, f) \leq \exp \left( \frac{\omega_2 (r)}{(1-r)^{1/p}} \right) \quad \forall f \in L.
\]
(91)
This concludes the proof. \( \square \)

**Remark 23.** The condition of Theorem 22 is not a sufficient condition for a set \( L \subset M^p \) to be bounded. To show this, define
\[
f_n (z) = a_n z^n, \quad a_n = \exp \left( \lambda_n r^{n/(p+1)} \right),
\]
(92)
where
\[ \lambda_n = n^{-1/2(p+1)}. \] (93)

Then as in the proof of Lemma 1 in [21] it is easy to verify that the set \( L = \{ f_n \} \subset M^p \) satisfies the condition of Theorem 22. Since
\[ \log |f_n^* (e^{i\theta})| = n^{-1/2(p+1)}, \] (94)
we see that \( L \) is not bounded in \( M^p \).

**Theorem 24.** There exist bounded subsets of \( M^p \) that are not relatively compact.

**Proof.** Define a sequence \( \{ h_n \} \) of functions on \([0, 2\pi]\) as
\[ h_n (t) = 1 + \sin(nt), \quad t \in [0, 2\pi], \] (95)
and set
\[ f_n (z) = \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} h_n (t) \, dt \right) \] (96)
\[ = \exp \left( 1 - iz^n \right), \quad z \in \mathbb{D}. \]

Obviously, \( \{ f_n \} \subset N^p \) and for each measurable set \( E \subset \mathbb{T} \) we have
\[ \int_0^{2\pi} h_n (t) \, dt = 2\pi, \] (97)
\[ 0 \leq \int_E h_n (t) \, dt \leq 2 |E|, \]
where \( |E| \) denotes the Lebesgue measure of \( E \). From this and Theorem 19, we see that the set \( L = \{ f_n \} \) is bounded in \( N^p \).

Now suppose that \( E \) is relatively compact. This means that there exists a subsequence \( \{ f_{nk} \} \) of \( \{ f_n \} \) and a function \( f \in N^p \) such that
\[ d_p (f_{nk}, f) \to 0 \quad \text{as} \quad k \to \infty, \] (98)
and thus
\[ f_{nk} (z) \to f (z), \] (99)
uniformly on each closed disk \( |z| \leq r < 1 \).

Therefore, by (96), it follows that \( f(z) \equiv e \) on \( \mathbb{D} \). On the other hand, from (98), it follows that
\[ f_{nk}^* (e^{i\theta}) \to f^* (e^{i\theta}) \quad \text{in measure on } \mathbb{T}. \] (100)

Therefore,
\[ \log^+ |f_{nk}^* (e^{i\theta})| = 1 + \sin (n_k \theta) \to \log^+ |f^* (e^{i\theta})| = 1 \]
in measure on \( \mathbb{T}. \) (101)

This contradiction shows that \( L \) is not relatively compact in \( N^p \). \( \square \)

**Conflict of Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.

**References**


