Research Article

Groups of Negations on the Unit Square

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The main results are about the groups of the negations on the unit square, which is considered as a bilattice. It is proven that all the automorphisms on it form a group; the set, containing the monotonic isomorphisms and the strict negations of the first (or the second or the third) kind, with the operator “composition,” is a group G_2 (or G_3 or G_4 , correspondingly). All these four kinds of mappings form a group G_5 . And all the groups $G_i, i = 2, 3, 4$ are normal subgroups of G_5 . Moreover, for G_5 , a generator set is given, which consists of all the involutive negations of the second kind and the standard negation of the first kind. As a subset of the unit square, the interval-valued set is also studied. Two groups are found: one group consists of all the isomorphisms on L^I , and the other group contains all the isomorphisms and all the strict negations on L^I , which keep the diagonal. Moreover, the former is a normal subgroup of the latter. And all the involutive negations on the interval-valued set form a generator set of the latter group.

1. Introduction

Negations, as a basic operation, play important roles in logic. In [1], the groups of the negations on the unit interval are studied. And in this paper, groups of negations on the bilattice $[0, 1]^2$ and on the interval-valued set L^I , which can be seen as a sublattice of the unit square, will be studied.

Bilattices, introduced by Ginsberg [2] as a uniform framework for inference in artificial intelligence, are algebraic structures that proved useful in many fields [3–5]. And the unit square $[0, 1]^2$ is a very special bilattice, which is a subset of the real plane. Thus it is of particular interest. In [6, 7], three kinds of negations on bilattices are introduced. Then one problem arises: are the properties of these negations similar to the negations on $[0, 1]$? In the third section of this paper, some results will be given.

All the vague sets, the interval-valued sets, the intuitionistic fuzzy sets and the grey sets are isomorphic to sublattice $L^I = \{(x_1, x_2) \in [0, 1]^2 \mid x_1 \leq x_2\}$, with the natural order $(x_1, x_2) \leq_k (y_1, y_2)$ if and only if $x_1 \leq y_1, x_2 \leq y_2$. Their properties could be found in [3, 7–23], and so forth. For sometimes, the sets are characterized as $L^* = \{(x_1, x_2) \in [0, 1]^2 \mid x_1 + x_2 \leq 1\}$, with the order $(x_1, x_2) \leq_t (y_1, y_2)$ if and only if $x_1 \leq y_1, x_2 \geq y_2$. Both L^I and L^* are sublattices of the bilattice $[0, 1]^2$, with orders $\leq_k$ and $\leq_t$. Therefore, similar

to [1], the groups of negations on the interval-valued sets can be discussed.

The contents will be arranged as follows. In the next section, some basic notions will be given. In the third section, the groups of the strict negations will be studied. And in the fourth section, we will study the negations on the interval-valued sets. At the end of this paper, a section of conclusion is shown.

2. Negations on the Bilattice $[0, 1]^2$

In [6, 7], three kinds of negations on the bilattice are given. As a particular bilattice, the unit square $[0, 1]^2$ also has these kinds of negations. These negations are based on two kinds of orders on the unit square $[0, 1]^2$, $\leq_t$ and $\leq_k$, defined as, for any $x = (x_1, x_2), y = (y_1, y_2) \in [0, 1]^2$,

$$\begin{aligned} (x_1, x_2) \leq_k (y_1, y_2) & \quad \text{if } x_1 \leq y_1, x_2 \leq y_2, \\ (x_1, x_2) \leq_t (y_1, y_2) & \quad \text{if } x_1 \leq y_1, x_2 \geq y_2, \end{aligned} \quad (1)$$

in which $\leq$ is the natural order on $[0, 1]$. Actually, $\leq_k$ is the natural order on the unit square.

Definition 1. The first kind negation N^- , called reflection, is a unary operator on the square, satisfying the following properties: $\forall x = (x_1, x_2), y = (y_1, y_2) \in [0, 1]^2$,

- (1) if $x \leq_k y$, then $N^-(x) \geq_k N^-(y)$;
- (2) if $x \leq_t y$, then $N^-(x) \leq_t N^-(y)$;
- (3) $N^-(0, 0) = (1, 1)$ and $N^-(1, 1) = (0, 0)$.

The second kind negation N^- , named conflation, is a unary operator satisfying

- (1') if $x \leq_k y$, then $N^-(x) \geq_k N^-(y)$;
- (2') if $x \leq_t y$, then $N^-(x) \geq_t N^-(y)$;
- (3') $N^-(0, 0) = (1, 1)$ and $N^-(1, 1) = (0, 0)$.

The last kind negation $N^\sim$ is a unary operator satisfying

- (1'') if $x \leq_k y$, then $N^\sim(x) \leq_k N^\sim(y)$;
- (2'') if $x \leq_t y$, then $N^\sim(x) \geq_t N^\sim(y)$;
- (3'') $N^\sim(0, 0) = (0, 0)$ and $N^\sim(1, 1) = (1, 1)$.

For convenience, in this paper, these three kinds of negations are collectively called negations N on the unit square, as a bilattice $([0, 1]^2, \leq_k, \leq_t)$.

The composition of two negations N, N' is defined as $(N \circ N')(x_1, x_2) = N(N'(x_1, x_2))$. Then, it is not hard to check that $N^- \circ N^-$ and $N^- \circ N^\sim$ are negations of the third kind; $N^- \circ N^\sim$ and $N^\sim \circ N^\sim$ are the second kind negations, and $N^- \circ N^-$ and $N^\sim \circ N^\sim$ are of the first kind.

Similar to the definition of strict interval-valued negations in [11], the strict negations on the unit square are given as follows.

Definition 2. A negation on the unit square is called strict, if it is continuous and both $\leq_k$ and $\leq_t$ are strict, that is, in Definition 1, if both $\leq_k$ and $\leq_t$ are replaced by $<_k$ and $<_t$ in the premise, then the conclusion will be changed to $<_k$ and $<_t$ correspondingly.

Example 3. Let $N_1(x_1, x_2) = (n_1(x_1), n_2(x_2))$ and $N_2(x_1, x_2) = (n_1(x_2), n_2(x_1))$, with

$$n_i(a) = \begin{cases} 0, & \text{if } a = 1, \\ c_i, & \text{if } a \in (0, 1), \\ 1, & \text{if } a = 0, \end{cases} \quad (2)$$

in which $c_i, i = 1, 2$ are constants in $[0, 1]$. Then both N_1 and N_2 are negations, but neither of them is strict.

Definition 4. If a negation N satisfies $N(N(x)) = x$, then it is called an involutive negation.

The mapping $N_{s,1}$ defined as $N_{s,1}(x_1, x_2) = (1 - x_2, 1 - x_1)$, for all $(x_1, x_2) \in [0, 1]^2$, is the standard negation of the first kind and named as the first standard negation. The mapping $N_{s,2}$ defined as $N_{s,2}(x_1, x_2) = (1 - x_1, 1 - x_2)$, for all $(x_1, x_2) \in [0, 1]^2$, is called the second standard negation. And $N_{s,3}(x_1, x_2) = (x_2, x_1)$ is called the third standard kind negation. It is obvious that $N_{s,1}$, $N_{s,2}$, and $N_{s,3}$ are involutive.

Obviously, each involutive negation is strict, but the converse is not valid.

Example 5. The following negation is strict but not involutive:

$$N_3(x_1, x_2) = (1 - x_2^2, 1 - x_1). \quad (3)$$

Lemma 6 (see [6]). Each of the first kind strict negations could be characterized as $N^-(x_1, x_2) = (n_1(x_2), n_2(x_1))$, with n_1, n_2 being strict negations on $[0, 1]$, and every second kind strict negation could be characterized as $N^-(x_1, x_2) = (n_1(x_1), n_2(x_2))$, with n_1 and n_2 being strict negations on $[0, 1]$.

Each of the first kind involutive negations could be characterized as $N^-(x_1, x_2) = (n_1(x_2), n_2(x_1))$, with $n_1 = n_2^{-1}$, negations on $[0, 1]$, and every second kind involutive negation could be characterized as $N^-(x_1, x_2) = (n_1(x_1), n_2(x_2))$, with n_1 and n_2 being involutive negations on $[0, 1]$.

Similarly, each of the third kind negations could be characterized as follows.

Lemma 7. $N^\sim$ is a strict negation of the third kind, if and only if there are two isomorphisms ϕ_1 and ϕ_2 on $[0, 1]$, s.t.

$$N^\sim(x_1, x_2) = (\phi_1(x_2), \phi_2(x_1)). \quad (4)$$

Proof. Since each $N^\sim$ could be represented as $N^\sim = N^- \circ N^-$, in which N^- is the first kind and N^- is the second kind. Then from Lemma 6, the result could be obtained. $\square$

From Lemmas 6 and 7, since n_i and $\phi_i, i = 1, 2$ are bijections on $[0, 1]$, we can know each strict negation N is a bijection on $[0, 1]^2$.

Note that, the composition of two involutive negations of different kinds is still a negation, but it may not be involutive. The following example shows it.

Example 8. $N_4(x_1, x_2) = (1 - x_2^2, \sqrt{1 - x_1})$ is an involutive negation of the first kind. The composition of it and the second standard negation $N_{2,s}$,

$$(N_4 \circ N_{2,s})(x_1, x_2) = \left(1 - (1 - x_2)^2, \sqrt{1 - (1 - x_1)}\right) = (2x_2 - x_2^2, \sqrt{x_1}), \quad (5)$$

is a strict but not involutive negation of the third kind.

Definition 9. A continuous mapping $\Phi : [0, 1]^2 \rightarrow [0, 1]^2$ is a monotonic isomorphism on the unit square, if it is bijective and preserves both the orders $\leq_k$ and $\leq_t$.

Obviously, each monotonic isomorphism is continuous. Define $N \circ \Phi$ and $\Phi' \circ \Phi$ as $(N \circ \Phi)(x_1, x_2) = N(\Phi(x_1, x_2))$ and $(\Phi' \circ \Phi)(x_1, x_2) = \Phi'(\Phi(x_1, x_2))$.

Lemma 10. If Φ is a monotonic isomorphism on the unit square, then there are isomorphisms ϕ_1 and ϕ_2 on the unit interval $[0, 1]$ such that

$$\Phi(x_1, x_2) = (\phi_1(x_1), \phi_2(x_2)). \quad (6)$$

Proof. Firstly, let us show $\Phi(1, 1) = (1, 1)$, $\Phi(0, 1) = (0, 1)$, $\Phi(1, 0) = (1, 0)$, and $\Phi(0, 0) = (0, 0)$. For any $(x_1, x_2) \in [0, 1]^2$, $(x_1, x_2) \leq_k (1, 1)$. Since Φ preserves $\leq_k$, we can know $\Phi(x_1, x_2) \leq_k \Phi(1, 1)$. Thus, the greatest element of $(\Phi([0, 1]^2), \leq_k)$ is $\Phi(1, 1)$. Because Φ is a bijection, we have $\Phi(1, 1) = (1, 1)$. Similarly, $\Phi(0, 0) = (0, 0)$, $\Phi(0, 1) = (0, 1)$, and $\Phi(1, 0) = (1, 0)$ can be proven.

Now, let us show that $\Phi \circ N_{s,1}$ is a strict negation of the first kind.

Suppose $(x_1, x_2) \leq_k (y_1, y_2) \in [0, 1]^2$, then $N_{s,1}(x_1, x_2) \geq_k N_{s,1}(y_1, y_2)$. Because Φ preserves $\leq_k$, we have

$$\begin{aligned} (\Phi \circ N_{s,1})(x_1, x_2) &= \Phi(N_{s,1}(x_1, x_2)) \geq_k \Phi(N_{s,1}(y_1, y_2)) \\ &= (\Phi \circ N_{s,1})(y_1, y_2). \end{aligned} \quad (7)$$

It shows that $\Phi \circ N_{s,1}$ reverses the order $\leq_k$. Similarly, since both Φ and $N_{s,1}$ keep the order $\leq_t$, we can get that $\Phi \circ N_{s,1}$ preserves the order $\leq_t$. Since Φ is a bijection, Φ strictly preserves the orders $\leq_k$ and $\leq_t$. Similarly, it can be proven that $\Phi \circ N_{s,1}$ strictly preserves the order $\leq_t$ and strictly reverses the order $\leq_k$.

Since both Φ and $N_{s,1}$ are continuous, $\Phi \circ N_{s,1}$ is continuous.

Since $\Phi(1, 1) = (1, 1)$, $\Phi(0, 0) = (0, 0)$, and $N_{s,1}(1, 1) = (0, 0)$, $N_{s,1}(0, 0) = (1, 1)$, the following hold:

$$\begin{aligned} (\Phi \circ N_{s,1})(0, 0) &= \Phi(N_{s,1}(0, 0)) = (1, 1), \\ (\Phi \circ N_{s,1})(1, 1) &= \Phi(N_{s,1}(1, 1)) = (0, 0). \end{aligned} \quad (8)$$

These two formulas, together with the facts that $\Phi \circ N_{s,1}$ is continuous, strictly reverses the order $\leq_k$, and strictly preserves the order $\leq_t$, show that $\Phi \circ N_{s,1}$ is a strict negation of the first kind, denoted by $N^\neg = \Phi \circ N_{s,1}$.

Then for any $(x_1, x_2) \in [0, 1]^2$, $(N^\neg \circ N_{s,1})(x_1, x_2) = ([\Phi \circ N_{s,1}] \circ N_{s,1})(x_1, x_2) = \Phi(N_{s,1}(N_{s,1}(x_1, x_2))) = \Phi(x_1, x_2)$, that is, $\Phi = N^\neg \circ N_{s,1}$. From Lemma 6, $N^\neg(x_1, x_2) = (n_1(x_2), n_2(x_1))$, with n_1, n_2 being negations on the unit interval $[0, 1]$. Since $N_{s,1}(x_1, x_2) = (1 - x_2, 1 - x_1)$, we could get $\Phi(x_1, x_2) = (n_1(1 - x_1), n_2(1 - x_2))$. Because $n_1(1 - x_1)$ and $n_2(1 - x_2)$ are isomorphisms on the unit interval, Φ can be characterized as $\Phi(x_1, x_2) = (\phi_1(x_1), \phi_2(x_2))$, in which $\phi_1(x_1) = n_1(1 - x_1)$, $\phi_2(x_2) = n_2(1 - x_2)$ are isomorphisms on the unit interval. $\square$

Lemma 11. If Φ is a monotonic isomorphism and N_s is a standard negation, then $\Phi \circ N_s$ and $N_s \circ \Phi$ are strict negations and of the same kind as N_s . Conversely, for each strict negation N , there exists a monotonic isomorphism Φ and a standard negation N_s of the same kind with N , s.t. $N = \Phi \circ N_s$.

Proof. If $N_s = N_{s,1}$, the proof of Lemma 10 has already shown that $\Phi \circ N_{s,1}$ is a strict negation. And by Lemma 10, $(N_{s,1} \circ \Phi)(x_1, x_2) = (1 - \phi(x_2), 1 - \phi(x_1))$. It shows that $N_{s,1} \circ \Phi$ is a continuous bijection on $[0, 1]^2$, which strictly preserves the order $\leq_k$ and strictly reverses $\leq_t$ and satisfies $(N_{s,1} \circ \Phi)(1, 1) = (0, 0)$ and $(N_{s,1} \circ \Phi)(0, 0) = (1, 1)$. Therefore, $N_{s,1} \circ \Phi$ is a strict negation of the first kind.

Similar for $N_s = N_{s,2}$ or $N_s = N_{s,3}$. Therefore, $\Phi \circ N_s$ and $N_s \circ \Phi$ are strict negations and of the same kind as N_s , for $N_s = N_{s,i}$, $i = 1, 2, 3$.

Next, let us show the second part. Suppose N is a negation and N_s is the standard negation of the same kind as N . Then, $N \circ N_s$ is a monotonic isomorphism Φ . As a result, for any $(x_1, x_2) \in [0, 1]^2$,

$$\begin{aligned} (\Phi \circ N_s)(x_1, x_2) &= ([N \circ N_s] \circ N_s)(x_1, x_2) \\ &= N[N_s(N_s(x_1, x_2))] = N(x_1, x_2), \end{aligned} \quad (9)$$

that is, $N = \Phi \circ N_s$. $\square$

3. Groups of the Negations on the Unit Square

Based on the notions in the above section, the groups of the negations on the unit square will be discussed.

In [1], the following theorem is obtained.

Theorem 12 (see [1, Theorem 4]). (1) For every strict negation n on the unit interval, there exist three involutive negations n_1, n_2 and n_3 s.t. $n = n_1 \circ n_2 \circ n_3$.

(2) For every isomorphism ϕ on the unit interval, there exist four involutive negations n_1, n_2, n_3 and n_4 s.t. $\phi = n_1 \circ n_2 \circ n_3 \circ n_4$.

This theorem shows that under the operator “composition,” all the involutive negations and isomorphisms on the unit interval cannot form a group, since it is not closed under the operator “composition.” But the set of all the strict negations and isomorphisms, together with the operator “composition,” is a group [1].

From this theorem and Lemmas 6 and 10, we could get the following result on the unit square.

Theorem 13. (1) For every strict negation N of the second kind on the unit square, there exist three involutive negations N_1, N_2 , and N_3 of the second kind, s.t. $N = N_1 \circ N_2 \circ N_3$.

(2) For every monotonic isomorphism Φ on the unit square, there exist four involutive negations N_1, N_2, N_3 , and N_4 of the second, s.t. $\Phi = N_1 \circ N_2 \circ N_3 \circ N_4$.

For the other two kinds of negations, we have not got similar results, since the composition operator is not commutative.

Next, let us discuss the groups of the negations on the unit square. The following sets are denoted by

$S_1 = \{\Phi : \Phi \text{ is a monotonic isomorphism on the unit square.}\}$

$S_2 = S_1 \cup \{N^\neg : N^\neg \text{ is a first kind strict negation on the unit square.}\}$

$S_3 = S_1 \cup \{N^- : N^- \text{ is a second kind strict negation on the unit square.}\}$

$S_4 = S_1 \cup \{N^\sim : N^\sim \text{ is a third kind strict negation on the unit square.}\}$

$S_5 = S_2 \cup S_3 \cup S_4$.

Theorem 14. $G_i = (S_i, \circ)$, $i = 1, 2, 3, 4, 5$ are groups, with the same unit element id . Moreover, $G_1 \triangleleft G_{2,3,4,5} \triangleleft G_5$, that is, G_1 is a

normal subgroup of G_i , $i = 2, 3, 4, 5$, and $G_{2,3,4,5}$ are normal subgroups of G_5 .

Proof. Obviously, for any element f in G_i , $i = 1, \dots, 5$, $\text{id} \circ f = f \circ \text{id} = f$, that is, id is the unit element of G_i , $i = 1, \dots, 5$.

For any two monotonic isomorphisms Φ_1, Φ_2 , the composition of them is still a monotonic isomorphism. Thus G_1 is closed under the operator " $\circ$ ". For two strict negations N_1, N_2 of the same kind, the composition of them is also a monotonic isomorphism. The composition of a strict negation N and a monotonic isomorphism Φ is still a strict negation of the same kind with N . Therefore, G_2, G_3 , and G_4 are closed under the operator " $\circ$ ". For any two strict negations N_1, N_2 of different kind, the composition is a strict negation of the other kind. So G_5 is closed.

Now, let us prove the associativity of G_5 . This proof also shows the associativity of G_i , $i = 1, 2, 3, 4$.

From Lemmas 6, 7, and 10, each of the negations and the monotonic isomorphisms can be characterized as $F(x_1, x_2) = (f_1(x_1), f_2(x_2))$ or $F(x_1, x_2) = (f_1(x_2), f_2(x_1))$, in which both of f_1 and f_2 are isomorphisms on $[0, 1]$ or both are strict negations on $[0, 1]$. Let F_1, F_2, F_3 be negations or monotonic isomorphisms on $[0, 1]^2$. Then,

$$\begin{aligned} (F_1 \circ [F_2 \circ F_3])(x_1, x_2) &= F_1 [(F_2 \circ F_3)(x_1, x_2)] \\ &= F_1 [F_2 (F_3(x_1, x_2))] \\ &= [F_1 \circ F_2] (F_3(x_1, x_2)) \\ &= ([F_1 \circ F_2] \circ F_3)(x_1, x_2), \end{aligned} \quad (10)$$

that is, $F_1 \circ [F_2 \circ F_3] = [F_1 \circ F_2] \circ F_3$, which shows the associativity of G_5 .

For any monotonic isomorphism Φ and any strict negation N , obviously Φ^{-1} and N^{-1} exist. Moreover, Φ^{-1} is also a monotonic isomorphism and N^{-1} is a negation of the same kind as N , that is, the inverse about the composition operator " $\circ$ " exists. Therefore, G_i , $i = 1, \dots, 5$ are groups.

Since $S_1 \subset S_{2,3,4} \subset S_5$, $G_1 \subset G_{2,3,4} \subset G_5$, we only need to prove that they are normal subgroups.

Firstly, let us show $G_1 \triangleleft G_2$. For any $\Phi \in S_1$, obviously, $\Phi \circ G_1 \circ \Phi^{-1} \subset G_1$, because G_1 is a group. For $N \in S_2 \setminus S_1$, from Lemma 6, we know $N(x_1, x_2) = (n_1(x_2), n_2(x_1))$ and $N^{-1}(x_1, x_2) = (n_1^{-1}(x_2), n_2^{-1}(x_1))$. For any $\Phi(x_1, x_2) = (\phi_1(x_1), \phi_2(x_2)) \in S_1$,

$$\begin{aligned} (N \circ \Phi \circ N^{-1})(x_1, x_2) \\ = (n_1(\phi_2(n_2^{-1}(x_1))), n_2(\phi_1(n_1^{-1}(x_2)))) \end{aligned} \quad (11)$$

Because both $n_1 \circ \phi_2 \circ n_2^{-1}$ and $n_2 \circ \phi_1 \circ n_1^{-1}$ are isomorphisms on the unit interval $[0, 1]$, $N \circ \Phi \circ N^{-1}$ is in S_1 . Thus, $N \circ G_1 \circ N^{-1} \subset G_1$, that is, G_1 is a normal subgroup of G_2 .

Similarly, we could prove $G_1 \triangleleft G_{3,4}$. Since, $S_5 = S_2 \cup S_3 \cup S_4$, we could know $G_1 \triangleleft G_5$.

Now, let us show $G_2 \triangleleft G_5$. Because G_2 is a group, for all $f \in G_2$ (f is a strict negation of the first kind or a monotonic isomorphism on $[0, 1]^2$), we have $f \circ G_2 \circ f^{-1} \subset G_2$. For any

$N \in G_5 \setminus G_2$, N is a negation of the second kind or the third kind. If N is of the second kind, then N^{-1} is also of the second kind. Thus for all $\Phi \in G_2$, $N \circ \Phi \circ N^{-1} \in G_2$, and for all $N' \in G_2$, $N \circ N' \circ N^{-1}$ is of the first kind, since $N' \circ N^{-1}$ is of the third kind. If N is of the third kind, then N^{-1} is also of the third kind. Thus for all $\Phi \in G_2$, $N \circ \Phi \circ N^{-1} \in G_2$, and for all $N' \in G_2$, $N \circ N' \circ N^{-1}$ is of the first kind, since $N' \circ N^{-1}$ is of the second kind. Therefore, for all $f \in G_5$, $f \circ G_2 \circ f^{-1} \subset G_2$, that is, G_2 is a normal subgroup of G_5 . Similarly, $G_3 \triangleleft G_5$, $G_4 \triangleleft G_5$ could be proven. $\square$

From Theorem 13, we immediately get the following result.

Theorem 15. The set $S_{I,2}$ of all the involutive negations of the second kind could generate the group G_3 , that is, for any $f \in G_3$, there is some involutive negations N_i , $i = 1, \dots, m$ of the second kind, such that $f = N_1 \circ N_2 \circ \dots \circ N_m$. And if f is a negation, m could be 3; if f is an isomorphism, m could be 4.

This theorem shows the relation between the involutive negations of the second kind and the group G_3 . However, the following problem is still unproven.

Problem 16. Could G_2 (or G_4) be generated by the set of all the involutive negations of the first (or the third) kind?

A generator set of G_5 is given in the following theorem.

Theorem 17. The set $S_{I,2} \cup \{N_{s,1}\}$ is a generator set of the group G_5 . And $S_{I,2} \cup \{N_{s,3}\}$ is also a the generator set of G_5 .

Proof. By Lemma 11, for any strict negation N of the first or the third kind, there exists a monotonic isomorphism Φ , s.t. $N = N_{s,1} \circ \Phi$ or $N = N_{s,2} \circ \Phi$. Since $N_{s,1} \circ N_{s,2} = N_{s,3}$, from Theorem 15, $S_{I,2} \cup \{N_{s,1}\}$ generates the group G_5 .

Because $N_{s,2} \circ N_{s,3} = N_{s,1}$, $S_{I,2} \cup \{N_{s,3}\}$ is also a the generator set of G_5 . $\square$

4. Negations on the Interval-Valued Set

The interval-valued set L^I is a sublattice of the unit square $[0, 1]^2$, so the negations of the interval-valued set could be defined as the restriction of the negations on the unit square.

Definition 18. A negation N^I on the interval-valued set is the restriction of some negation N of the first kind on the unit square, and satisfies that

$$\forall x \in L^I, \quad N^I(x) = N|_{L^I}(x) \in L^I. \quad (12)$$

Definition 19. An interval-valued negation is strict, if it is the restriction of some strict negations of the first kind on the unit square and satisfies (12).

An interval-valued negation N^I is involutive, if it satisfies

$$N^I(N^I(x)) = x, \quad \forall x \in L^I. \quad (13)$$

From this definition, each strict interval-valued negation is an injection and each involutive negation is strict [6]. Also,

in [6], it is proven that for each involutive interval-valued negation N^I , it keeps the diagonal $\Delta = \{(x_1, x_2) \in L^I : x_1 = x_2\}$, that is,

$$\forall x \in \Delta, \quad N^I(x) \in \Delta. \quad (14)$$

Definition 20. A mapping Φ^I on the interval-valued set is an isomorphism, if it is a bijection and keeps the natural order.

Actually, if Φ^I is an isomorphism on the interval-valued set, then Φ^I keeps both the orders $\leq_k$ and $\leq_t$ [6]. Thus, there exist some isomorphisms Φ on the unit square, s.t. $\Phi^I = \Phi|_{L^I}$. Moreover, every Φ keeps the diagonal Δ [6].

Different from the unit square, all the strict negations and the isomorphisms on the interval-valued set, together with the composition operator, do not consist of a group. The following is a counter example.

Example 21. There is no strict negation on the interval-valued set, which is an inverse of the following negation N_1^I :

$$N_1^I(x_1, x_2) = (1 - x_2, 1 - x_1^2), \quad \forall (x_1, x_2) \in L^I. \quad (15)$$

It is not hard to check that N_1^I is a strict negation on the interval-valued set.

It seems that there is a mapping $F_1(x_1, x_2) = (\sqrt{1 - x_2^2}, 1 - x_1)$, s.t. F_1 is its inverse. However, not all of the points of the interval-valued set are well defined under F_1 , such as the point $(0.5, 0.5)$. The “image” of it is $(0.75, 0.5)$, which is out of the interval-valued set.

Also, the following mapping F_2 is also not the inverse of N_1^I , because it is not an injection, thus not a strict negation on L^I . Consider

$$F_2(x_1, x_2) = \begin{cases} (\sqrt{1 - x_2^2}, 1 - x_1), & \text{if } (1 - x_1)^2 \geq 1 - x_2, \\ (x_1, x_1), & \text{otherwise.} \end{cases} \quad (16)$$

Now, let us give the proof of Example 21.

Suppose N_2^I is the inverse negation of N_1^I . Clearly, N_1^I maps the interval-valued set L^I to the set $S = \{(x_1, x_2) : 0 \leq 1 - x_2 \leq (1 - x_1)^2 \leq 1\}$. Then $(N_2^I)|_S$ is a surjection S to L^I . Since for the points in $L^I \setminus S$, their images under N_2^I also should be in L^I , we can get that N_2^I is not an injection, thus not a strict negation on L^I . So N_1^I has no inverse.

Denote that

$$S_1^I = \{\Phi^I : \Phi \text{ is an isomorphism on } L^I\}$$

$$S_2^I = S_1^I \cup \{N^I : N^I \text{ is a strict negation on } L^I \text{ and keeps the diagonal } \Delta\}$$

Then we have the following theorem.

Theorem 22. $G_1^I = (S_1^I, \circ)$ and $G_2^I = (S_2^I, \circ)$ are groups, with $\circ$ the composition of the mappings. Moreover, $G_1^I \triangleleft G_2^I$.

Proof. Obviously, the unit element is the identity mapping id , and the operation is closed and associative.

Suppose Φ^I is an isomorphism in S_1^I . Then Φ^I can be represented as $\Phi^I(x_1, x_2) = (\phi(x_1), \phi(x_2))$, with ϕ an isomorphism on the unit interval $[0, 1]$, because Φ^I keeps the diagonal. Define the mapping $(\Phi^I)^{-1}$ as $(\Phi^I)^{-1}(x_1, x_2) = (\phi^{-1}(x_1), \phi^{-1}(x_2))$. Then $(\Phi^I)^{-1}$ is also an isomorphism on L^I , that is, $(\Phi^I)^{-1} \in S_1^I$, and

$$\left[(\Phi^I)^{-1} \circ \Phi^I \right] (x_1, x_2) = (x_1, x_2) = \left[\Phi^I \circ (\Phi^I)^{-1} \right] (x_1, x_2), \quad (17)$$

that is, $(\Phi^I)^{-1}$ is the inverse of Φ^I . Thus, G_1^I is a group.

Let N^I be a strict negation on L^I , which keeps the diagonal Δ . From Lemma 6, there exist some strict negations n_1, n_2 on $[0, 1]$, s.t. $N^I(x_1, x_2) = (n_1(x_2), n_2(x_1))$. From (14), $n_1 = n_2$. Then $(N^I)^{-1}(x_1, x_2) = (n_1^{-1}(x_2), n_1^{-1}(x_1))$ is also a strict negation on L^I and keeps the diagonal. Also we can check that

$$\left[(N^I)^{-1} \circ N^I \right] (x_1, x_2) = (x_1, x_2) = \left[N^I \circ (N^I)^{-1} \right] (x_1, x_2); \quad (18)$$

that is, $(N^I)^{-1}$ is the inverse of N^I . Thus, G_2^I is a group.

Similar to the proof of Theorem 14, $G_1^I \triangleleft G_2^I$ could be proven. $\square$

This theorem could be extended to the unit square.

Theorem 23. (1) All the monotonic isomorphisms on the unit square, which keep the diagonal, form a group, called G_6 .

(2) All the strict negations of the first kind and the monotonic isomorphisms on the unit square, which keep the diagonal, form a group, called G_7 .

The proof is similar to Theorem 22.

From Theorems 12, 22, and 23, we could obtain the following theorem.

Theorem 24. (1) The set of all the involutive negations on L^I generates the group G_2^I .

(2) The set of all the involutive negations of the first kind on the unit square, which keep the diagonal, generates the group G_7 .

5. Conclusion

In this paper, we firstly study the negations on the unit square. The main results are Theorems 14 and 23, which show the groups that are formed by the strict negations and the monotonic isomorphisms. Then we discuss the negations on the interval-valued set. The main result is Theorem 22, that is, all the strict negations and isomorphisms on L^I , which keep the diagonal, form a group. Moreover, some generator sets of the groups are given.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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