Research Article

1-Quasiconformal Mappings and CR Mappings on Goursat Groups

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We show that 1-quasiconformal mappings on Goursat groups are CR or anti-CR mappings. This can reduce the determination of 1-quasiconformal mappings to the determination of CR automorphisms of CR manifolds, which is a fundamental problem in the theory of several complex variables.

1. Introduction

The well-known Liouville theorem tells us that 1-quasiconformal mappings on $\mathbb{R}^n$ are elements of group $SO(n+1, 1)$. Quasiconformal mappings on the Heisenberg group were introduced by Mostow [1] in his study of rigidity theorem. Since then, quasiconformal mappings on nilpotent Lie group and on more general Carnot-Carathéodory spaces had drawn more and more concern. Korányi and Reimann [2] and Korányi and Reimann [3] showed that C^4 orientation preserving 1-quasiconformal mappings between domains of Heisenberg group $\mathbb{H}^n$ are CR or anti-CR mappings and are obtained by the restriction of $SU(n+1, 1)$ group actions (see also [4]). Cowling et al. [5] extended Liouville theorem to conformal mappings on G/P , where G is a semisimple Lie group and P is a minimal parabolic subgroup of G . Ottazzi and Warhurst [6, 7] obtained a Liouville type theorem for all Carnot groups which stated that 1-quasiconformal maps form finite dimensional Lie groups. Xie [8] investigated 1-quasiconformal mappings on Carnot groups with reducible first layer equipped with left-invariant sub-Riemannian Carnot metrics and showed that each 1-quasiconformal map defined on a domain of such groups is the restriction of the composition of a standard dilation, a left translation, and an isometric graded isomorphism. In [9], we showed the CR property of 1-quasiconformal mappings on the Engel group and obtained the identity component of the group

of 1-quasiconformal mappings. For other deformations of CR mappings between the nilpotent Lie group of step two and their Beltrami equations, one can refer to [10]. In this paper, we will investigate the 1-quasiconformal mappings on Goursat groups, which are stratified nilpotent Lie groups including Engel groups and a large class of high step Carnot groups.

A *Goursat group* is a nilpotent group endowed with a Goursat distribution D . A distribution D of corank $s \geq 2$ on a manifold M is *Goursat* if the subsheaves D^i of the tangent bundle defined inductively by $D^{i+1} = [D^i, D^i]$ ($i = 1, 2, \dots, s$; $D^1 = D$; $[D^i, D^i]$ denotes the sheaf of vector fields generated by D^i and the Lie brackets $[X, Y]$, $X, Y \in D^i$) correspond to distributions; that is, they have constant rank, and this rank is $\text{rank } D^{i+1} = \text{rank } D^i + 1$, $i = 1, 2, \dots, s$ [11]. By a distribution of rank d we understand any subbundle of linear dimension d in the tangent bundle TM ; its corank is $\dim M - d$.

Our model of Goursat groups is G_n in [12], which takes $\mathbb{R}^{n+1}$ as the underlying space with the group multiplication

$$\begin{aligned} (x_0, x_1, \dots, x_n) (x'_0, x'_1, \dots, x'_n) \\ = \left(x_0 + x'_0, x_1 + x'_1, x_2 + x'_2 \right. \\ \left. + x_0 x'_1, \dots, x_n + x'_n + x_0 x'_{n-1} \right. \\ \left. + \frac{1}{2} x_0^2 x'_{n-2} + \dots + \frac{1}{(n-1)!} x_0^{n-1} x'_1 \right). \end{aligned} \quad (1)$$

It is easy to see that its Lie algebra of left-invariant vector fields is given by

$$X_0 = \frac{\partial}{\partial x_0},$$

$$X_j = \frac{\partial}{\partial x_j} + x_0 \frac{\partial}{\partial x_{j+1}} + \cdots + \frac{x_0^{n-j}}{(n-j)!} \frac{\partial}{\partial x_n}, \quad j = 1, \dots, n. \quad (2)$$

Then $X_0, X_1, \dots, X_n$ span the tangent space everywhere and the only nontrivial commutation relations among these vector fields are

$$[X_0, X_j] = X_{j+1}, \quad j = 1, \dots, n-1. \quad (3)$$

We use $\mathfrak{g}$ to denote the Lie algebra and it splits into the direct sum

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_n, \quad (4)$$

where the vector space V_1 is spanned by the fields X_0, X_1 and the space $V_j, j = 2, \dots, n$, is spanned by X_j , respectively. It is clear to see that when $n = 3$, G_n is Engel group.

Let $d_c(\cdot, \cdot)$ be the Carnot-Carathéodory distance on the Goursat group. A homeomorphism $f : \Omega \rightarrow \Omega'$ between domains on the Goursat group G is *quasiconformal* if

$$H(p) = \limsup_{r \rightarrow 0} H(p, r)$$

$$= \limsup_{r \rightarrow 0} \frac{\max_{d_c(p,q)=r} d_c(f(p), f(q))}{\min_{d_c(p,q)=r} d_c(f(p), f(q))}, \quad p \in \Omega, \quad (5)$$

is uniformly bounded. If in addition $\|H\|_\infty = \sup_{p \in \Omega} H(p) \leq K$ the homeomorphism f is *K-quasiconformal*.

On $\mathbb{R}^2$, a 1-quasiconformal mapping is holomorphic or antiholomorphic. If we identify the Heisenberg group with a quadratic hypersurface in $\mathbb{C}^n$, 1-quasiconformal mapping on the quadratic hypersurface is the restriction of a holomorphic or antiholomorphic mapping on $\mathbb{C}^n$ [3]. This can be viewed as the higher dimensional case of the holomorphicity of 1-quasiconformal mapping. This phenomenon happens for Goursat group.

A *CR manifold* is a smooth manifold M equipped with a distribution $H \subset TM$ of even rank $2n$ and a complex structure as endomorphisms $J_p : H_p \rightarrow H_p$ with $J_p^2 = -\text{id}$. Denote

$$T_{1,0}M = \ker(J_p - \sqrt{-1} \text{id}),$$

$$T_{0,1}M = \ker(J_p + \sqrt{-1} \text{id}). \quad (6)$$

A CR manifold is supposed to satisfy the following integrability condition: $[Z_1, Z_2] \in C^\infty(T_{1,0}M)$, if $Z_1, Z_2 \in C^\infty(T_{1,0}M)$. The number n is called *CR dimension* of M and $k = \dim M - 2n$ is called the *codimension* of M . A smooth mapping $f : M_1 \rightarrow M_2$ is called CR if

$$f_* T_{1,0}M_1 \subset T_{1,0}M_2. \quad (7)$$

CR manifolds appear naturally as embedded real submanifolds of complex manifolds. The distribution H is then defined as $H = TM \cap iTM$ and J is the restriction of the complex structure in the ambient complex manifold to H . We recall that, for a manifold $M = \{\rho_1 = 0, \rho_2 = 0, \dots, \rho_k = 0\}$ with $\bar{\partial}\rho_1 \wedge \bar{\partial}\rho_2 \wedge \cdots \wedge \bar{\partial}\rho_k \neq 0$, the induced CR structure on M is given by $T_{1,0}M = T_{1,0}\mathbb{C}^n \cap CTM = \{Z \in T_{1,0}\mathbb{C}^n \mid Z\rho_1 = 0, Z\rho_2 = 0, \dots, Z\rho_k = 0\}$, and $\dim T_{1,0}M = n - k$.

The Goursat group G_n can be realized as the real submanifold of $\mathbb{C}^n$:

$$S_n = \left\{ (z, w_2, \dots, w_n) \in \mathbb{C}^n \mid \text{Im } w_k = -\frac{1}{2^k k!} (z^k + \bar{z}^k) \right. \\ \left. - \frac{1}{2^k} \sum_{j=1}^{k-1} \frac{1}{j! (k-j)!} z^j \bar{z}^{k-j}, \right. \\ \left. k = 2, \dots, n \right\}, \quad (8)$$

where $z = x + iy, w_j = u_j + iv_j, j = 2, \dots, n$, by the diffeomorphism $T : S_n \rightarrow G_n$ defined by

$$T(z, w_2, \dots, w_n) = (z, \text{Re } w_2, \dots, \text{Re } w_n). \quad (9)$$

It is easy to check that

$$Z = \frac{\partial}{\partial z} - i \text{Re } z \frac{\partial}{\partial w_2} - \cdots - \frac{i}{(n-1)!} (\text{Re } z)^{n-1} \frac{\partial}{\partial w_n}, \quad (10)$$

where $\partial/\partial z = (1/2)((\partial/\partial x) - i(\partial/\partial y))$, $\partial/\partial w_j = (1/2)((\partial/\partial u_j) - i(\partial/\partial v_j))$, $j = 2, \dots, n$, is the complex tangent vector of S_n , and we can define the standard CR structure on S_n by $T_{1,0}S_n = \mathbb{C}Z$ and $\dim T_{1,0}S_n = 1$. We can also define quasiconformal mappings on S_n by (5) with respect to its Carnot-Carathéodory distance defined in Section 2. There is one to one corresponding between 1-quasiconformal mappings on G_n and S_n via the diffeomorphism T . Therefore, if we want to study 1-quasiconformal mappings on G_n , we can just study the 1-quasiconformal mappings on S_n . For further details see Section 2.

Denote

$$\mathcal{P} = \{(x_0, x_1, \dots, x_n) \in G_n \mid x_2 = x_3 = \cdots = x_n = 0\}. \quad (11)$$

We show that 1-quasiconformal mappings between domains on Goursat group G_n whose Pansu differentials preserve $\mathcal{P}$ are CR or anti-CR.

Theorem 1. Let Ω be a domain in S_n . Suppose that $f = (f_1, \dots, f_n) : \Omega \rightarrow f(\Omega)$ is a 1-quasiconformal mapping with $D\tilde{f}(p)$ preserving $\mathcal{P}$; then f is CR or anti-CR; that is, either

$$\bar{Z}f_k = 0, \quad k = 1, 2, \dots, n, \quad (12)$$

or

$$Zf_k = 0, \quad k = 1, 2, \dots, n, \quad (13)$$

where $\tilde{f} = T \circ f \circ T^{-1}$ and p is some P -differential point.

Remark 2. By this theorem the determination of 1-quasiconformal mappings is reduced to the determination of CR automorphisms of a submanifold in $\mathbb{C}^n$. According to the extension theorem (Theorem 1.4 in [13]), locally, the smooth CR automorphisms of S_n are holomorphic automorphisms of S_n , which is also an interesting problem.

Remark 3. In the cases of Heisenberg group and Engel group, the condition of $D\tilde{f}(p)$ preserving $\mathcal{P}$ in Theorem 1 can be omitted. However, by a similar method as in those cases, we cannot remove it, since it is inconvenient to use the expressions of Baker-Campbell-Hausdorff formula for groups of higher step.

2. Preliminaries

There is a natural homogeneous norm $\|\cdot\|$ on G_n , given by

$$\|(x_0, x_1, \dots, x_n)\|^{2nl} = (x_0^2 + x_1^2)^{nl} + \sum_{k=2}^n |x_k|^{2nl/k}, \quad (14)$$

and an associated pseudometric d , given by

$$d(p, q) = \|q^{-1}p\|, \quad (15)$$

for $p, q \in G_n$. The metric is left-invariant and is related to the dilations by the formula

$$d(\delta_c p, \delta_c q) = cd(p, q), \quad (16)$$

where

$$\delta_c(x_0, x_1, \dots, x_n) = (cx_0, cx_1, c^2x_2, \dots, c^n x_n), \quad c \in \mathbb{R}^+, \quad (17)$$

denotes the dilation on G_n .

The horizontal tangent space at $p \in G_n$ is a subspace $H_p G_n$ of the tangent space $T_p G_n$, and it is spanned by the vector fields $X_0(p)$ and $X_1(p)$. An absolutely continuous curve $\gamma : [0, 1] \rightarrow G_n$ is horizontal if its tangent vectors $\dot{\gamma}(t)$, $t \in [0, 1]$, lie in the horizontal tangent space $H_{\gamma(t)} G_n$. By Chow [14], any given two points $p, q \in G_n$ can be connected by a horizontal curve.

On $H_p G_n$, we fix a quadric form $\langle \cdot, \cdot \rangle_G$ with respect to which the vector fields $X_0(p)$, $X_1(p)$ are orthonormal at every point $p \in G_n$ and satisfy

$$\begin{aligned} & |\alpha X_0(p) + \beta X_1(p)|_G^2 \\ &= \langle \alpha X_0(p) + \beta X_1(p), \alpha X_0(p) + \beta X_1(p) \rangle_G \\ &= \alpha^2 + \beta^2, \end{aligned} \quad (18)$$

where $\alpha, \beta \in \mathbb{R}$. $\langle \cdot, \cdot \rangle_G$ is called *infinitesimal Carnot-Carathéodory (C-C) metric*. Then the length $l(\gamma)$ of the curve γ is defined by

$$l(\gamma) = \int_0^1 \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_G^{1/2} dt. \quad (19)$$

The *Carnot-Carathéodory distance (C-C distance)* $d_c(p, q)$ is the infimum of the lengths of all absolutely continuous horizontal curves connecting p and q :

$$d_c(p, q) = \inf_{\gamma} l(\gamma). \quad (20)$$

According to [15], this metric is equivalent to the pseudometric; that is,

$$C_1 d(x, y) \leq d_c(x, y) \leq C_2 d(x, y), \quad \forall x, y \in G_n, \quad (21)$$

for suitable constants C_1, C_2 .

For the manifold S_n , the horizontal tangent space at $x \in S_n$ is $H_x S_n = \text{span}_{\mathbb{R}} \{\text{Re } Z(x), \text{Im } Z(x)\}$. On $H_x S_n$, we fix a quadric form $\langle \cdot, \cdot \rangle_S$ with respect to which $2 \text{Re } Z(x)$, $2 \text{Im } Z(x)$ are orthonormal at every point $x \in S_n$ and satisfy

$$\begin{aligned} & |a \text{Re } Z(x) + b \text{Im } Z(x)|_S^2 \\ &= \langle a \text{Re } Z(x) + b \text{Im } Z(x), a \text{Re } Z(x) + b \text{Im } Z(x) \rangle_S \\ &= \frac{a^2 + b^2}{4}, \end{aligned} \quad (22)$$

where $a, b \in \mathbb{R}$. The C-C distance on S_n can be defined as above.

By definition (8), S_n in $\mathbb{C} \times \mathbb{C}^{n-1}$ corresponding to G_n is defined by the equation

$$\rho_1 = \rho_2 = \dots = \rho_{n-1} = 0, \quad (23)$$

where

$$\begin{aligned} \rho_1(z, w_2, \dots, w_n) &= \frac{w_2 - \bar{w}_2}{2i} + \frac{1}{4} \text{Re } z^2 + \frac{1}{4} |z|^2, \\ \rho_2(z, w_2, \dots, w_n) &= \frac{w_3 - \bar{w}_3}{2i} + \frac{1}{8} \text{Re } z^2 \bar{z} + \frac{1}{24} \text{Re } z^3, \\ &\vdots \\ \rho_{n-1}(z, w_2, \dots, w_n) &= \frac{w_n - \bar{w}_n}{2i} + \frac{1}{2^n} \sum_{j=1}^{n-1} \frac{1}{j! (n-j)!} z^j \bar{z}^{n-j} \\ &\quad + \frac{1}{2^n n!} (z^n + \bar{z}^n). \end{aligned} \quad (24)$$

It is direct to check that

$$\bar{\partial} \rho_1 \wedge \bar{\partial} \rho_2 \wedge \dots \wedge \bar{\partial} \rho_{n-1} \neq 0. \quad (25)$$

And the vector field Z defined by (10) satisfies

$$Z \rho_j = 0, \quad j = 1, \dots, n-1. \quad (26)$$

That is, Z is a complex tangent vector field of S_n , and S_n is a CR manifold with the induced CR structure $T_{1,0} S_n = \mathbb{C} Z$. We obtain the following relation between S_n and G_n .

Proposition 4. S_n is CR diffeomorphic and isometric to G_n via T defined by (9); that is, T and T^{-1} are CR and isometric.

Proof. We can denote the point in G_n as $(z, x_2, x_3, \dots, x_n)$ where $z = x_0 + ix_1$, and then the group multiplication of G_n can be written in the following form:

$$\begin{aligned} & (z, x_2, \dots, x_n) (z', x'_2, \dots, x'_n) \\ &= \left(z + z', x_2 + x'_2 + (\operatorname{Re} z) \operatorname{Im} z', \dots, x_n \right. \\ & \quad \left. + x'_n + (\operatorname{Re} z) x'_{n-1} + \frac{1}{2} (\operatorname{Re} z)^2 x'_{n-2} \right. \\ & \quad \left. + \dots + \frac{1}{(n-1)!} (\operatorname{Re} z)^{n-1} \operatorname{Im} z' \right). \end{aligned} \quad (27)$$

Then the complex horizontal space is spanned by $Z_G, \bar{Z}_G$, where

$$\begin{aligned} Z_G &= \frac{1}{2} (X_0 - iX_1) \\ &= \frac{\partial}{\partial z} - \frac{i}{2} (\operatorname{Re} z) \frac{\partial}{\partial x_2} - \dots - \frac{i}{2(n-1)!} (\operatorname{Re} z)^{n-1} \frac{\partial}{\partial x_n}; \end{aligned} \quad (28)$$

here $\partial/\partial z = (1/2)((\partial/\partial x_0) - i(\partial/\partial x_1))$.

We claim that, under this diffeomorphism T defined by (9), the complex tangent vector fields $Z, \bar{Z}$ of S_n are mapped onto Z_G and $\bar{Z}_G$, respectively. In fact, for any $f \in C^k(G_n)$ and $q = (z, w_2, \dots, w_n) \in S_n$,

$$\begin{aligned} & (T_* Z) f(q) \\ &= Z(f \circ T)(q) \\ &= \left[\frac{\partial}{\partial z} - i \operatorname{Re} z \frac{\partial}{\partial w_2} - \dots - \frac{i}{(n-1)!} (\operatorname{Re} z)^{n-1} \frac{\partial}{\partial w_n} \right] \\ & \quad \left[f \left(z, \bar{z}, \frac{w_2 + \bar{w}_2}{2}, \dots, \frac{w_n + \bar{w}_n}{2} \right) \right] \\ &= \frac{\partial f}{\partial z} - i \frac{\operatorname{Re} z}{2} \frac{\partial f}{\partial x_2} - \dots - \frac{i}{2(n-1)!} (\operatorname{Re} z)^{n-1} \frac{\partial f}{\partial x_n} \\ &= Z_G f(T(q)). \end{aligned} \quad (29)$$

Therefore,

$$T_* Z = Z_G. \quad (30)$$

Similarly

$$T_* \bar{Z} = \bar{Z}_G. \quad (31)$$

It is clear to see that $T^{-1} : G_n \rightarrow S_n$ is

$$T^{-1}(z, x_2, \dots, x_n) = (z, x_2 + iv_2, \dots, x_n + iv_n), \quad (32)$$

where

$$\begin{aligned} v_k &= -\frac{1}{2^k k!} (z^k + \bar{z}^k) - \frac{1}{2^k} \sum_{j=1}^{k-1} \frac{1}{j! (k-j)!} z^j \bar{z}^{k-j}, \\ & \quad k = 2, \dots, n. \end{aligned} \quad (33)$$

Similarly, we find

$$T_*^{-1} Z_G = Z, \quad T_*^{-1} \bar{Z}_G = \bar{Z}. \quad (34)$$

Hence, T, T^{-1} are CR and by (30), (31), and (34) together with (18) and (22), we get $|T_* X|_G = |X|_S$ for any $X \in H_q S_n$ and $|T_*^{-1} Y|_S = |Y|_G$ for any $Y \in H_p G_n$, so T, T^{-1} preserve C-C distance. This proves the proposition. $\square$

3. P-Differential of 1-Quasiconformal Mappings on G_n

The notion of differential of a map between domains in Carnot group is due to Pansu [16]. Let $\tau_p(h) = ph$ denote left-translation on G_n . The mapping $f : \Omega \rightarrow \Omega'$ (Ω, Ω' are domains in G_n) is called p -differentiable at $p \in \Omega$ if, for $c \rightarrow 0$, the mappings

$$\delta_c^{-1} \circ \tau_{f(p)}^{-1} \circ f \circ \tau_p \circ \delta_c \quad (35)$$

converge locally uniformly to a homomorphism $Df(p)$ of the group G_n , which is called *Pansu differential* (P -differential).

The Pansu differential $Df(p)$ is a Carnot homomorphism. Denote by $Df_*(p)$ the induced Lie algebra homomorphism of the group homomorphism $Df(p)$. According to [17], $Df_*(p)$ is grading-preserving. As a result of [16] or [18], quasiconformal mappings between domains in Carnot group are P -differentiable a.e. The result is valid on the group G_n as it is a Carnot group.

Define the bilinear form B on $\mathfrak{g}$ by

$$[X_j, X_k] = B(X_j, X_k) X_2, \quad j, k = 0, 1. \quad (36)$$

Then by $[X_0, X_1] = X_2$, the corresponding matrix is

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (37)$$

Let f be a 1-quasiconformal mapping between domains in G_n ; since $Df_*(p)$ is grading-preserving, then $Df_*(p)$ maps V_j ($j = 1, \dots, n$) onto itself, respectively. In particular $V_1 = \operatorname{span}\{X_0, X_1\}$, so for any $X \in V_1$,

$$Df_*(p) X = SX, \quad (38)$$

for some 2×2 matrix S . Since $V_2 = \mathbb{R}X_2$, then

$$Df_*(p) X_2 = \lambda X_2 \quad (39)$$

for some λ .

Proposition 5. The matrix S above can be written as the following form:

$$S = \begin{pmatrix} c \cos \theta & c \sin \theta \\ -\frac{\lambda}{c} \sin \theta & \frac{\lambda}{c} \cos \theta \end{pmatrix}, \quad (40)$$

for some $\theta \in [0, 2\pi)$, where $c = |Df_*(p)X|_G$ for any $X \in V_1$ with $|X|_G = 1$, $\lambda^2/c^4 = 1$; λ, θ, c depend on $p \in G_n$.

Proof. Since $Df_*(p)X = SX$, $X \in V_1$, now suppose

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}; \quad (41)$$

then

$$Df_*(p)X_k = s_{1k}X_0 + s_{2k}X_1, \quad k = 0, 1. \quad (42)$$

Because $Df_*(p)$ is Lie algebra homomorphism, that is,

$$[Df_*(p)X_k, Df_*(p)X_l] = Df_*(p)[X_k, X_l], \quad (43)$$

together with $Df_*(p)X_2 = \lambda X_2$, we find that

$$B(s_{1k}X_0 + s_{2k}X_1, s_{1l}X_0 + s_{2l}X_1) = \lambda B(X_k, X_l), \quad (44)$$

namely,

$$S^T B S = \lambda B. \quad (45)$$

By Lemma 3.3 in [19], for the 1-quasiconformal mapping f , we have

$$\frac{\max_{X \in V_1, |X|_G=1} |Df_*(p)X|_G}{\min_{X \in V_1, |X|_G=1} |Df_*(p)X|_G} = 1, \quad (46)$$

where $|\cdot|_G$ is defined by (18), that is,

$$\frac{\max_{X \in V_1, |X|_G=1} |SX|_G}{\min_{X \in V_1, |X|_G=1} |SX|_G} = 1. \quad (47)$$

Therefore, there exists $c > 0$, such that

$$|SX|_G = c, \quad (48)$$

for any $X \in V_1$, $|X|_G = 1$. Then $(1/c)S$ is an orthogonal matrix. Together with (45), we have

$$BS = \frac{\lambda}{c^2} SB, \quad (49)$$

that is,

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = \frac{\lambda}{c^2} \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (50)$$

Thus

$$\begin{aligned} s_{21} &= -\frac{\lambda}{c^2} s_{12}, \\ s_{22} &= \frac{\lambda}{c^2} s_{11}, \\ s_{11} &= \frac{\lambda}{c^2} s_{22}, \\ s_{12} &= -\frac{\lambda}{c^2} s_{21}. \end{aligned} \quad (51)$$

Therefore, $\lambda^2/c^4 = 1$. Note that $((1/c)S)^T((1/c)S) = I$, where I is the 2×2 unit matrix; hence

$$\begin{aligned} s_{11}^2 + s_{12}^2 &= c^2, \\ s_{21}^2 + s_{22}^2 &= c^2, \\ s_{11}s_{21} + s_{12}s_{22} &= 0. \end{aligned} \quad (52)$$

From (51), (52) and setting $s_{11} = c \cos \theta$, $\theta \in [0, 2\pi)$, we have (40). The result follows. $\square$

We get the following relation between P -differential and the usual derivatives. For the corresponding result on Heisenberg group see p.31 of [3]. In the following, the letter C will be used to denote various constants, and the various uses of the letter do not, however, denote the same constant.

Proposition 6. Suppose f is a quasiconformal mapping between domains in G_n with $f(0) = 0$, and if f is P -differentiable at 0 and $Df(0)$ preserves the plan $\mathcal{P}$ in (11), then f is differentiable in the usual sense at 0 along the horizontal directions and

$$Df_*(0)\partial_{x_j} = f_{*0}\partial_{x_j}, \quad j = 0, 1, \quad (53)$$

where $f_{*0}\partial_{x_j} = \sum_{k=0}^n (\partial f_k / \partial x_j)(0)\partial_{x_k}$ and $Df_*(0)$ is the P -derivative of f at 0.

Proof. Since f is P -differentiable at 0, by definition, there exists $0 < \delta < 1$ such that, for $c < \delta$ and $\|(z, x_2, \dots, x_n)\| \leq 1$,

$$\begin{aligned} &\|(Df(0)(z, x_2, \dots, x_n))^{-1} \\ &\quad \times (\delta_c^{-1} \circ \tau_{f(0)}^{-1} \circ f \circ \tau_0 \circ \delta_c(z, x_2, \dots, x_n))\| \\ &= \|(Df(0)(z, x_2, \dots, x_n))^{-1} \\ &\quad \times (\delta_c^{-1} \circ f \circ \delta_c(z, x_2, \dots, x_n))\| < \varepsilon. \end{aligned} \quad (54)$$

Recall that the homomorphism $Df(0)$ commutes with δ_c ; then for points $(z/|z|, 0, \dots, 0)$ in particular, we have

$$\begin{aligned} &\left\| \delta_c^{-1} \left(\left(Df(0) \left(c \frac{z}{|z|}, 0, \dots, 0 \right) \right)^{-1} f \left(c \frac{z}{|z|}, 0, \dots, 0 \right) \right) \right\| \\ &= \frac{1}{c} \left\| \left(Df(0) \left(c \frac{z}{|z|}, 0, \dots, 0 \right) \right)^{-1} f \left(c \frac{z}{|z|}, 0, \dots, 0 \right) \right\| < \varepsilon. \end{aligned} \quad (55)$$

Therefore

$$\frac{1}{|z|} \|(Df(0)(z, 0, \dots, 0))^{-1} f(z, 0, \dots, 0)\| < \varepsilon, \quad (56)$$

if $|z| < \delta$.

Now compare this with Euclidean distances. Since $Df(0)$ preserves the plane $\mathcal{P}$, that is,

$$Df(0)(z, 0, \dots, 0) = (\tilde{z}, 0, \dots, 0) = (\tilde{x}_0, \tilde{x}_1, 0, \dots, 0) \quad (57)$$

for some $\tilde{z} = (\tilde{x}_0, \tilde{x}_1) \in \mathbb{R}^2$, then

$$\begin{aligned} & (Df(0)(z, 0, \dots, 0))^{-1} \\ &= (\tilde{x}_0, \tilde{x}_1, 0, \dots, 0)^{-1} \\ &= \left(-\tilde{x}_0, -\tilde{x}_1, \tilde{x}_0\tilde{x}_1, -\frac{\tilde{x}_0^2}{2}\tilde{x}_1, \dots, \frac{(-1)^n \tilde{x}_0^{n-1}}{(n-1)!}\tilde{x}_1 \right). \end{aligned} \quad (58)$$

Assume that $(x'_0, x'_1, x'_2, \dots, x'_n) = f(z, 0, \dots, 0)$; we get

$$\begin{aligned} & \|(Df(0)(z, 0, \dots, 0))^{-1} f(z, 0, \dots, 0)\|^{2n!} \\ &= \left\| \left(-\tilde{x}_0, -\tilde{x}_1, \tilde{x}_0\tilde{x}_1, -\frac{\tilde{x}_0^2}{2}\tilde{x}_1, \dots, \frac{(-1)^n \tilde{x}_0^{n-1}}{(n-1)!}\tilde{x}_1 \right) \right. \\ & \quad \times (x'_0, x'_1, x'_2, \dots, x'_n) \left. \right\|^{2n!} \\ &= \left(|x'_0 - \tilde{x}_0|^2 + |x'_1 - \tilde{x}_1|^2 \right)^{n!} + |x'_2 - \tilde{x}_0(x'_1 - \tilde{x}_1)|^{2n!/2} \\ & \quad + \left| x'_3 - \tilde{x}_0x'_2 + \frac{(-\tilde{x}_0)^2}{2}(x'_1 - \tilde{x}_1) \right|^{2n!/3} + \dots \\ & \quad + \left| x'_n - \tilde{x}_0x'_{n-1} + \frac{(-\tilde{x}_0)^2}{2}x'_{n-2} + \dots \right. \\ & \quad \left. + \frac{(-\tilde{x}_0)^{n-1}}{(n-1)!}(x'_1 - \tilde{x}_1) \right|^{2n!/n} \\ & < \varepsilon^{2n!} |z|^{2n!}. \end{aligned} \quad (59)$$

It follows that

$$\begin{aligned} & |x'_0 - \tilde{x}_0|^2 + |x'_1 - \tilde{x}_1|^2 < \varepsilon^2 |z|^2, \\ & |x'_2 - \tilde{x}_0(x'_1 - \tilde{x}_1)| < \varepsilon^2 |z|^2, \\ & \left| x'_3 - \tilde{x}_0x'_2 + \frac{(-\tilde{x}_0)^2}{2}(x'_1 - \tilde{x}_1) \right| < \varepsilon^3 |z|^3, \\ & \vdots \\ & \left| x'_n - \tilde{x}_0x'_{n-1} + \frac{(-\tilde{x}_0)^2}{2}x'_{n-2} + \dots \right. \\ & \quad \left. + \frac{(-\tilde{x}_0)^{n-1}}{(n-1)!}(x'_1 - \tilde{x}_1) \right| < \varepsilon^n |z|^n. \end{aligned} \quad (60)$$

Therefore,

$$|x'_0 - \tilde{x}_0| < \varepsilon |z|, \quad |x'_1 - \tilde{x}_1| < \varepsilon |z|, \quad (61)$$

$$|x'_2| < \varepsilon^2 |z|^2 + \varepsilon |\tilde{x}_0| |z|,$$

$$|x'_3| < \varepsilon^3 |z|^3 + |\tilde{x}_0| |x'_2| + \frac{(-\tilde{x}_0)^2}{2} \varepsilon |z|$$

$\vdots$

$$|x'_n| < \varepsilon^n |z|^n + |\tilde{x}_0| |x'_{n-1}| + \frac{(-\tilde{x}_0)^2}{2} |x'_{n-2}| + \dots$$

$$+ \varepsilon \left| \frac{(-\tilde{x}_0)^{n-1}}{(n-1)!} \right| |z|.$$

From Lemma 3.3 in [19], $Df(0)$ is Lipschitz; then by the equivalence of d_c and d , we have

$$\|Df(0)(z, 0, \dots, 0)\| \leq C \text{Lip}_f(0) \|(z, 0, \dots, 0)\|, \quad (63)$$

for some $C > 0$, namely,

$$|\tilde{x}_0| \leq \|Df(0)(z, 0, \dots, 0)\| \leq C \text{Lip}_f(0) |z| = M_1 |z|. \quad (64)$$

Since f is continuous, then when $|z| < \delta < 1$, we have

$$|(x'_0, x'_1, x'_2, \dots, x'_n)| \leq |f(z, 0, \dots, 0)| < M_2, \quad (65)$$

where M_2 is a positive constant. Now substituting (64) and (65) into (62) and then by iteration, we obtain

$$|x'_2| < \varepsilon^2 |z|^2 + \varepsilon M_1 |z|^2 < C\varepsilon |z|^2,$$

$$|x'_3| < \varepsilon^3 |z|^3 + |\tilde{x}_0| |x'_2| + \frac{(-\tilde{x}_0)^2}{2} \varepsilon |z|$$

$$< \varepsilon^3 |z|^3 + C\varepsilon |\tilde{x}_0| |z|^2 + \frac{(-\tilde{x}_0)^2}{2} \varepsilon |z| < C\varepsilon |z|^3,$$

$\vdots$

$$|x'_n| < \varepsilon^n |z|^n + |\tilde{x}_0| |x'_{n-1}| + \frac{(-\tilde{x}_0)^2}{2} |x'_{n-2}| + \dots \quad (66)$$

$$+ \varepsilon \left| \frac{(-\tilde{x}_0)^{n-1}}{(n-1)!} \right| |z|$$

$$< \varepsilon^2 |z|^n + C\varepsilon |\tilde{x}_0| |z|^{n-1} + C\varepsilon \frac{(-\tilde{x}_0)^2}{2} |z|^{n-2} + \dots$$

$$+ C\varepsilon \left| \frac{(-\tilde{x}_0)^{n-1}}{(n-1)!} \right| |z|$$

$$< C\varepsilon |z|^n.$$

Then by (61), for any $|z| < \delta < 1$, we have

$$\begin{aligned} & \frac{1}{|z|} |Df(0)(z, 0, \dots, 0) - f(z, 0, \dots, 0)| \\ &= \frac{1}{|z|} |(\tilde{x}_0, \tilde{x}_1, 0, \dots, 0) - (x'_0, x'_1, x'_2, \dots, x'_n)| \\ &= \frac{1}{|z|} \left(|x'_0 - \tilde{x}_0|^2 + |x'_1 - \tilde{x}_1|^2 + \sum_{k=2}^n |x'_k|^2 \right)^{1/2} \\ &< C \frac{1}{|z|} \left(\varepsilon^2 |z|^2 + \sum_{k=2}^n \varepsilon^2 |z|^{2k} \right)^{1/2} \\ &< C\varepsilon. \end{aligned} \quad (67)$$

If we let $z = (x_0, 0)$, then

$$\begin{aligned} & \frac{1}{|x_0|} |Df(0)(x_0, 0, \dots, 0) - f(x_0, 0, \dots, 0)| \\ &= \left| Df(0)(1, 0, \dots, 0) - \frac{f(x_0, 0, \dots, 0)}{x_0} \right| \\ &< C\varepsilon, \end{aligned} \quad (68)$$

namely,

$$Df(0)(1, 0, \dots, 0) = \lim_{x_0 \rightarrow 0} \frac{f(x_0, 0, \dots, 0) - f(0)}{x_0} = \frac{\partial f}{\partial x_0}(0). \quad (69)$$

Therefore,

$$Df_*(0) \partial_{x_0} = \sum_{j=0}^n \frac{\partial f_j}{\partial x_0}(0) \partial_{x_j} = f_{*0} \partial_{x_0}. \quad (70)$$

Similarly

$$Df_*(0) \partial_{x_1} = \sum_{j=0}^n \frac{\partial f_j}{\partial x_1}(0) \partial_{x_j} = f_{*0} \partial_{x_1}. \quad (71)$$

The proposition is proved. $\square$

We need the following relation between P -differential and the usual tangential mapping. See Proposition 3.2 in [10] for the corresponding result on the Heisenberg group.

Proposition 7. Let $f : \Omega_1 \rightarrow \Omega_2$ be a quasiconformal mapping between domains in the Goursat group G_n . Suppose f is P -differentiable at p and $Df(p)$ preserves the plan $\mathcal{P}$ in (11); then

$$f_*(p)X(p) = (\tau_{f(p)})_{*0} [Df_*(p)X(0)], \quad (72)$$

for $X \in V_1$, where $f_*(p)X(p) = \sum_{k=0}^n Xf_k(p) \partial_{x_k}$ and τ_{q*} are the tangential mappings of τ_q for $q \in G_n$.

Proof. Suppose f is P -differentiable at $p \in G_n$; let $F = \tau_{f(p)}^{-1} \circ f \circ \tau_p$; then F is also a quasiconformal mapping and $F(0) = 0$. Moreover, by Chain rule (Lemma 3.7 in [19]), F is P -differentiable at the origin 0, and

$$DF(0) = D\tau_{f(p)}^{-1}(f(p)) Df(p) D\tau_p(0). \quad (73)$$

Since $D\tau_p$ is the identity automorphism, then

$$DF(0) = Df(p). \quad (74)$$

Therefore, $DF(0)$ preserves the plan $\mathcal{P}$.

By Proposition 6, we have

$$DF_*(0) \partial_{x_j} = F_{*0} \partial_{x_j}, \quad j = 0, 1. \quad (75)$$

From (2) we can see that $X_0(0) = \partial_{x_0}$ and $X_1(0) = \partial_{x_1}$. Therefore,

$$DF_*(0)X_j(0) = F_{*0}X_j(0), \quad j = 0, 1. \quad (76)$$

Consequently,

$$\begin{aligned} & f_*(p)X_j(p) \\ &= (\tau_{f(p)} \circ F \circ \tau_p^{-1})_{*p} X_j(p) \\ &= (\tau_{f(p)})_{*0} F_{*0} X_j(0) \\ &= (\tau_{f(p)})_{*0} [DF_*(0)X_j(0)] \\ &= (\tau_{f(p)})_{*0} [Df_*(p)X_j(0)], \\ & \quad j = 0, 1. \end{aligned} \quad (77)$$

This completes the proof. $\square$

We need the following proposition due to Capogna ([19], Theorem 1.1).

Proposition 8. Suppose that Ω_1 and Ω_2 are open subsets of Carnot groups G_1 and G_2 , and $f : \Omega_1 \rightarrow \Omega_2$ is 1-quasiconformal. Then f is smooth.

Proof of Theorem 1. Let f be a 1-quasiconformal mapping satisfying the hypothesis in Theorem 1. By Proposition 4, T, T^{-1} defined by (9) and (32) are CR and isometric; then $\tilde{f} = T \circ f \circ T^{-1}$ is a 1-quasiconformal mapping between domains in G_n . Suppose $p \in G_n$ is a P -differentiable point of $\tilde{f}$; then $D\tilde{f}_*(p)$ is grading-preserving; that is, $D\tilde{f}_*(p)X_2 = \lambda X_2$ for some λ , and $D\tilde{f}_*X = SX, X \in V_1$. From Proposition 5, at $p \in G_n$, S has form (40), and $\lambda = \pm c^2$, where $c = |D\tilde{f}_*(p)X|_G$ for any $X \in V_1$ with $|X|_G = 1$.

In the complexified subspace $\mathbb{C}V_1$, the corresponding complex basis is $Z_G, \bar{Z}_G$, where Z_G is as (28). On the matrix level

$$S \longrightarrow S_c = \Theta S \Theta^{-1}, \quad (78)$$

with $\Theta = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$. Then

$$S_c = \begin{pmatrix} \frac{c^2 + \lambda}{2c} e^{-i\theta} & \frac{c^2 - \lambda}{2c} e^{i\theta} \\ \frac{c^2 - \lambda}{2c} e^{-i\theta} & \frac{c^2 + \lambda}{2c} e^{i\theta} \end{pmatrix}, \quad \theta \in [0, 2\pi). \quad (79)$$

We claim that $D\tilde{f}_*(p)Z_G = S_c Z_G$, $D\tilde{f}_*(p)\bar{Z}_G = S_c \bar{Z}_G$. In fact, by (28) and (40), we have

$$\begin{aligned} D\tilde{f}_*(p)Z_G &= D\tilde{f}_*(p) \left[\frac{1}{2} (X_0 - iX_1) \right] \\ &= \frac{1}{2} \left[cX_0 \cos \theta - \frac{\lambda}{c} X_1 \sin \theta - i \left(cX_0 \sin \theta + \frac{\lambda}{c} X_1 \cos \theta \right) \right] \\ &= \frac{1}{2} \left[ce^{-i\theta} X_0 - i \frac{\lambda}{c} e^{-i\theta} X_1 \right] \\ &= \frac{1}{2} e^{-i\theta} \left[c(Z_G + \bar{Z}_G) + \frac{\lambda}{c} (Z_G - \bar{Z}_G) \right] \\ &= \frac{1}{2} e^{-i\theta} \left[\left(c + \frac{\lambda}{c} \right) Z_G + \left(c - \frac{\lambda}{c} \right) \bar{Z}_G \right] = S_c Z_G. \end{aligned} \quad (80)$$

Similarly, $D\tilde{f}_*(p)\bar{Z}_G = S_c \bar{Z}_G$.

From Proposition 8, $\tilde{f}$ is smooth, and therefore $f = T^{-1} \circ \tilde{f} \circ T$ is also smooth. We write the 1-quasiconformal mapping f in the form

$$\begin{aligned} f(z, w_2, \dots, w_n) &= (f_1, f_2, \dots, f_n)(z, w_2, \dots, w_n) \\ &= (\omega_1, \omega_2, \dots, \omega_n) \end{aligned} \quad (81)$$

with

$$\begin{aligned} \operatorname{Im} f_k &= -\frac{1}{2^k k!} \left(f_1^k + \bar{f}_1^k \right) \\ &\quad - \frac{1}{2^k} \sum_{j=1}^{k-1} \frac{1}{j! (k-j)!} f_1^j \bar{f}_1^{k-j}, \end{aligned} \quad (82)$$

where $k = 2, 3, \dots, n$. Then

$$f_* Z = U + \bar{V}, \quad (83)$$

where

$$U = \sum_{k=1}^n (Z f_k) \frac{\partial}{\partial \omega_k}, \quad \bar{V} = \sum_{k=1}^n (Z \bar{f}_k) \frac{\partial}{\partial \bar{\omega}_k}. \quad (84)$$

In fact, for any $\varphi \in C^k(S_n)$,

$$\begin{aligned} (f_* Z) \varphi &= Z(\varphi \circ f) \\ &= Z[\varphi(f_1, f_2, \dots, f_n, \bar{f}_1, \bar{f}_2, \dots, \bar{f}_n)] \\ &= \sum_{k=1}^n \frac{\partial \varphi}{\partial \omega_k} Z f_k + \sum_{k=1}^n \frac{\partial \varphi}{\partial \bar{\omega}_k} Z \bar{f}_k \\ &= \left[\sum_{k=1}^n (Z f_k) \frac{\partial}{\partial \omega_k} + \sum_{k=1}^n (Z \bar{f}_k) \frac{\partial}{\partial \bar{\omega}_k} \right] \varphi. \end{aligned} \quad (85)$$

Let $q = T^{-1}(p) \in S_n$; by (30), (34), (72), and (80) together with the left-invariant property of Z_G ,

$$\begin{aligned} f_{*q}(Z(q)) &= (T^{-1} \circ \tilde{f} \circ T)_{*q}(Z(q)) \\ &= T_*^{-1} \circ \tilde{f}_* \circ T_{*q}(Z(q)) \\ &= T_*^{-1} \circ \tilde{f}_{*p}(Z_G(p)) \\ &= T_*^{-1} \circ \tau_{\tilde{f}(p)*0} \circ [D\tilde{f}_*(p)Z_G(0)] \\ &= T_{*\tilde{f}(p)}^{-1} \circ \tau_{\tilde{f}(p)*0} \\ &\quad \times \left(\frac{c^2 + \lambda}{2c} e^{-i\theta} Z_G(0) + \frac{c^2 - \lambda}{2c} e^{i\theta} \bar{Z}_G(0) \right) \\ &= T_{*\tilde{f}(p)}^{-1} \left(\frac{c^2 + \lambda}{2c} e^{-i\theta} Z_G(\tilde{f}(p)) \right. \\ &\quad \left. + \frac{c^2 - \lambda}{2c} e^{i\theta} \bar{Z}_G(\tilde{f}(p)) \right) \\ &= \frac{c^2 + \lambda}{2c} e^{-i\theta} Z(f(q)) + \frac{c^2 - \lambda}{2c} e^{i\theta} \bar{Z}(f(q)). \end{aligned} \quad (86)$$

Since f is P -differentiable a.e., we have

$$f_* Z = \frac{c^2 + \lambda}{2c} e^{-i\theta} Z + \frac{c^2 - \lambda}{2c} e^{i\theta} \bar{Z}, \quad \text{a.e.} \quad (87)$$

Consequently, $U, \bar{V}$ in (83) have the following forms

$$\begin{aligned} U &= \sum_{k=1}^n (Z f_k) \frac{\partial}{\partial \omega_k} = \frac{c^2 + \lambda}{2c} e^{-i\theta} Z, \quad \text{a.e.}, \\ \bar{V} &= \sum_{k=1}^n (Z \bar{f}_k) \frac{\partial}{\partial \bar{\omega}_k} = \frac{c^2 - \lambda}{2c} e^{i\theta} \bar{Z}, \quad \text{a.e.} \end{aligned} \quad (88)$$

Hence, if $\lambda = c^2$, then $\bar{Z} f_k = 0, k = 1, 2, \dots, n$; if $\lambda = -c^2$, then $Z f_k = 0, k = 1, 2, \dots, n$. Since Z and f are smooth, Theorem 1 follows. $\square$

Remark 9. The CR automorphism of G_n ($n > 3$) is unknown now. It is also very interesting to determine it, since if this can be solved, we can get the 1-quasiconformal mappings on these groups.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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