Research Article

Pick's Theorem in Two-Dimensional Subspace of $\mathbb{R}^3$

Lin Si

College of Science, Beijing Forestry University, Beijing 100083, China

Correspondence should be addressed to Lin Si; silincd@163.com

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In the Euclidean space $\mathbb{R}^3$, denote the set of all points with integer coordinate by $\mathbb{Z}^3$. For any two-dimensional simple lattice polygon P , we establish the following analogy version of Pick's Theorem, $k(I(P) + (1/2)B(P) - 1)$, where $B(P)$ is the number of lattice points on the boundary of P in $\mathbb{Z}^3$, $I(P)$ is the number of lattice points in the interior of P in $\mathbb{Z}^3$, and k is a constant only related to the two-dimensional subspace including P .

1. Introduction

In the Euclidean plane $\mathbb{R}^2$, a lattice point is one whose coordinates are both integers. A lattice polygon is a polygon with all vertices on integer coordinates. The area $A(P)$ of a simple lattice polygon P can be given by celebrated Pick's theorem [1]

$$A(P) = I(P) + \frac{1}{2}B(P) - 1, \quad (1)$$

where $B(P)$ is the number of lattice points on the boundary of P and $I(P)$ is the number of lattice points in the interior of P .

Pick's formula can be used to compute the area of a lattice polygon conveniently.

For example, in Figure 1, $I(P) = 60$, $B(P) = 15$. Then, the area of the polygon is $A(P) = 60 + 15 - 1 = 74$.

There are many papers concerning Pick's theorem and its generalizations [2–5], which mostly be discussed in two dimensions.

Unfortunately, Pick theorem is failed in three dimensions. In 1957, John Reeve found a class of tetrahedra, named as Reeve tetrahedra later, whose vertices are

$$(0, 0, 0)^T, (1, 0, 0)^T, (0, 1, 0)^T, (1, 1, r)^T, \quad (2)$$

where r is a positive integer.

All Reeve tetrahedra contain the same number of lattice points, but their volumes are different.

In this note, we discussed Pick's theorem in two-dimensional subspace of $\mathbb{R}^3$. For any $(a, b, c)^T \in \mathbb{Z}^3$ with $(a, b, c) = 1$, that is, the greatest common factor of a, b, c is one, denote by K , $ax + by + cz = 0$, the two-dimensional subspace of $\mathbb{R}^3$. Then we established the following theorem.

Theorem 1. *If P is simple lattice polygon in the K , then the area of P is*

$$k \left(I(P) + \frac{1}{2}B(P) - 1 \right), \quad (3)$$

where $B(P)$ is the number of lattice points on the boundary of P in $\mathbb{Z}^3$, $I(P)$ is the number of lattice points in the interior of P in $\mathbb{Z}^3$, and k is the constant $(a^3 + ab^2)\sqrt{a^2 + b^2 + c^2}$.

Remark 2. Although the simple lattice polygon P is in the two-dimensional subspace K , the lattice points in P belong to $\mathbb{Z}^3$.

Let $(a, b, c)^T = (1, 0, 0)^T$ in the Theorem; then we can get Pick's theorem in some coordinate plane of $\mathbb{R}^3$.

Corollary 3. *If P is simple lattice polygon in the K , whose normal vector is $(1, 0, 0)^T$, then the area of P is*

$$I(P) + \frac{1}{2}B(P) - 1. \quad (4)$$

FIGURE 1

2. Proof of Main Result

For any $(a, b, c)^T \in \mathbb{Z}^3$ with $(a, b, c) = 1$, there is two-dimensional subspace of $\mathbb{R}^3$

$$ax + by + cz = 0, \quad (5)$$

whose normal vector is just $(a, b, c)^T$. We denote this two-dimensional subspace by K .

By the theory of linear equations system, $(-b, a, 0)^T$ and $(-c, 0, a)^T$ are two linearly independent solutions of (5). We denote $(-b, a, 0)^T$ by α and $(-c, 0, a)^T$ by β . Obviously, α and β are also the basis of K .

Lemma 4. For any $(a, b, c)^T \in \mathbb{Z}^3$ with $(a, b, c) = 1$, there exists the lattice basis with the minimal area in the two-dimensional subspace K .

Proof. The area of parallelogram generated by α and β is

$$\begin{aligned} & \frac{1}{\sqrt{a^2 + b^2 + c^2}} \begin{vmatrix} a & -b & -c \\ b & a & 0 \\ c & 0 & a \end{vmatrix} \\ &= \frac{a^3 + ab^2 + ac^2}{\sqrt{a^2 + b^2 + c^2}} = a\sqrt{a^2 + b^2 + c^2}. \end{aligned} \quad (6)$$

Denote $(a, b, c)^T$ by n . For any lattice basis in K , $k_1\alpha + k_2\beta$ and $l_1\alpha + l_2\beta$, where $k_i, l_i \in \mathbb{Z}$ ($i = 1, 2$) and $\begin{vmatrix} k_1 & l_1 \\ k_2 & l_2 \end{vmatrix} \neq 0$. The area of parallelogram generated by $k_1\alpha + k_2\beta$ and $l_1\alpha + l_2\beta$ is

$$\begin{aligned} & \frac{1}{\sqrt{a^2 + b^2 + c^2}} \left| n : k_1\alpha + k_2\beta : l_1\alpha + l_2\beta \right| \\ &= \frac{1}{\sqrt{a^2 + b^2 + c^2}} \left| (n, \alpha, \beta) \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & k_1 & l_1 \\ 0 & k_2 & l_2 \end{pmatrix} \right|, \end{aligned} \quad (7)$$

where $\left| n : k_1\alpha + k_2\beta : l_1\alpha + l_2\beta \right|$ denote the determinant of n , $k_1\alpha + k_2\beta$, and $l_1\alpha + l_2\beta$.

Thus the lattice basis $k_1\alpha + k_2\beta$ and $l_1\alpha + l_2\beta$ have the minimal area if and only if $\begin{vmatrix} k_1 & l_1 \\ k_2 & l_2 \end{vmatrix} = 1$.

Let $k_1 = 1$, $k_2 = 0$, $l_1 = 0$, and $l_2 = 1$, and α, β are the lattice basis with the minimal area in the two-dimensional subspace K . $\square$

Lemma 5. For any $(a, b, c)^T \in \mathbb{Z}^3$ with $(a, b, c) = 1$, there exists the orthogonal lattice basis in the two-dimensional subspace K .

Proof. By Lemma 4, α, β are the lattice basis with the minimal area in the two-dimensional subspace K . By Schmidt orthogonalization, let

$$\begin{aligned} \gamma_1 &= \alpha = (-b, a, 0)^T, \\ \gamma_2 &= \beta - \frac{(\beta, \gamma_1)}{(\gamma_1, \gamma_1)} \gamma_1 = (-c, 0, a)^T - \frac{bc}{a^2 + b^2} \gamma_1 \\ &= (-c, 0, a)^T - \left(\frac{-b^2c}{a^2 + b^2}, \frac{abc}{a^2 + b^2}, 0 \right)^T \\ &= \left(\frac{-a^2c}{a^2 + b^2}, \frac{-abc}{a^2 + b^2}, \frac{a^3 + ab^2}{a^2 + b^2} \right)^T, \end{aligned} \quad (8)$$

where (β, γ_1) denote the usual inner product of β, γ_1 in $\mathbb{R}^3$. Thus

$$\begin{aligned} \eta_1 &= \gamma_1 = \alpha = (-b, a, 0)^T, \\ \eta_2 &= (a^2 + b^2) \gamma_2 = (-a^2c, -abc, a^3 + ab^2)^T \end{aligned} \quad (9)$$

are the orthogonal lattice basis in the two-dimensional subspace K . $\square$

Proof of Theorem. By Lemma 5, η_1, η_2 are the orthogonal lattice basis in the two-dimensional subspace K .

The area of parallelogram generated by η_1, η_2 is

$$\begin{aligned} & \frac{1}{\sqrt{a^2 + b^2 + c^2}} \begin{vmatrix} a & -b & -a^2c \\ b & a & -abc \\ c & 0 & a^3 + ab^2 \end{vmatrix} \\ &= \frac{a^5 + a^3b^2 + ab^2c^2 + a^3c^2 + a^3b^2 + ab^4}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{a(a^4 + 2a^2b^2 + b^4 + (a^2 + b^2)c^2)}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{a(a^2 + b^2)(a^2 + b^2 + c^2)}{\sqrt{a^2 + b^2 + c^2}} \\ &= (a^3 + ab^2) \sqrt{a^2 + b^2 + c^2}, \end{aligned} \quad (10)$$

which just is the constant k in the theorem. $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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