Research Article
A Class of Nonlocal Coupled Semilinear Parabolic System with Nonlocal Boundaries

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We investigate the positive solutions of the semilinear parabolic system with coupled nonlinear nonlocal sources subject to weighted nonlocal Dirichlet boundary conditions. The blow-up and global existence criteria are obtained.

1. Introduction

In this paper, we consider the positive solutions of the semilinear parabolic system with coupled nonlinear nonlocal sources subject to weighted nonlocal Dirichlet boundary conditions:

\[ u_{it} = \Delta u_i + \int_\Omega u_i^q u_{i+1}^p (x,t) \, dx, \]
\[ i = 1, 2, \ldots, k, \quad u_{k+1} = u_1, \quad x \in \Omega, \quad t > 0, \]
\[ u_j (x,t) = \int_\Omega \phi_i (x,y) u_i (y,t) \, dy, \]
\[ i = 1, 2, \ldots, k, \quad x \in \partial \Omega, \quad t > 0, \]
\[ u_i (x,0) = u_{i0} (x), \quad i = 1, 2, \ldots, k, \quad x \in \Omega, \]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \), \( N \geq 1 \), with smooth boundary \( \partial \Omega \). The exponents \( p_1 > 0, q_i \geq 0 \). The weighted functions \( \phi_i \) in the boundary conditions are continuous, nonnegative on \( \partial \Omega \times \Omega \) and \( \int_\Omega \phi_i (x,y) \, dy > 0 \) on \( \partial \Omega \). The initial data \( u_{i0} (x) \in C^{2+\gamma} (\bar{\Omega}) \) with \( 0 < \gamma < 1, u_{i0} (x) \geq 0, \neq 0 \), and satisfy the compatibility conditions.

Many physical phenomena were formulated into nonlocal mathematical models and studied by many authors [1–13]. For example, in [1], Bebernes and Bressan studied an ignition model for a compressible reactive gas which is a nonlocal reaction-diffusion equation. Furthermore, Bebernes et al. [14] considered a more general model:

\[ u_t - \Delta u = f (u) + g (t), \quad x \in \Omega, \quad t > 0, \]
\[ u (x,0) = u_0 (x), \quad x \in \Omega, \]
\[ u (x,t) = 0, \quad x \in \partial \Omega, \quad t > 0, \]

where \( u_0 (x) \geq 0, g(t) > 0 \) or \( g(t) = (k/|\Omega|) \int_\Omega u_i (x,t) \, dx \) with \( k > 0 \). Chadam et al. [15] studied another form of (2) with \( f (u) = 0 \) and \( g(t) = \int_\Omega \psi (u(x,t)) \, dx \), proved that the blow-up set is the whole region (including the homogeneous Neumann boundary conditions). Souplet [16, 17] considered (2) with the general function \( g(t) \). Pao [18] discussed a nonlocal reaction-diffusion equation arising from the combustion theory.

The problems with both nonlocal sources and nonlocal boundary conditions have been studied as well. To motivate our study, we give a short review of examples of such parabolic equations or systems studied in the literature. For example, Lin and Liu [19] studied the following problem:

\[ u_t - \Delta u = \int_\Omega f (u (y,t)) \, dy, \quad x \in \Omega, \quad t > 0, \]
\[ u (x,t) = \int_\Omega \phi (x,y) u (y,t) \, dy, \quad x \in \partial \Omega, \quad t > 0, \]
\[ u (x,0) = u_0 (x), \quad x \in \Omega; \]
they established local existence, global existence, and nonexistence of solutions and discussed the blow-up properties of solutions.

Gladkov and Kim [20] considered the problem of the form
\[
\begin{align*}
  u_t &= \Delta u + c(x,t)u^p, \quad x \in \Omega, \quad t > 0, \\
  u(x,0) &= u_0(x), \quad x \in \Omega,
\end{align*}
\]
with \( p, l > 0 \). And some criteria for the existence of the global solution as well as for the solution to blow up in finite time were obtained.

In [21], Kong and Wang studied system (1) when \( k = 2 \):
\[
\begin{align*}
  u_t &= \Delta u + \int_\Omega u^m(x,t)\psi(x,t)\,dx, \quad x \in \Omega, \quad t > 0, \\
  v_t &= \Delta v + \int_\Omega u^p(x,t)\phi(x,t)\,dx, \quad x \in \Omega, \quad t > 0, \\
  u(x,0) &= u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega,
\end{align*}
\]
and the condition (H) holds. Then the solution of (1) blows up in finite time for any positive initial data.

The main purpose of this paper is to get the blow-up criterion of problem (1) for any positive integer \( k \).

Recently, Zheng and Kong [22] also studied the following problem:
\[
\begin{align*}
  u_t - \Delta u &= u^m(x,t)\int_\Omega \psi(x,t)\,dx, \quad x \in \Omega, \quad t > 0, \\
  v_t - \Delta v &= v^p(x,t)\int_\Omega u^q(x,t)\,dx, \quad x \in \Omega, \quad t > 0,
\end{align*}
\]
with the same initial and boundary conditions as (5), and they established similar conditions for global and nonglobal solutions and also blow-up solutions.

Theorem 1. Problem (1) has a positive classical solution \((u_1,u_2,\ldots,u_k) \in [C^{2+\hat{\alpha},1+\frac{\hat{\alpha}}{2}}(Q_T) \cap C(\overline{Q_T})]^k\) for some \( \hat{\alpha} : 0 < \hat{\alpha} < 1 \).

Theorem 2. If \( p_i, q_i, i = 1,2,\ldots,k \) satisfy
\[
  q_i < 1, \quad i = 1,2,\ldots,k,
\]
and if \( p_1p_2\cdots p_k \leq (1-q_1)(1-q_2)\cdots (1-q_k) \),
then the solution \((u_1,u_2,\ldots,u_k)\) of (1) exists globally for any nontrivial nonnegative initial data.

Theorem 3. If \( p_i, q_i, i = 1,2,\ldots,k \) satisfy one of the following:
\[
  (a) \quad q_i > 1, \quad r \in \{1,2,\ldots,k\},
\]
\[
  (b) \quad p_1p_2\cdots p_k > (1-q_1)(1-q_2)\cdots (1-q_k)
\]
and if \( \int_\Omega \phi_i(x,y)\,dy < 1, \quad i = 1,2,\ldots,k, \) for all \( x \in \partial \Omega \), then the solution of (1) exists globally for small nonnegative initial data.

Theorem 4. If \( p_i, q_i, i = 1,2,\ldots,k \) satisfy one of the following:
\[
  (a) \quad q_r > 1, \quad r \in \{1,2,\ldots,k\},
\]
\[
  (b) \quad p_1p_2\cdots p_k > (1-q_1)(1-q_2)\cdots (1-q_k),
\]
then the solution of (1) blows up in finite time for large initial data.

If the initial data \( u_{i0}(x) \) satisfies
\[
  (H) \quad \Delta u_{i0} + \int_\Omega \phi_{i0}(x)\,dx \geq 0, \quad i = 1,2,\ldots,k,
\]
we have another blow-up result.

Theorem 5. Assume that
\[
  q_r > 1, \quad \int_\Omega \phi_r(x,y)\,dy \geq 1, \quad r \in \{1,2,\ldots,k\}
\]
and the condition (H) holds. Then the solution of (1) blows up in finite time for any positive initial data.
This paper is organized as follows. Section 2 is devoted to some comparison principles. In Section 3, we prove two global existence results. The blow-up results are proved in the final section.

2. Comparison Principle

Before proving the main results, we give the maximum and comparison principles related to the problem. First, we give the following definition of the upper and lower solutions.

**Definition 6.** A pair of functions $(\overline{u}_i(x,t), \underbar{u}_i(x,t))$ is called an upper solution of (1), if, for every $i = 1, 2, \ldots, k$, $\overline{u}_i(x,t) \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ and satisfies

$$
\overline{u}_i \geq \Delta \overline{u}_i + \int_{\Omega} \overline{q}_i(x,y) \overline{u}_i(y,t) \, dy, \quad x \in \Omega, \ t > 0,
$$

$$
\overline{u}_i(x,0) \geq \overline{\alpha}_i(0), \quad x \in \Omega.
$$

Similarly, a lower solution of (1) is defined by the opposite inequalities.

**Lemma 7.** Suppose that $a_i, b_i, f_i \in C(\overline{Q}_T)$ and $f_i \geq 0$, $c_i$, $d_i \geq 0$ in $Q_T$, $g_i(x,y) \geq 0$ on $\partial \Omega \times \overline{\Omega}$, $\int_{\Omega} g_i(x,y) \, dy > 0$ on $\partial \Omega$, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, N$. If, for every $i = 1, 2, \ldots, k$, $w_i \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ and satisfies

$$
\frac{\partial w_i}{\partial x_j} \geq w_i + f_i(x,t) \int_{\Omega} (c_i w_i + d_i w_{i+1}) \, dx, \quad (x,t) \in Q_T,
$$

$$
w_i(x,t) \geq \int_{\Omega} g_i(x,y) w_i(x,y) \, dy, \quad (x,t) \in S_T,
$$

$$
w_i(x,0) \geq 0, \quad x \in \Omega,
$$

where $w_{k+1} = w_1$, then $w_i(x,t) > 0$, $i = 1, 2, \ldots, on \overline{Q}_T$.

**Proof.** Set $\overline{\beta}_i = \sup_{Q_T} \{ |z_i| \}$, $z_i = e^{-Kt}w_i$ with $K > \max \{ \overline{\beta}_i, i = 1, 2, \ldots, k \}$. Then

$$
z_i - \Delta z_i + (K - b_i) z_i \geq \sum_{j=1}^{N} a_{ij} \frac{\partial z_i}{\partial x_j} + f_i(x,t) \int_{\Omega} (c_i z_i + d_i z_{i+1}) \, dx, \quad (x,t) \in Q_T,
$$

$$
z_i(x,t) \geq \int_{\Omega} g_i(x,y) z_i(x,y) \, dy, \quad (x,t) \in S_T,
$$

$$
z_i(x,0) \geq 0, \quad x \in \Omega.
$$

Since $z_i(x,0) > 0$, $i = 1, 2, \ldots$, there exists $\theta > 0$ such that $z_i > 0$ for $(x,t) \in \overline{\Omega} \times (0, \delta)$. Suppose for a contradiction that $\tau = \sup \{ t \in (0,T) : z_i > 0 \text{ on } \overline{\Omega} \times (0, \tau), i = 1, 2, \ldots, k \} < T$. Then $z_i \geq 0$ on $\overline{\Omega}$, and at least one of $z_i$ vanishes at $(\overline{x}, \tau)$ for some $\overline{x} \in \overline{\Omega}$. Without loss of generality, suppose that $z_i(\overline{x}, \tau) = 0 = \inf z_i(\overline{x}, \tau)$. If $(\overline{x}, \tau) \in Q_T$, by virtue of the first inequality of (16), we find that

$$
z_d - \Delta z_d + (K - b_d) z_d \geq \sum_{j=1}^{N} a_{dj} \frac{\partial z_d}{\partial x_j}, \quad (x,t) \in \overline{Q}_T.
$$

This leads to the conclusion that $z_d \equiv 0$ in $Q_T$ by the strong maximum principle, a contradiction. If $(\overline{x}, \tau) \in S_T$, this results in a contradiction too, that

$$
0 = z_1(\overline{x}, \tau) = \int_{\Omega} g_1(x,y) z_1(x,y) \, dy > 0
$$

due to $\int_{\Omega} g_1(x,y) \, dy > 0$ on $\partial \Omega$. This proves that $z_1 > 0$ and consequently $w_1 > 0$. We complete the proof.

**Lemma 8.** Suppose that, for every $i = 1, 2, \ldots, k$, $w_i \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ and satisfies

$$
\frac{\partial w_i}{\partial x_j} \geq w_i + f_i(x,t) \int_{\Omega} (c_i w_i + d_i w_{i+1}) \, dx, \quad (x,t) \in Q_T,
$$

$$
w_i(x,t) \geq \int_{\Omega} g_i(x,y) w_i(x,y) \, dy, \quad (x,t) \in S_T,
$$

$$
w_i(x,0) \geq 0, \quad x \in \Omega,
$$

where $w_{k+1} = w_1$ and $a_i(x,t), b_i(x,t)$ are continuous, non-negative functions in $\overline{Q}_T$, $g_i(x,y) \geq 0$ on $\partial \Omega \times \overline{\Omega}$ such that $\int_{\Omega} g_i(x,y) \, dy < 1$ on $\partial \Omega$, and there exist positive constants $C_i$ such that $\int_{\Omega} (a_i(x,t) + b_i(x,t)) \, dx \leq C_i$. Then $w_i(x,t) \geq 0$, $i = 1, 2, \ldots, on \overline{Q}_T$.

**Proof.** Suppose that the strict inequalities of (19) hold; by Lemma 7, we have $w_i(x,t) > 0$. Now we consider the general case. Set $v_i = w_i + \varepsilon e^{Kt}$,

$$
v_i - \Delta v_i - \int_{\Omega} (a_i(x,t) v_i + b_i(x,t) v_{i+1}) \, dx
$$

$$
\geq \varepsilon e^{Kt} \left( K - \int_{\Omega} (a_i(x,t) + b_i(x,t)) \, dx \right) > 0,
$$

$$
v_i(x,t) - \int_{\Omega} g_i(x,y) v_i(x,y) \, dy
$$

$$
\geq \varepsilon e^{Kt} \left( 1 - \int_{\Omega} g_i(x,y) \, dy \right) > 0, \quad (x,t) \in S_T,
$$

$$
v_i(x,0) \geq \varepsilon e^{Kt} > 0, \quad x \in \Omega,
$$

where $\varepsilon$ is any fixed positive constant, and $K = 1 + \max \{ \varepsilon_i(x,t), i = 1, 2, \ldots, k \}$. By (19), we get, for $i = 1, 2, \ldots, k$,
Therefore, we have \( v_i(x, t) \geq 0 \) on \( Q_T \). Letting \( \epsilon \to 0^+ \), we get the desired result.

If the boundary condition \( \int_{\Omega} g_i(x, y) \, dy < 1 \) is not necessarily valid, we have the following result. The argument of its proof can be referred to [22, Lemma 2.2].

**Lemma 9.** Suppose that \( a_{ij}, b_i, f_i \in C(Q_T), f_i \geq 0, c_i, d_i, \) are nonnegative and bounded in \( Q_T \), \( g(x, y) \geq 0 \) on \( \partial \Omega \times \bar{\Omega} \), \( \int_{\Omega} g_i(x, y) \, dy > 0 \) on \( \partial \Omega, i = 1, 2, \ldots, k, j = 1, 2, \ldots, N \). If, for every \( i = 1, 2, \ldots, k \), \( u_i \in C^{2,1}(Q_T) \cap C(\overline{Q_T}) \) and satisfies

\[
\frac{\partial u_i}{\partial \nu} \geq \sum_{j=1}^{N} a_{ij} \frac{\partial u_i}{\partial x_j} + b_i u_i
\]

\[
+ f_i(x, t) \int_{\Omega} \left( c_i u_i + d_i u_{i+1} \right) \, dx, \quad (x, t) \in Q_T,
\]

\[
u_i(x, t) \geq \int_{\Omega} g_i(x, y) \, \nu_i(x, y) \, dy, \quad (x, t) \in Q_T,
\]

\[
u_i(x, 0) \geq 0, \quad x \in \Omega,
\]

where \( u_{k+1} = u_1 \), then \( \nu_i(x, t) \geq 0, \) \( i = 1, 2, \ldots, k \), on \( \bar{Q}_T \).

By Lemma 9, we can easily get the following result.

**Lemma 10.** Let \( \left( \overline{u}_1, \overline{u}_2, \ldots, \overline{u}_k \right) \) and \( \left( u_1, u_2, \ldots, u_k \right) \) be nonnegative upper and lower solution of system (1) on \( Q_T \), respectively. If one assumes that, for some \( r \in \{1, 2, \ldots, k\} \),

(i) \( \overline{u}_{r+1} > \delta \) or \( u_{r+1} > \delta \) when \( p_r < 1 \),

(ii) \( \overline{u}_r > \delta \) or \( u_r > \delta \) when \( q_r < 1 \),

then \( \left( \overline{u}_1, \overline{u}_2, \ldots, \overline{u}_k \right) \geq \left( u_1, u_2, \ldots, u_k \right) \) on \( Q_T \).

### 3. Global Existence Results

Before proving Theorem 2, we give a global existence result for a scalar equation.

**Lemma 11.** Let \( u_0(x) \) and \( q(x, y) \) be continuous, nonnegative functions on \( \Omega \) and \( \partial \Omega \times \bar{\Omega} \), respectively, and let the nonnegative constants \( \theta_{ij} \) satisfy \( 0 < \theta_{ij} + \theta_{i2} \leq 1 \). Then the solutions of the nonlocal problem

\[
\overline{w}_i - \Delta \overline{w} = \sum_{j=1}^{k} w^{\theta_{ij}}(x, t) \int_{\Omega} w^{\theta_{ij}}(x, t) \, dx, \quad x \in \Omega, \quad t > 0,
\]

\[
\overline{w}(x, t) = \int_{\Omega} q(x, y) \, \overline{w}(y, t) \, dy, \quad x \in \partial \Omega, \quad t > 0,
\]

\[
\overline{w}(x, 0) = u_0(x), \quad x \in \Omega
\]

exist globally.

**Proof.** The augment is similar to the proof of [22, Lemma 3.1] or [21, Lemma 6]. For the reader's convenience, we complete it. It is easy to prove that there exists a positive function \( \psi \in C^2(\overline{\Omega}) \) such that

\[
\min_{\Omega} \psi(x) > \max_{\Omega} w_0^2(x),
\]

\[
\psi(x) \geq \int_{\Omega} q^2(x, y) \, dy \int_{\Omega} \psi(y) \, dy,
\]

where \( \psi \) is defined by (24).

Let \( \theta > 0 \) be large enough such that

\[
2\theta \min_{\Omega} \psi(x)
\]

\[
\geq (2k + 1) \max_{\Omega} \left| \Delta \psi(x) \right|, \quad (i = 1, 2, \ldots, k) |\Omega|
\]

\[
\int_{\Omega} \psi(x) \bigg( \theta_{i1} + \theta_{i2} + 1 \bigg) / 2 \bigg)
\]

Setting \( z(x, t) = e^{2\theta \psi(x)} \) for \( (x, t) \in \Omega \times (0, \infty) \), one readily checks that

\[
z(x, t) \geq 2z \int_{\Omega} z^{\theta_{i1} + \theta_{i2} - 1} \, dx, \quad x \in \Omega, \quad t > 0,
\]

\[
z(x, t) \geq \int_{\Omega} q^2(x, y) \, dy \int_{\Omega} z(y, t) \, dy, \quad x \in \partial \Omega, \quad t > 0,
\]

\[
z(x, 0) \geq u_0^2(x) + 1, \quad x \in \Omega
\]

Let \( \overline{w} = z^{1/2}(x, t) \); it follows that

\[
\overline{w}_i - \Delta \overline{w} \geq \sum_{i=1}^{k} \overline{w}^{\theta_{ij}}(x, t) \int_{\Omega} \overline{w}^{\theta_{ij}}(x, t) \, dx, \quad x \in \Omega, \quad t > 0,
\]

\[
\overline{w}(x, t) \geq \int_{\Omega} q^2(x, y) \, dy \int_{\Omega} \overline{w}(y, t) \, dy, \quad x \in \partial \Omega, \quad t > 0,
\]

\[
\overline{w}(x, 0) = u_0(x), \quad x \in \Omega
\]

(27)

This implies that \( \overline{w} \) is a global upper solution of (23). Clearly, \( 0 \) is a lower solution of it. So we complete the proof.

**Proof of Theorem 2.** By (11), we know that there exists \( a_i \in (0, 1), i = 1, 2, \ldots, k \), such that

\[
\frac{p_i}{1 - q_i} \leq \frac{a_i}{a_{i+1}}, \quad i = 1, 2, \ldots, k, \quad a_{k+1} = a_i.
\]

(28)
Define \( \alpha = \sum_{i=1}^{k} 1/a_i \). Let \( \Phi(x, y) \geq \max \{ \varphi_i(x, y), i = 1, 2, \ldots, k \} \) be a continuous function defined for \((x, y) \in \partial \Omega \times \Omega\). Suppose that \( z \) solves

\[
z_t - \Delta z = \alpha \sum_{i=1}^{k} z^{1-a_i}(x, t) \int_{\Omega} z^{a_i}(x, t) \, dx, \quad x \in \Omega, \quad t > 0,
\]

\[
z(x, t) = \sum_{i=1}^{k} g_i(x) \int_{\Omega} \Phi(x, y) z(y, t) \, dy, \quad x \in \partial \Omega, \quad t > 0,
\]

\[
z(x, 0) = 1 + \sum_{i=1}^{k} u_{i,0}^{1/a_i}(x), \quad x \in \Omega,
\]

where

\[
g_i(x) = \left( \int_{\Omega} \Phi(x, y) \, dy \right)^{(1-a_i)/a_i}.
\]

In view of Lemma II, we know that \( z \) is global. Moreover, \( z > 1 \) in \( \Omega \times [0, \infty) \) by the maximum principle. Set \( \bar{u}_i = z_i, i = 1, 2, \ldots, k \). By (28) and (29) and using Hölder’s inequality, we get

\[
\bar{u}_i - \Delta \bar{u}_i - \int_{\Omega} \frac{\partial}{\partial t} \bar{u}_i^p \, dx
\]

\[
= a_i z_i^{a_i-1} z_t - a_i z_i^{a_i-1} \Delta z - a_i (a_i - 1) |\nabla z|^2
\]

\[
\geq a_i z_i^{a_i-1} (z_t - \Delta z) - \int_{\Omega} z_i^{a_i-1} \, dx \geq (a_i - 1) \int_{\Omega} z_i^{a_i-1} \, dx
\]

\[
\geq 0, \quad (x, t) \in Q_T,
\]

\[
\bar{u}_i - \int_{\Omega} \varphi_i(x, y) \bar{u}_i(y, t) \, dy
\]

\[
= z_i - \int_{\Omega} \varphi_i(x, y) z_i^{a_i}(y, t) \, dy
\]

\[
\geq \left( \int_{\Omega} \varphi_i(x, y) \, dy \right)^{1-a_i} \left( \int_{\Omega} \varphi_i(x, y) z(y, t) \, dy \right)^{a_i}
\]

\[
- \int_{\Omega} \varphi_i(x, y) z_i^{a_i}(y, t) \, dy
\]

\[
\geq 0, \quad (x, t) \in S_T,
\]

\[
\bar{u}_i(x, 0) \geq u_{i,0}(x), \quad x \in \Omega.
\]

This means that \( (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k) \) is a global upper solution of (1).

**Proof of Theorem 3.** Define

\[
\max \left\{ \sup_{\partial \Omega} \int_{\Omega} \varphi_i(x, y) \, dy, i = 1, 2, \ldots, k \right\} = \delta_0 \in (0, 1).
\]

Let \( w \) be the unique solution of the elliptic problem

\[
-\Delta w = 1, \quad x \in \Omega; \quad w = C_0, \quad x \in \partial \Omega.
\]

Then there exists a constant \( M > 0 \) such that \( C_0 \leq w(x) \leq C_0 + M \) in \( \Omega \). We choose \( C_0 \) to be large enough such that

\[
\frac{1 + C_0}{1 + C_0 + M} \geq \delta_0.
\]

Set \( \bar{u}_i(x, t) = b_i(1 + w(x)) \). When \( (x, t) \in S_T \), it follows that

\[
\bar{u}_i - \int_{\Omega} \varphi_i(x, y) \bar{u}_i(y, t) \, dy
\]

\[
= b_i(1 + C_0) - b_i \int_{\Omega} \varphi_i(x, y) (1 + w(y)) \, dy
\]

\[
\geq b_i \left( 1 + C_0 - (1 + C_0 + M) \delta_0 \right) \geq 0.
\]

Now we investigate \( (x, t) \in Q_T \). Set \( L_i = (1 + C_0 + M)^{q_i} \). For convenience. A simple computation yields

\[
\bar{u}_i - \Delta \bar{u}_i - \int_{\Omega} \frac{\partial}{\partial t} \bar{u}_i^p \, dx
\]

\[
= b_i - b_i^p \int_{\Omega} \left( 1 + w(x) \right)^{p_i} \, dx
\]

\[
\geq b_i^{p_i} \left( b_i^{-q_i} - b_i^{-q_i} L_i \right).
\]

(a) If \( q_i > 1 \), no matter \( q_{i+1} > 1 \) or \( q_{i+1} \leq 1 \), we can choose \( b_i \) to be small enough such that \( b_i^{-q_i} \geq b_i^{p_i} L_i \). For fixed \( b_i \), there exist \( b_i, i = 1, 2, \ldots, r-1, r+1, \ldots, k \), satisfying \( b_i^{-q_i} \geq b_i^{p_i} L_i \). It follows that

\[
\bar{u}_i - \Delta \bar{u}_i - \int_{\Omega} \frac{\partial}{\partial t} \bar{u}_i^p \, dx \geq 0, \quad i = 1, 2, \ldots, k.
\]

(b) If \( q_i \leq 1 \), \( i = 1, 2, \ldots, k \) and \( p_1 p_2 \cdots p_k > (1 - q_1)(1 - q_2) \cdots (1 - q_k) \), we can choose \( b_i \) to be small enough such that

\[
b_i^{1-(q_i)(1-q_i)-1} \geq b_i^{p_i p_2 \cdots p_k L_i} \geq b_i^{p_i p_2 \cdots p_k L_i} \cdots b_i^{p_i p_2 \cdots p_k L_i}
\]

Consequently, there exist \( b_i > 0, i = 2, 3, \ldots, k \), satisfying \( b_i^{-q_i} \geq b_i^{p_i} L_i \). Hence (37) holds too.

By (35) and (37), in any case (a) or (b), we know that the solution of (1) must be global for small data \( u_{i,0}(x) \leq b_i(1 + w(x)), i = 1, 2, \ldots, k \) for \( x \in \Omega \).

**4. Blow-Up Results**

In this section, we assume that \( (u(x, t), v(x, t)) \) is a positive solution of (1) on \( \Omega \times (0, T) \), where \( T \) is the maximal existence time.
Proof of Theorem 4. We denote by $\lambda_1, \phi_1(x)$ the first eigenvalue and the corresponding eigenfunction of the linear elliptic problem:

$$-\Delta \varphi(x) = \lambda \varphi(x), \quad x \in \Omega; \quad \varphi(x) = 0, \quad x \in \partial \Omega,$$

and $\phi_1(x)$ satisfies

$$\varphi_1(x) > 0, \quad x \in \Omega; \quad \max_{\Omega} \phi_1(x) = 1. \quad (40)$$

Define $\gamma = \min_{\Omega} \{\alpha_i(q_i - 1) + \alpha_i + 1, i = 1, 2, \ldots, k\}$.

(a) If $q_r \geq 1$, we claim that there exist positive constants $\alpha_i > 1, i = 1, 2, \ldots, k$, such that the inequality

$$\alpha_i (q_i - 1) + \alpha_i + 1 > 0 \quad (41)$$

holds. First, when $i = r$, (41) holds for any $\alpha_r > 1$. When $i = r + 1$, (41) holds for any $\alpha_\ast > 1$; if $q_{r+1} \leq 1$, we can choose $\alpha_\ast > \max_{\Omega} \{1, \alpha_{r+1} - 1, q_{r+1} / p_{r+1}\}$. That is, (41) holds too. When $i = r - 1$, if $q_{r-1} \geq 1$, (41) holds for any $\alpha_\ast > 1$; if $q_{r-1} < 1$, we can choose $\alpha_i < 1 < \alpha_i < (\alpha_i + 1 - q_{r-1})$ such that (41) holds too.

(b) If $q_i < 1, i = 1, 2, \ldots, k$, and $p_1 p_2 \cdots p_k > (1 - q_i) (1 - q_2) \cdots (1 - q_k)$, we can choose $\alpha_i > 1$ such that

$$\frac{p_1}{1 - q_1} > \frac{\alpha_1}{\alpha_2}, \quad \frac{p_2}{1 - q_2} > \frac{\alpha_2}{\alpha_3}, \ldots, \frac{p_k}{1 - q_k} > \frac{\alpha_k}{\alpha_1}. \quad (42)$$

Hence (41) holds too.

Hence, for the case (a) or (b), we all have $\gamma > 1$. Now let $s(t)$ be the unique solution of the ODE problem

$$s'(t) = -\lambda s(t) + l s^\gamma(t), \quad t > 0,$$

$$s(0) = s_0 > 1, \quad (43)$$

where $l = \min_{\Omega} \{1 / \alpha_i \} \int_\Omega \phi_1^{\alpha_i + 1 - q_i} d\Omega, \quad i = 1, 2, \ldots, k$. Then $s(t)$ blows up in finite time $T(s_0)$ with $s_0$ being large enough.

Set

$$u_r = s^\gamma(t) \phi_1^\gamma(x), \quad (x,t) \in \overline{\Omega} \times [0, T(s_0)), \quad i = 1, 2, \ldots, k. \quad (44)$$

We will show that $(u_r, v)$ is a lower solution of problem (1). A direct computation yields

$$u_r \geq \alpha \int_\Omega \phi_1^{\alpha_i + 1 - q_i} \phi_1^\gamma \phi_i^\gamma d\Omega = 0, \quad (x,t) \in \overline{\Omega} \times [0, T(s_0)), \quad (45)$$

by the compatibility conditions, and College Students’ Innovative Projects, 2013.
References


