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Research Article

A Class of Nonlocal Coupled Semilinear Parabolic System with Nonlocal Boundaries

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We investigate the positive solutions of the semilinear parabolic system with coupled nonlinear nonlocal sources subject to weighted nonlocal Dirichlet boundary conditions. The blow-up and global existence criteria are obtained.

1. Introduction

In this paper, we consider the positive solutions of the semilinear parabolic system with coupled nonlinear nonlocal sources subject to weighted nonlocal Dirichlet boundary conditions:

$$\begin{gathered} u_{it} = \Delta u_i + \int_\Omega u_i^{q_i} u_{i+1}^{p_i}(x,t)\,dx, \\ i = 1,2,\ldots,k, \quad u_{k+1} = u_1, \quad x \in \Omega, \quad t > 0, \\ u_i(x,t) = \int_\Omega \varphi_i(x,y)\,u_i(y,t)\,dy, \\ i = 1,2,\ldots,k, \quad x \in \partial\Omega, \quad t > 0, \\ u_i(x,0) = u_{i,0}(x), \quad i = 1,2,\ldots,k, \ x \in \Omega, \end{gathered} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \ge 1$, with smooth boundary $\partial\Omega$. The exponents $p_i > 0$, $q_i \ge 0$. The weighted functions φ_i in the boundary conditions are continuous, nonnegative on $\partial\Omega \times \overline{\Omega}$ and $\int_\Omega \varphi_i(x,y)dy > 0$ on $\partial\Omega$. The initial data $u_{i0}(x) \in C^{2+\nu}(\overline{\Omega})$ with $0 < \nu < 1$, $u_{i0}(x) \ge 0, \not\equiv 0$, and satisfy the compatibility conditions.

Many physical phenomena were formulated into nonlocal mathematical models and studied by many authors [1–13]. For example, in [1], Bebernes and Bressan studied an ignition model for a compressible reactive gas which is a nonlocal reaction-diffusion equation. Furthermore, Bebernes et al. [14] considered a more general model:

$$\begin{gathered} u_t - \Delta u = f(u) + g(t), \quad x \in \Omega, \ t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega, \\ u(x,t) = 0, \quad x \in \partial\Omega, \ t > 0, \end{gathered} \tag{2}$$

where $u_0(x) \ge 0, g(t) > 0$ or $g(t) = (k/|\Omega|)\int_\Omega u_t(x,t)dx$ with $k > 0$. Chadam et al. [15] studied another form of (2) with $f(u) = 0$ and $g(t) = \int_\Omega \psi(u(x,t))dx$ and proved that the blow-up set is the whole region (including the homogeneous Neumann boundary conditions). Souplet [16, 17] considered (2) with the general function $g(t)$. Pao [18] discussed a nonlocal reaction-diffusion equation arising from the combustion theory.

The problems with both nonlocal sources and nonlocal boundary conditions have been studied as well. To motivate our study, we give a short review of examples of such parabolic equations or systems studied in the literature. For example, Lin and Liu [19] studied the following problem:

$$\begin{gathered} u_t - \Delta u = \int_\Omega f(u(y,t))\,dy, \quad x \in \Omega, \ t > 0, \\ u(x,t) = \int_\Omega \varphi(x,y)\,u(y,t)\,dy, \quad x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega; \end{gathered} \tag{3}$$

they established local existence, global existence, and nonexistence of solutions and discussed the blow-up properties of solutions.

Gladkov and Kim [20] considered the problem of the form

$$\begin{gathered} u_t = \Delta u + c(x,t)\,u^p, \quad x \in \Omega,\ t > 0, \\ u(x,t) = \int_\Omega \varphi(x,y)\,u^l(y,t)\,dy, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{gathered} \tag{4}$$

with $p, l > 0$. And some criteria for the existence of global solution as well as for the solution to blow up in finite time were obtained.

In [21], Kong and Wang studied system (1) when $k = 2$:

$$\begin{gathered} u_t = \Delta u + \int_\Omega u^m(x,t)\,v^n(x,t)\,dx, \quad x \in \Omega,\ t > 0, \\ v_t = \Delta v + \int_\Omega u^p(x,t)\,v^q(x,t)\,dx, \quad x \in \Omega,\ t > 0, \\ u(x,t) = \int_\Omega \varphi(x,y)\,u(y,t)\,dy, \\ v(x,t) = \int_\Omega \psi(x,y)\,v(y,t)\,dy, \\ x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega; \end{gathered} \tag{5}$$

they obtained the following results, and we extend them as follows.

(i) Assume that $m, q < 1$ and $np \le (1-m)(1-q)$ hold; then the solution of (5) exists globally.

(ii) If one of the following conditions holds:

$$\begin{gathered} \text{(a)}\ m > 1, \\ \text{(b)}\ q > 1, \\ \text{(c)}\ np > (1-m)(1-q), \end{gathered} \tag{6}$$

then the solution of (5) blows up in a finite time for the sufficiently large initial data.

(iii) Assume that $\int_\Omega \varphi(x,y)dy \ge 1$ and $\int_\Omega \psi(x,y)dy \ge 1$ for all $x \in \partial\Omega$ and one of (6) holds; then the solution of problem (5) blows up in a finite time for any positive initial data.

Recently, Zheng and Kong [22] also studied the following problem:

$$\begin{gathered} u_t - \Delta u = u^m(x,t) \int_\Omega v^n(x,t)\,dx, \quad x \in \Omega,\ t > 0, \\ v_t - \Delta v = v^q(x,t) \int_\Omega u^p(x,t)\,dx, \quad x \in \Omega,\ t > 0, \end{gathered} \tag{7}$$

with the same initial and boundary conditions as (5), and they established similar conditions for global and nonglobal solutions and also blow-up solutions.

The main purpose of this paper is to get the blow-up criterion of problem (1) for any positive integer k.

In the following, we set $Q_T = \Omega \times (0,T)$, and $S_T = \partial\Omega \times (0,T)$ with $0 < T < \infty$ for convenience.

It is known by the standard theory [16, 23] that there exists a local positive solution to (1). Moreover, by the comparison principle (see Lemma 10 in the next section), the uniqueness of solutions holds if $p_i, q_i \ge 1$, $i = 1, 2, \ldots, k$.

Theorem 1. *Problem* (1) *has a positive classical solution* $(u_1, u_2, \ldots, u_k) \in [C^{2+\widehat{\alpha}, 1+\widehat{\alpha}/2}(Q_T) \cap C(\overline{Q}_T)]^k$ *for some* $\widehat{\alpha} : 0 < \widehat{\alpha} < 1$. *Moreover, if* $T < \infty$, *then*

$$\lim_{t \to T} \left(\|u_1(\cdot,t)\|_\infty + \cdots + \|u_k(\cdot,t)\|_\infty \right) = \infty. \tag{8}$$

Theorem 2. *If exponents* p_i, q_i, $i = 1, 2, \ldots, k$ *satisfy*

$$\begin{gathered} q_i < 1, \quad i = 1, 2, \ldots, k, \\ p_1 p_2 \cdots p_k \le (1-q_1)(1-q_2)\cdots(1-q_k), \end{gathered} \tag{9}$$

the solution $(u_1, u_2, \ldots, u_k)$ *of* (1) *exists globally for any nontrivial nonnegative initial data.*

Theorem 3. *If exponents* p_i, q_i, $i = 1, 2, \ldots, k$ *satisfy one of the following:*

$$\begin{gathered} (a)\ q_r > 1, \quad r \in \{1, 2, \ldots, k\}, \\ (b)\ p_1 p_2 \cdots p_k > (1-q_1)(1-q_2)\cdots(1-q_k) \end{gathered} \tag{10}$$

and if $\int_\Omega \varphi_i(x,y)dy < 1$, $i = 1, 2, \ldots, k$, *for all* $x \in \partial\Omega$, *then the solution of* (1) *exists globally for small nonnegative initial data.*

Theorem 4. *If exponents* p_i, q_i, $i = 1, 2, \ldots, k$ *satisfy one of the following:*

$$\begin{gathered} (a)\ q_r > 1, \quad r \in \{1, 2, \ldots, k\}, \\ (b)\ p_1 p_2 \cdots p_k > (1-q_1)(1-q_2)\cdots(1-q_k), \end{gathered} \tag{11}$$

then the solution of (1) *blows up in finite time for large initial data.*

If the initial data $u_{i,0}(x)$ satisfies

$$(H)\ \Delta u_{i,0} + \int_\Omega u_{i,0}^{q_i} u_{i,0+1}^{q_i} \ge 0, \quad i = 1, 2, \ldots, k, \tag{12}$$

we have another blow-up result.

Theorem 5. *Assume that*

$$q_r > 1, \qquad \int_\Omega \varphi_r(x,y)\,dy \ge 1, \quad r \in \{1, 2, \ldots, k\} \tag{13}$$

and the condition (H) holds. Then the solution of (1) *blows up in finite time for any positive initial data.*

This paper is organized as follows. Section 2 is devoted to some comparison principles. In Section 3, we prove two global existence results. The blow-up results are proved in the final section.

2. Comparison Principle

Before proving the main results, we give the maximum and comparison principles related to the problem. First, we give the following definition of the upper and lower solutions.

Definition 6. A pair of functions $(\overline{u}_1(x,t),\ldots,\overline{u}_k(x,t))$ is called an upper solution of (1), if, for every $i = 1,2,\ldots,k$, $\overline{u}_i(x,t) \in C^{2,1}(Q_T)\cap C(\overline{Q}_T)$ and satisfies

$$\begin{gathered}\overline{u}_{it} \geq \Delta\overline{u}_i + \int_\Omega \overline{u}_i^{q_i}\overline{u}_{i+1}^{p_i}(x,t)\,dx, \qquad \overline{u}_{k+1} = \overline{u}_1, \\ x\in\Omega,\ t>0, \\ \overline{u}_i(x,t) \geq \int_\Omega \varphi_i(x,y)\,\overline{u}_i(y,t)\,dy, \quad x\in\partial\Omega,\ t>0, \\ \overline{u}_i(x,0) \geq u_{i,0}(x), \quad x\in\Omega. \end{gathered} \tag{14}$$

Similarly, a lower solution of (1) is defined by the opposite inequalities.

Lemma 7. *Suppose that $a_{ij}, b_i, f_i \in C(\overline{Q}_T)$ and $f_i \geq 0$, c_i, $d_i \geq 0$ in Q_T, $g_i(x,y) \geq 0$ on $\partial\Omega\times\overline{\Omega}$, $\int_\Omega g_i(x,y)dy > 0$ on $\partial\Omega$, $i = 1,2,\ldots,k$, $j = 1,2,\ldots,N$. If, for every $i = 1,2,\ldots,k$, $w_i \in C^{2,1}(Q_T)\cap C(\overline{Q}_T)$ and satisfies*

$$\begin{gathered} w_{it} - \Delta w_i \geq \sum_{j=1}^N a_{ij}\frac{\partial w_i}{\partial x_j} + b_i w_i \\ + f_i(x,t)\int_\Omega (c_i w_i + d_i w_{i+1})\,dx, \quad (x,t)\in Q_T, \\ w_i(x,t) \geq \int_\Omega g_i(x,y)\,w_i(y,t)\,dy, \quad (x,t)\in S_T, \\ w_i(x,0) > 0, \quad x\in\Omega, \end{gathered} \tag{15}$$

where $w_{k+1} = w_1$, then $w_i(x,t) > 0$, $i = 1,2$, on $\overline{Q}_T$.

Proof. Set $\overline{b}_i = \sup_{\overline{Q}_T}|b_i|$, $z_i = e^{-Kt}w_i$ with $K > \max\{\overline{b}_i, i = 1,2,\ldots,k\}$. Then

$$\begin{gathered} z_{it} - \Delta z_i + (K-b_i)\,z_i \\ \geq \sum_{j=1}^N a_{ij}\frac{\partial z_i}{\partial x_j} + f_i(x,t)\int_\Omega (c_i z_i + d_i z_{i+1})\,dx, \quad (x,t)\in Q_T, \\ z_i(x,t) \geq \int_\Omega g_i(x,y)\,z_i(y,t)\,dy, \quad (x,t)\in S_T, \\ z_i(x,0) > 0, \quad x\in\Omega. \end{gathered} \tag{16}$$

Since $z_i(x,0) > 0$, $i = 1,2,\ldots$, there exists $\delta > 0$ such that $z_i > 0$ for $(x,t)\in\overline{\Omega}\times(0,\delta)$. Suppose for a contradiction that $\overline{t} = \sup\{t\in(0,T) : z_i > 0 \text{ on } \overline{\Omega}\times[0,t), i = 1,2,\ldots,k\} < T$. Then $z_i \geq 0$ on $\overline{Q}_{\overline{t}}$, and at least one of z_i vanishes at $(\overline{x},\overline{t})$ for some $\overline{x}\in\overline{\Omega}$. Without loss of generality, suppose that $z_1(\overline{x},\overline{t}) = 0 = \inf_{\overline{Q}_{\overline{t}}} z_1$. If $(\overline{x},\overline{t})\in Q_{\overline{t}}$; by virtue of the first inequality of (16), we find that

$$z_{1t} - \Delta z_1 + (K-b_1)\,z_1 - \sum_{j=1}^N a_{ij}\frac{\partial z_i}{\partial x_j} \geq 0, \quad (x,t)\in\overline{Q}_{\overline{t}}. \tag{17}$$

This leads to the conclusion that $z_1 \equiv 0$ in $\overline{Q}_{\overline{t}}$ by the strong maximum principle, a contradiction. If $(\overline{x},\overline{t})\in S_{\overline{t}}$, this results in a contradiction too, that

$$0 = z_1(\overline{x},\overline{t}) = \int_\Omega g_1(x,y)\,z_1(y,t)\,dy > 0 \tag{18}$$

due to $\int_\Omega g_1(x,y)dy > 0$ on $\partial\Omega$. This proves that $z_1 > 0$ and consequently $w_1 > 0$. We complete the proof. □

Lemma 8. *Suppose that, for every $i = 1,2,\ldots,k$, $w_i \in C^{2,1}(Q_T)\cap C(\overline{Q}_T)$ and satisfies*

$$\begin{gathered} w_{it} - \Delta w_i \geq \int_\Omega (a_i(x,t)\,w_i + b_i(x,t)\,w_{i+1})\,dx, \quad (x,t)\in Q_T, \\ w_i(x,t) \geq \int_\Omega g_i(x,y)\,w_i(y,t)\,dy, \quad (x,t)\in S_T, \\ w_i(x,0) \geq 0, \quad x\in\Omega, \end{gathered} \tag{19}$$

where $w_{k+1} = w_1$ and $a_i(x,t), b_i(x,t)$ are continuous, nonnegative functions in $\overline{Q}_T$, $g_i(x,y) \geq 0$ on $\partial\Omega\times\overline{\Omega}$ such that $\int_\Omega g_i(x,y)dy < 1$ on $\partial\Omega$, and there exist positive constants C_i such that $\int_\Omega (a_i(x,t) + b_i(x,t))dx \leq C_i$. Then $w_i(x,t) \geq 0$, $i = 1,2$, on $\overline{Q}_T$.

Proof. Suppose that the strict inequalities of (19) hold; by Lemma 7, we have $w_i(x,t) > 0$. Now we consider the general case. Set

$$v_i = w_i + \varepsilon e^{Kt}, \tag{20}$$

where ε is any fixed positive constant, and $K = 1 + \max\{\int_\Omega (a_i(x,t) + b_i(x,t))\,dx, i = 1,2,\ldots,k\}$. By (19), we get, for $i = 1,2,\ldots,k$,

$$\begin{gathered} v_{it} - \Delta v_i - \int_\Omega (a_i(x,t)\,v_i + b_i(x,t)\,v_{i+1})\,dx \\ \geq \varepsilon e^{Kt}\left(K - \int_\Omega (a_i(x,t) + b_i(x,t))\,dx\right) > 0, \\ (x,t)\in Q_T, \\ v_i(x,t) - \int_\Omega g_i(x,y)\,v_i(y,t)\,dy \\ \geq \varepsilon e^{Kt}\left(1 - \int_\Omega g_i(x,y)\,dy\right) > 0, \quad (x,t)\in S_T, \\ v_i(x,0) \geq \varepsilon e^{Kt} > 0, \quad x\in\Omega, \end{gathered} \tag{21}$$

Therefore, we have $v_i(x,t) \geq 0$ on Q_T. Letting $\varepsilon \to 0^+$, we get the desired result. □

If the boundary condition $\int_\Omega g_i(x,y)\mathrm{d}y < 1$ is not necessarily valid, we have the following result. The argument of its proof can be referred to [22, Lemma 2.2].

Lemma 9. *Suppose that $a_{ij}, b_i, f_i \in C(\overline{Q}_T)$, $f_i \geq 0$, c_i, d_i, are nonnegative and bounded in Q_T, $g_i(x,y) \geq 0$ on $\partial\Omega \times \overline{\Omega}$, $\int_\Omega g_i(x,y)\mathrm{d}y > 0$ on $\partial\Omega$, $i = 1,2,\ldots,k$, $j = 1,2,\ldots,N$. If, for every $i = 1,2,\ldots,k$, $w_i \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ and satisfies*

$$\begin{gathered} w_{it} - \Delta w_i \geq \sum_{j=1}^{N} a_{ij}\frac{\partial w_i}{\partial x_j} + b_i w_i \\ + f_i(x,t)\int_\Omega (c_i w_i + d_i w_{i+1})\,\mathrm{d}x, \quad (x,t) \in Q_T, \\ w_i(x,t) \geq \int_\Omega g_i(x,y)\,w_i(y,t)\,\mathrm{d}y, \quad (x,t) \in S_T, \\ w_i(x,0) \geq 0, \quad x \in \Omega, \end{gathered} \tag{22}$$

where $w_{k+1} = w_1$, then $w_i(x,t) \geq 0$, $i = 1,2$, on $\overline{Q}_T$.

By Lemma 9, we can easily get the following result.

Lemma 10. *Let $(\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_k)$ and $(\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_k)$ be nonnegative upper and lower solution of system (1) on $\overline{Q}_t$, respectively. If one assumes that, for some $r \in \{1,2,\ldots,k\}$,*

(i) *$\overline{u}_{r+1} > \delta$ or $\underline{u}_{r+1} > \delta$ when $p_r < 1$,*

(ii) *$\overline{u}_r > \delta$ or $\underline{u}_r > \delta$ when $q_r < 1$,*

then $(\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_k) \geq (\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_k)$ on Q_T.

3. Global Existence Results

Before proving Theorem 2, we give a global existence result for a scalar equation.

Lemma 11. *Let $w_0(x)$ and $\varphi(x,y)$ be continuous, nonnegative functions on $\overline{\Omega}$ and $\partial\Omega\times\overline{\Omega}$, respectively, and let the nonnegative constants θ_{ij} satisfy $0 < \theta_{i1} + \theta_{i2} \leq 1$. Then the solutions of the nonlocal problem*

$$\begin{gathered} w_t - \Delta w = \sum_{i=1}^{k} w^{\theta_{i1}}(x,t)\int_\Omega w^{\theta_{i2}}(x,t)\,\mathrm{d}x, \quad x \in \Omega,\ t > 0, \\ w(x,t) = \int_\Omega \varphi(x,y)\,w(y,t)\,\mathrm{d}y, \quad x \in \partial\Omega,\ t > 0, \\ w(x,0) = w_0(x), \quad x \in \Omega \end{gathered} \tag{23}$$

exist globally.

Proof. The augment is similar to the proof of [22, Lemma 3.1] or [21, Lemma 6]. For the reader's convenience, we complete it. It is easy to prove that there exists a positive function $\psi \in C^2(\overline{\Omega})$ such that

$$\begin{gathered} \min_{\overline{\Omega}} \psi(x) > \max_{\overline{\Omega}} w_0^2(x), \\ \psi(x) \geq \int_\Omega \varphi^2(x,y)\,\mathrm{d}y \int_\Omega \psi(y)\,\mathrm{d}y, \\ x \in \partial\Omega. \end{gathered} \tag{24}$$

Let $\theta > 0$ be large enough such that

$$\begin{aligned} 2\theta \min_{\overline{\Omega}} \psi(x) \geq (2k+1)\max\Big\{ &\max_{\overline{\Omega}} |\Delta\psi(x)|, \\ &|\Omega| \Big[\max_{\overline{\Omega}} \psi(x)\Big]^{(\theta_{i1}+\theta_{i2}+1)/2} \\ &(i = 1,2,\ldots,k)\,|\Omega| \Big\}. \end{aligned} \tag{25}$$

Setting $z(x,t) = e^{2\theta t}\psi(x)$ for $(x,t) \in \Omega \times (0,\infty)$, one readily checks that

$$\begin{gathered} z_t - \Delta z \geq 2\sum_{i=1}^{k} z^{(\theta_{i1}+1)/2}(x,t)\int_\Omega z^{\theta_{i2}/2}(x,t)\,\mathrm{d}x, \\ x \in \Omega,\ t > 0, \\ z(x,t) \geq \int_\Omega \varphi^2(x,y)\,\mathrm{d}y \int_\Omega z(y,t)\,\mathrm{d}y, \quad x \in \partial\Omega,\ t > 0, \\ z(x,0) \geq w_0^2(x) + 1, \quad x \in \Omega, \end{gathered} \tag{26}$$

Let $\overline{w} = z^{1/2}(x,t)$; it follows that

$$\begin{gathered} \overline{w}_t - \Delta\overline{w} \geq \sum_{i=1}^{k} \overline{w}^{\theta_{i1}}(x,t)\int_\Omega \overline{w}^{\theta_{i2}}(x,t)\,\mathrm{d}x, \quad x \in \Omega,\ t > 0, \\ \overline{w}(x,t) \geq \int_\Omega \varphi^2(x,y)\,\mathrm{d}y \int_\Omega \overline{w}(y,t)\,\mathrm{d}y, \quad x \in \partial\Omega,\ t > 0, \\ \overline{w}(x,0) > w_0(x), \quad x \in \Omega. \end{gathered} \tag{27}$$

This implies that $\overline{w}$ is a global upper solution of (23). Clearly, 0 is a lower solution of it. So we complete the proof. □

Proof of Theorem 2. By (11), we know that there exists $a_i \in (0,1)$, $i = 1,2,\ldots,k$, such that

$$\frac{p_i}{1-q_i} \leq \frac{a_i}{a_{i+1}}, \quad i = 1,2,\ldots,k,\ a_{k+1} = a_1. \tag{28}$$

Define $\alpha = \sum_{i=1}^{k} 1/a_i$. Let $\Phi(x,y) \ge \max\{\varphi_i(x,y),\ i = 1,2,\ldots,k\}$ be a continuous function defined for $(x,y) \in \partial\Omega\times\overline{\Omega}$. Suppose that z solves

$$z_t - \Delta z = \alpha \sum_{i=1}^{k} z^{1-a_i}(x,t)\int_\Omega z^{a_i}(x,t)\,\mathrm{d}x, \quad x \in \Omega,\ t > 0,$$
$$z(x,t) = \sum_{i=1}^{k} g_i(x)\int_\Omega \Phi(x,y)\,z(y,t)\,\mathrm{d}y, \quad x \in \partial\Omega,\ t > 0,$$
$$z(x,0) = 1 + \sum_{i=1}^{k} u_{i,0}^{1/a_i}(x), \quad x \in \Omega, \tag{29}$$

where

$$g_i(x) = \left(\int_\Omega \Phi(x,y)\,\mathrm{d}y\right)^{(1-a_i)/a_i}. \tag{30}$$

In view of Lemma 11, we know that z is global. Moreover, $z > 1$ in $\overline{\Omega}\times[0,\infty)$ by the maximum principle. Set $\overline{u}_i = z^{a_i}$, $i = 1,2,\ldots,k$. By (28) and (29) and using Hölder's inequality, we get

$$\begin{aligned}
\overline{u}_{it} - \Delta\overline{u}_i - \int_\Omega \overline{u}_i^{q_i}\overline{u}_{i+1}^{p_i}\,\mathrm{d}x
&= a_i z^{a_i-1} z_t - a_i z^{a_i-1}\Delta z - a_i(a_i-1)|\nabla z|^2 \\
&\quad - \int_\Omega z^{a_i q_i + a_{i+1}p_i}\,\mathrm{d}x \\
&\ge a_i z^{a_i-1}(z_t - \Delta z) - \int_\Omega z^{a_i}\,\mathrm{d}x \ge (\alpha a_i - 1)\int_\Omega z^{a_i}\,\mathrm{d}x \\
&\ge 0, \quad (x,t) \in Q_T, \\
\overline{u}_i - \int_\Omega \varphi_i(x,y)\,\overline{u}_i(y,t)\,\mathrm{d}y
&= z^{a_i} - \int_\Omega \varphi_i(x,y)\,z^{a_i}(y,t)\,\mathrm{d}y \\
&\ge \left(\int_\Omega \varphi_i(x,y)\,\mathrm{d}y\right)^{1-a_i}\left(\int_\Omega \varphi_i(x,y)\,z(y,t)\,\mathrm{d}y\right)^{a_i} \\
&\quad - \int_\Omega \varphi_i(x,y)\,z^{a_i}(y,t)\,\mathrm{d}y \\
&\ge 0, \quad (x,t) \in S_T, \\
\overline{u}_i(x,0) &\ge u_{i,0}(x), \quad x \in \Omega.
\end{aligned} \tag{31}$$

This means that $(\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_k)$ is a global upper solution of (1). □

Proof of Theorem 3. Define

$$\max\left\{\sup_{\partial\Omega}\int_\Omega \varphi_i(x,y)\,\mathrm{d}y,\ i = 1,2,\ldots,k\right\} = \delta_0 \in (0,1). \tag{32}$$

Let w be the unique solution of the elliptic problem

$$-\Delta w = 1, \quad x \in \Omega; \quad w = C_0, \quad x \in \partial\Omega. \tag{33}$$

Then there exists a constant $M > 0$ such that $C_0 \le w(x) \le C_0 + M$ in $\overline{\Omega}$. We choose C_0 to be large enough such that

$$\frac{1 + C_0}{1 + C_0 + M} \ge \delta_0. \tag{34}$$

Set $\overline{u}_i(x,t) = b_i(1 + w(x))$. When $(x,t) \in S_T$, it follows that

$$\begin{aligned}
\overline{u}_i - \int_\Omega \varphi_i(x,y)\,\overline{u}_i(y,t)\,\mathrm{d}y
&= b_i(1 + C_0) - b_i\int_\Omega \varphi_i(x,y)(1 + w(y))\,\mathrm{d}y \\
&\ge b_i\left[1 + C_0 - (1 + C_0 + M)\delta_0\right] \\
&\ge 0.
\end{aligned} \tag{35}$$

Now we investigate $(x,t) \in Q_T$. Set $L_i = (1 + C_0 + M)^{p_i+q_i}|\Omega|$ for convenience. A simple computation yields

$$\begin{aligned}
\overline{u}_{it} - \Delta\overline{u}_i - \int_\Omega \overline{u}_i^{q_i}\overline{u}_{i+1}^{p_i}\,\mathrm{d}x
&= b_i - b_i^{q_i} b_{i+1}^{p_i}\int_\Omega (1 + w(x))^{p_i+q_i}\,\mathrm{d}x \\
&\ge b_i^{q_i}\left(b_i^{1-q_i} - b_{i+1}^{p_i} L_i\right).
\end{aligned} \tag{36}$$

(a) If $q_r > 1$, no matter $q_{r+1} > 1$ or $q_{r+1} \le 1$, we can choose b_r to be small enough such that $b_r^{1-q_r} \ge b_{r+1}^{p_r} L_r$. For fixed b_r, there exist b_i, $i = 1,2,\ldots,r-1,r+1,\ldots,k$, satisfying $b_i^{1-q_i} \ge b_{i+1}^{p_i} L_i$, $i = 1,2,\ldots,k$. It follows that

$$\overline{u}_{it} - \Delta\overline{u}_i - \int_\Omega \overline{u}_i^{q_i}\overline{u}_{i+1}^{p_i}\,\mathrm{d}x \ge 0, \quad i = 1,2,\ldots,k. \tag{37}$$

(b) If $q_i \le 1$, $i = 1,2,\ldots,k$ and $p_1 p_2 \cdots p_k > (1 - q_1)(1 - q_2)\cdots(1 - q_k)$, we can choose b_1 to be small enough such that

$$\begin{aligned}
&b_1^{(1-q_1)(1-q_2)\cdots(1-q_k)} \\
&\quad > b_1^{p_1 p_2 \cdots p_k} L_1^{(1-q_2)\cdots(1-q_k)} L_2^{p_1(1-q_3)\cdots(1-q_k)} \\
&\qquad \cdots L_{k-1}^{p_1 p_2 \cdots p_{k-2}(1-q_{k-1})} L_k^{p_1 p_2 \cdots p_{k-1}}.
\end{aligned} \tag{38}$$

Consequently, there exist $b_i > 0$, $i = 2,3,\ldots,k$, $b_{k+1} = b_1$ satisfying $b_i^{1-q_i} \ge b_{i+1}^{p_i} L_i$, $i = 1,2,\ldots,k$. Hence (37) holds too.

By (35) and (37), in any case (a) or (b), we know that the solution of (1) must be global for small data $u_{i,0}(x) \le b_i(1 + w(x))$, $i = 1,2,\ldots,k$ for $x \in \Omega$. □

4. Blow-Up Results

In this section, we assume that $(u(x,t), v(x,t))$ is a positive solution of (1) on $\overline{\Omega}\times[0,T)$, where T is the maximal existence time.

Proof of Theorem 4. We denote by $\lambda_1, \phi_1(x)$ the first eigenvalue and the corresponding eigenfunction of the linear elliptic problem:

$$-\Delta\varphi(x) = \lambda\varphi(x), \quad x \in \Omega; \qquad \varphi(x) = 0, \quad x \in \partial\Omega, \tag{39}$$

and $\phi_1(x)$ satisfies

$$\varphi_1(x) > 0, \quad x \in \Omega, \quad \max_{\overline{\Omega}} \phi_1(x) = 1. \tag{40}$$

Define $\gamma = \min\{\alpha_i(q_i - 1) + \alpha_{i+1}p_i + 1, i = 1, 2, \ldots, k\}$.

(a) If $q_r \geq 1$, we claim that there exist positive constants $\alpha_i > 1, i = 1, 2, \ldots, k$, such that the inequality

$$\alpha_i(q_i - 1) + \alpha_{i+1}p_i > 0 \tag{41}$$

holds. First, when $i = r$, (41) holds for any $\alpha_r, \alpha_{r+1} > 1$. When $i = r + 1$, if $q_{r+1} \geq 1$, (41) holds for any $\alpha_{r+2} > 1$; if $q_{r+1} \leq 1$ we can choose $\alpha_{r+2} > \max\{1, \alpha_{r+1}(1-q_{r+1})/p_{r+1}\}$. That is, (41) holds too. When $i = r-1$, if $q_{r-1} \geq 1$, (41) holds for any $\alpha_{r-1} > 1$; if $q_{r-1} < 1$, we can choose $1 < \alpha_{r-1} < (\alpha_i p_{r-1}/1-q_{r-1})$ such that (41) holds too.

(b) If $q_i < 1, i = 1, 2, \ldots, k$, and $p_1 p_2 \cdots p_k > (1 - q_1)(1 - q_2) \cdots (1 - q_k)$, we can choose $\alpha_i > 1$ such that

$$\frac{p_1}{1-q_1} > \frac{\alpha_1}{\alpha_2}, \qquad \frac{p_2}{1-q_2} > \frac{\alpha_2}{\alpha_3}, \ldots, \frac{p_k}{1-q_k} > \frac{\alpha_k}{\alpha_1}. \tag{42}$$

Hence (41) holds too.

Hence, for the case (a) or (b), we all have $\gamma > 1$. Now let $s(t)$ be the unique solution of the ODE problem

$$\begin{aligned} s'(t) &= -\lambda s(t) + l s^{\gamma}(t), \quad t > 0, \\ s(0) &= s_0 > 1, \end{aligned} \tag{43}$$

where $l = \min\{(1/\alpha_i)\int_\Omega \phi_1^{\alpha_i q_i + \alpha_{i+1} p_i}, i = 1, 2, \ldots, k\}$. Then $s(t)$ blows up in finite time $T(s_0)$ with s_0 being large enough.

Set

$$\begin{aligned} \underline{u}_i = s^{\alpha_i}(t)\phi_1^{\alpha_i}(x), \quad (x,t) \in \overline{\Omega} \times [0, T(s_0)), \\ i = 1, 2, \ldots, k. \end{aligned} \tag{44}$$

We will show that $(\underline{u}, \underline{v})$ is a lower solution of problem (1). A direct computation yields

$$\begin{aligned} &\underline{u}_{it} - \Delta\underline{u}_i - \int_\Omega \underline{u}_i^{q_i}\underline{u}_{i+1}^{p_i}\,dx \\ &= \alpha_i l s^{\alpha_i - 1 + \gamma}\phi_1^{\alpha_i} - \alpha_i(\alpha_i - 1)s^{\alpha_i}\phi_1^{\alpha_i - 2}|\nabla\phi_1|^2 \\ &\quad - \int_\Omega s^{\alpha_i q_i + a_{i+1}p_i}\phi_1{}^{\alpha_i q_i + a_{i+1}p_i}\,dx \\ &\leq \alpha_i l s^{\alpha_i - 1 + \gamma} - s^{\alpha_i q_i + a_{i+1}p_i}\int_\Omega \phi_1{}^{\alpha_i q_i + a_{i+1}p_i}\,dx \\ &\leq 0, \quad (x,t) \in \overline{\Omega} \times [0, T(s_0)), \\ &\underline{u}_i - \int_\Omega \varphi_i(x,y)\underline{u}_i(y,t)\,dy \\ &= 0 - s^{\alpha_i}(t)\int_\Omega \varphi_i(x,y)\phi_1^{\alpha_i}(y)\,dy \\ &\leq 0, \quad (x,t) \in \partial\Omega \times (0, T(s_0)). \end{aligned} \tag{45}$$

$(\underline{u}_1, \ldots, \underline{u}_k)$ is a blowing up lower solution of (1) provided the initial data are so large that $u_{i,0}(x) \geq s^{\alpha_i}(0)\phi_1^{\alpha_i}(x)$, $i = 1, 2, \ldots, k$ for $x \in \Omega$. We complete the proof. □

Proof of Theorem 5. Since $u_{i,0} > 0$ in Ω, $\int_\Omega \varphi_r(x,y)dy > 0$ on $\partial\Omega$, and

$$u_{i,0}(x) = \int_\Omega \varphi_r(x,y)u_{i,0}(y)\,dy, \quad x \in \partial\Omega, \tag{46}$$

by the compatibility conditions, we have $u_{i,0} > 0$ on $\partial\Omega$. Denote by η the positive constant such that $u_{i,0} > \eta$ on $\overline{\Omega}$. The assumption (H) implies that $(u_i)_t > 0$ by the comparison principle, and in turn $u_i > \eta$, $i = 1, 2, \ldots, k$ on $\overline{\Omega} \times [0, T)$. Furthermore, u_r satisfies

$$\begin{aligned} (u_r)_t &\geq \Delta u_r + |\Omega|\eta^{p_r}u_r^{q_r}, \quad (x,t) \in Q_T, \\ u_r &= \int_\Omega \varphi_r(x,y)u_r(y,t)\,dy, \quad (x,t) \in S_T, \\ u_r(x,0) &= u_{r,0}(x), \quad x \in \Omega. \end{aligned} \tag{47}$$

Let $z_r(t)$ be the solution of the following Cauchy problem:

$$\begin{aligned} z_r'(t) &= |\Omega|\eta^{p_r}z_r^{q_r}, \\ z_r(0) &= \frac{1}{2}\eta > 0. \end{aligned} \tag{48}$$

Clearly, $z_r(t)$ blows up under the condition

$$q_r > 1. \tag{49}$$

On the other hand, since $\int_\Omega \varphi_r(x,y)dy \geq 1$, by Lemma 9, we have $u_r \geq z_r$ as long as both u_r and z_r exist, and thus u_r blows up for any positive initial data. The proof now is completed. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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