Research Article

(Fuzzy) Ideals of BN-Algebras

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The notions of an ideal and a fuzzy ideal in BN-algebras are introduced. The properties and characterizations of them are investigated. The concepts of normal ideals and normal congruences of a BN-algebra are also studied, the properties of them are displayed, and a one-to-one correspondence between them is presented. Conditions for a fuzzy set to be a fuzzy ideal are given. The relationships between ideals and fuzzy ideals of a BN-algebra are established. The homomorphic properties of fuzzy ideals of a BN-algebra are provided. Finally, characterizations of Noetherian BN-algebras and Artinian BN-algebras via fuzzy ideals are obtained.

1. Introduction

In 1966, Imai and Iséki [1] introduced the notion of a BCK-algebra. There exist several generalizations of BCK-algebras, such as BCI-algebras [2], BCH-algebras [3], BCC-algebras [4], BH-algebras [5], and d-algebras [6]. Neggers et al. defined B/BM/BG-algebras [7–9] and showed that the class of all B-algebras is a proper subclass of the class of all BG-algebras. They also proved that an algebra is a BM-algebra if and only if it is a 0-commutative B-algebra (therefore, every BM-algebra is a B-algebra). In [10], it is shown that the class of 0-commutative B-algebras is the class of p -semisimple BCI-algebras (and hence any BM-algebra is a BCI-algebra). The class of BM-algebras contains Coxeter algebras (see [8]). Some other connections between BM-algebras and its related topics are studied in [11]. Walendziak introduced in [12] the concept of BF-algebras, which is a generalization of B-algebras and BN-algebras defined by C. B. Kim and H. S. Kim [13]. An interesting result of [13] states that an algebra is a BN-algebra if and only if it is a 0-commutative BF-algebra.

We will denote by **BCK** (resp., **BCI/BCH/BH/B/BM/BG/BF/BN**) the class of all BCK-algebras (resp., BCI/BCH/BH/B/BM/BG/BF/BN-algebras). The interrelationships

between some classes of algebras mentioned before are visualized in Figure 1 (an arrow indicates proper inclusion; that is, if $\mathbf{X}$ and $\mathbf{Y}$ are classes of algebras, then $\mathbf{X} \rightarrow \mathbf{Y}$ means $\mathbf{X} \subset \mathbf{Y}$).

In this paper we consider ideals and fuzzy ideals in BN-algebras. In Section 2, similarly to BCK/BCI/BCH/BH/BF-algebras (see [5, 12, 14–16]), we define the concept of an ideal. We also introduce the notions of normal ideals and normal congruences. We investigate the properties of them and prove that there is a one-to-one correspondence between normal ideals and normal congruences of a BN-algebra. Moreover we obtain the isomorphism theorem for BN-algebras. In this section we also give all what is necessary, to make the paper self-contained. In Section 3 we define fuzzy ideals in BN-algebras (fuzzy ideals of BCK/BCI/BCC/BF-algebras are considered in [17–20]) and provide conditions for a fuzzy set to be a fuzzy ideal. Given a fuzzy set μ , we make the least fuzzy ideal containing μ . This leads us to show that the set of fuzzy ideals of a BN-algebra is a complete lattice. Moreover, the homomorphic properties of fuzzy ideals are provided. Finally, characterizations of Noetherian BN-algebras and Artinian BN-algebras in terms of fuzzy ideals are given in Section 4. Noetherian BCK/BCI/BM/BF-algebras are studied in [21–26].

FIGURE 1

2. On BN-Algebras: Ideals

An algebra $(A; *, 0)$ of type $(2, 0)$ is called a *Coxeter algebra* [27] if for all $x, y, z \in A$ the following identities hold:

- (B1) $x * x = 0$,
- (B2) $x * 0 = x$,
- (C) $(x * y) * z = x * (y * z)$.

Remark 1. Kim et al. [27] showed that a Coxeter algebra is equivalent to an abelian group all of whose elements have order 2. Therefore, if $(A; *, 0)$ is a Coxeter algebra, then the operation $*$ is commutative and associative.

We say that $(A; *, 0)$ is a *BM-algebra* [8] if it satisfies (B2) and

$$(BM) (z * x) * (z * y) = y * x.$$

Proposition 2 (see [8]). *If $(A; *, 0)$ is a BM-algebra, then, for any $x, y, z \in A$,*

- (i) $0 * (x * y) = y * x$,
- (ii) $(x * y) * z = (x * z) * y$.

Proposition 3. *If $(A; *, 0)$ is a BM-algebra, then, for all $x, y \in A$,*

- (i) $x * (x * y) = y$,
- (ii) if $x * y = 0$, then $x = y$.

Proof. (i) Substituting $z = x$ and $x = 0$ in (BM), we have $(x * 0) * (x * y) = y * 0$. Applying (B2) we get (i). □

(ii) follows from (i) and (B2). □

An algebra $(A; *, 0)$ of type $(2, 0)$ is called a *BF-algebra* [12] if it satisfies (B1), (B2), and

$$(BF) 0 * (x * y) = y * x.$$

Definition 4. An algebra $(A; *, 0)$ of type $(2, 0)$ is called a *BN-algebra* [13] if (B1), (B2), and

$$(BN) (x * y) * z = (0 * z) * (y * x)$$

hold for all $x, y, z \in A$.

Let $(A; *, 0)$ be a BN-algebra. We define a binary relation $\leq$ on A by $x \leq y$ if and only if $x * y = 0$. It is easy to see that, for any $x \in A$, if $x \leq 0$, then $x = 0$.

Example 5. Let $\mathbb{R}$ be the set of real numbers and let $\mathcal{R} = (\mathbb{R}; *, 0)$ be the algebra with the operation $*$ defined by

$$x * y = \begin{cases} x, & \text{if } y = 0, \\ y, & \text{if } x = 0, \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

Then $\mathcal{R}$ is a BN-algebra.

Example 6 (see [13]). Let $A = \{0, 1, 2, 3\}$ and $*$ be defined by the following:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	1	1
2	2	1	0	1
3	3	1	1	0

(2)

Then $(A; *, 0)$ is a BN-algebra.

Example 7. Let $(G; +, 0)$ be an abelian group. If we define $x * y = x - y$, for all $x, y \in G$, then, by Theorem 2.15 of [13], $(G; *, 0)$ is a BN-algebra.

Proposition 8 (see [13]). *If $(A; *, 0)$ is a BN-algebra, then, for all $x, y \in A$,*

- (i) $0 * (0 * x) = x$,
- (ii) $0 * (x * y) = y * x$,
- (iii) $y * x = (0 * x) * (0 * y)$,
- (iv) $x * y = 0 \Rightarrow y * x = 0$,
- (v) $0 * x = 0 * y \Rightarrow x = y$.

From [13] we obtain the following relation: **BM** $\subset$ **BN** $\subset$ **BF**.

From now on, $\mathcal{A}$ always denotes a BN-algebra $(A; *, 0)$. We introduce the notion of an ideal in BN-algebras.

Definition 9. A subset I of A is called an *ideal* of $\mathcal{A}$ if it satisfies

- (I₁) $0 \in I$,
- (I₂) $x * y \in I$ and $y \in I$ imply $x \in I$ for any $x, y \in A$.

We will denote by $\text{Id}(\mathcal{A})$ the set of all ideals of a BN-algebra $\mathcal{A}$.

Example 10. Let $\mathcal{R} = (\mathbb{R}; *, 0)$ be the BN-algebra given in Example 5. Observe that $\text{Id}(\mathcal{R}) = \{\{0\}, \mathbb{R}\}$. Indeed, let I be an

ideal of $\mathcal{A}$ and suppose that $I \neq \{0\}$. Then $a \in I$ for some $a \in \mathbb{R} - \{0\}$. Let $b \in \mathbb{R}$. Obviously, $b * a \in I$ and hence $b \in I$. Therefore, $I = \mathbb{R}$.

Proposition 11. Let $I \in \text{Id}(\mathcal{A})$ and $x, y \in A$. If $x \leq y$ and $y \in I$, then $x \in I$.

Proof. Let $x \leq y$ and $y \in I$. Then $x * y = 0 \in I$ and $y \in I$. Hence $x \in I$. $\square$

A nonempty subset S of A is called a *subalgebra* of $\mathcal{A}$ if $x * y \in S$ for any $x, y \in S$. It is easy to see that if S is a subalgebra of $\mathcal{A}$, then $0 \in S$.

Example 12. Let $\mathcal{Z} = (\mathbb{Z}; -, 0)$, where $\mathbb{Z}$ is the set of all integers. We know that $\mathcal{Z}$ is a BN-algebra (see Example 7). Obviously, a nonempty subset $S \subseteq \mathbb{Z}$ is a subalgebra of $\mathcal{Z}$ if and only if $a - b \in S$ for all $a, b \in S$; that is, S is a subgroup of the group of integers. Therefore,

$$\begin{aligned} \emptyset \neq S \subseteq \mathbb{Z} \text{ is a subalgebra of } \mathcal{Z} &\Leftrightarrow S \\ &= \{mk : k \in \mathbb{Z}\} \text{ for some } m \in \mathbb{N} \cup \{0\}. \end{aligned} \quad (3)$$

Remark 13. Consider the BN-algebra $\mathcal{A} = (A; *, 0)$ given in Example 6. We have $\text{Id}(\mathcal{A}) = \{\{0\}, \{0, 2\}, \{0, 3\}, \{0, 2, 3\}, A\}$. Observe that the ideal $I = \{0, 2, 3\}$ is not a subalgebra of $\mathcal{A}$ and $S = \{0, 1\}$ is a subalgebra of $\mathcal{A}$ but it is not an ideal.

Remark 14. It is easy to prove that the intersection of an arbitrary number of ideals of a BN-algebra $\mathcal{A}$ is an ideal of $\mathcal{A}$. It is also not hard to show that the union of an ascending sequence of ideals of $\mathcal{A}$ is an ideal of $\mathcal{A}$.

A nonempty subset N of A is said to be *normal* in $\mathcal{A}$ if

$$(x * a) * (y * b) \in N \quad (4)$$

for any $x * y, a * b \in N$. We say that an ideal I of $\mathcal{A}$ (resp., a subalgebra S of $\mathcal{A}$) is *normal* if the set I (resp., the set S) is normal. We will denote by $\text{Id}_N(\mathcal{A})$ the set of all normal ideals of $\mathcal{A}$.

Remark 15. It is easy to see that $\{0\}, A \in \text{Id}(\mathcal{A})$. The ideal A is normal, but $\{0\}$ is not normal in general (see the example below).

Example 16. Consider the set $A = \{0, 1, 2, 3\}$ with the operation $*$ defined by the following:

$$\begin{array}{c|cccc} * & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 2 & 3 \\ 1 & 1 & 0 & 3 & 0 \\ 2 & 2 & 3 & 0 & 2 \\ 3 & 3 & 0 & 2 & 0 \end{array} \quad (5)$$

It is easily seen that $(A; *, 0)$ is a BN-algebra. The ideal $I = \{0\}$ is not normal because $1 * 3 = 0 \in I$, but $(2 * 1) * (2 * 3) = 3 * 2 = 2 \notin I$.

Proposition 17. If I is a normal ideal of $\mathcal{A}$, then I is a subalgebra of $\mathcal{A}$.

Proof. Let $x, y \in I$. Since I is normal, we conclude that $(x * y) * (x * 0) \in I$; that is, $(x * y) * x \in I$. From the definition of an ideal we have $x * y \in I$. Thus I is a subalgebra of $\mathcal{A}$. $\square$

Remark 18. The BN-algebra $(A; *, 0)$ given in Example 16 shows that the converse of Proposition 17 does not hold. Indeed, $\{0\}$ is a subalgebra of $\mathcal{A}$ but it is not a normal ideal.

Proposition 19. Every ideal of a Coxeter algebra is normal.

Proof. Let $\mathcal{A} = (A; *, 0)$ be a Coxeter algebra and let I be an ideal of $\mathcal{A}$. Let $x, y, a, b \in A$ and suppose that $x * y, a * b \in I$. Since I is a subalgebra, we have $(x * y) * (a * b) \in I$. From Remark 1 it follows that $(x * a) * (y * b) \in I$. Thus I is normal. $\square$

Lemma 20. Let I be a normal ideal of a BN-algebra $\mathcal{A}$ and $x, y \in A$. Then

- (a) $x \in I \Rightarrow 0 * x \in I$,
- (b) $x * y \in I \Rightarrow y * x \in I$.

Proof. (a) Let $x \in I$. Then $x = x * 0 \in I$. Since I is normal, $(0 * x) * (0 * 0) \in I$. Hence, $0 * x \in I$.

(b) Let $x * y \in I$. By (a), $0 * (x * y) \in I$. Applying Proposition 8(ii) we have $y * x \in I$. $\square$

Proposition 21. Let $\mathcal{A}$ be a BN-algebra and let $S \subseteq A$. Then S is a normal subalgebra of $\mathcal{A}$ if and only if S is a normal ideal.

Proof. Let S be a normal subalgebra of $\mathcal{A}$. Clearly, $0 \in S$. Suppose that $x * y \in S$ and $y \in S$. Then $0 * y \in S$. Since S is normal, we have $(x * 0) * (y * y) \in S$. By (B1) and (B2), $(x * 0) * (y * y) = x$. Therefore $x \in S$, and thus S is an ideal. The converse follows from Proposition 17. $\square$

Remark 22. Let I be a normal ideal of a BN-algebra $\mathcal{A}$. For any $x, y \in A$, we define

$$x \sim_I y \iff x * y \in I. \quad (6)$$

Then $\sim_I$ is a congruence of $\mathcal{A}$ by the proof of Theorem 3.5 of [13].

Definition 23. Let θ be a congruence of a BN-algebra $\mathcal{A}$. One says that θ is *normal* if

$$(x * y)\theta, (a * b)\theta \text{ imply } ((x * a) * (y * b))\theta \quad (7)$$

for arbitrary $x, y, a, b \in A$.

Example 24. Let $A = \{0, 1, 2, 3\}$ and $*$ be defined by the following:

$$\begin{array}{c|cccc} * & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 2 & 3 \\ 1 & 1 & 0 & 3 & 2 \\ 2 & 2 & 3 & 0 & 1 \\ 3 & 3 & 2 & 1 & 0 \end{array} \quad (8)$$

It is easy to check that $\mathcal{A} = (A; *, 0)$ is a BN-algebra and $\theta = \{(0, 0), (1, 1), (2, 2), (3, 3), (1, 0), (0, 1), (2, 3), (3, 2)\}$ is a normal congruence of $\mathcal{A}$.

By $C_N(\mathcal{A})$ we denote the set of all normal congruences of $\mathcal{A}$.

Remark 25. Let $\mathcal{A}$ be a BN-algebra. It is easily seen that $\sim_I \in C_N(\mathcal{A})$, where I is a normal ideal of $\mathcal{A}$. In particular, $A^2 = \sim_A \in C_N(\mathcal{A})$.

Proposition 26. In BM-algebras all congruences are normal.

Proof. Let $\mathcal{A}$ be a BM-algebra and let θ be a congruence of $\mathcal{A}$. Let $x, y, a, b \in A$ and suppose that $(x * y)\theta 0$ and $(a * b)\theta 0$. Hence, $((x * y) * (a * b))\theta 0$. Observe that

$$(x * y) * (a * b) = (x * a) * (y * b). \quad (9)$$

Indeed, we have

$$\begin{aligned} & (x * y) * (a * b) \\ &= (0 * (y * x)) * (a * b) \quad [\text{by Proposition 2 (i)}] \\ &= (0 * (a * b)) * (y * x) \quad [\text{by Proposition 2 (ii)}] \\ &= (b * a) * (y * x) \quad [\text{by Proposition 2 (i)}] \\ &= ((y * a) * (y * b)) * (y * x) \quad [\text{by (BM)}] \\ &= ((y * a) * (y * x)) * (y * b) \quad [\text{by Proposition 2 (ii)}] \\ &= (x * a) * (y * b) \quad [\text{by (BM)}]. \end{aligned} \quad (10)$$

Therefore, $((x * a) * (y * b))\theta 0$ and thus θ is normal. $\square$

Remark 27. Let $\mathcal{A} = (A; *, 0)$ be the BN-algebra from Example 16. Observe that the least congruence on $\mathcal{A}$, namely, the identity relation $\omega = \{(a, a) : a \in A\}$, is not normal. Indeed, we get $(1 * 3, 0) \in \omega$ and $(2 * 2, 0) \in \omega$ but $((1 * 2) * (3 * 2), 0) = (2, 0) \notin \omega$.

Let $\mathcal{A}$ be a BN-algebra and let θ be a congruence on $\mathcal{A}$. For $x \in A$, by $[x]_\theta$ we denote the congruence class containing x ; that is, $[x]_\theta = \{y \in A : x\theta y\}$.

Proposition 28. Let θ be a congruence on $\mathcal{A}$. Then $\theta \in C_N(\mathcal{A})$ if and only if $[0]_\theta$ is a normal ideal of $\mathcal{A}$.

Proof. Set $I = [0]_\theta$ and let $\theta \in C_N(\mathcal{A})$. It follows easily that $0 \in I$. Let x and y be elements of A such that $x * y, y \in I$. Then, $(x * y)\theta 0$ and $y\theta 0$. Since $y\theta 0$, we obtain $(x * y)\theta(x * 0)$ and hence $x = x * 0 \in I$. Consequently, I is an ideal. Now, suppose that $x * y, a * b \in I$, where $x, y, a, b \in A$. Then, $(x * y)\theta 0$ and $(a * b)\theta 0$. By the definition of a normal congruence, $((x * a) * (y * b))\theta 0$; that is, $(x * a) * (y * b) \in I$. Thus, $I = [0]_\theta$ is normal.

The converse is obvious. $\square$

Theorem 29. There is a bijection between the set of normal ideals and the set of normal congruences of a BN-algebra.

Proof. Let $\mathcal{A}$ be a BN-algebra. We consider functions $f : \text{Id}_N(\mathcal{A}) \rightarrow C_N(\mathcal{A})$ and $g : C_N(\mathcal{A}) \rightarrow \text{Id}_N(\mathcal{A})$ given as follows:

$$\begin{aligned} f(I) &= \sim_I \quad \text{for any } I \in \text{Id}_N(\mathcal{A}), \\ g(\theta) &= [0]_\theta \quad \text{for any } \theta \in C_N(\mathcal{A}). \end{aligned} \quad (11)$$

Since $\sim_I \in C_N(\mathcal{A})$ and $[0]_\theta \in \text{Id}_N(\mathcal{A})$, we conclude that f and g are well-defined. Next, note that $I = [0]_{\sim_I}$. Indeed,

$$x \in I \iff x * 0 \in I \iff x \sim_I 0 \iff x \in [0]_{\sim_I}. \quad (12)$$

Hence,

$$(g \circ f)(I) = g(\sim_I) = [0]_{\sim_I} = I \quad \forall I \in \text{Id}_N(\mathcal{A}). \quad (13)$$

Further, observe that

$$x\theta y \iff (x * y)\theta 0 \iff x * y \in [0]_\theta \quad (14)$$

and from this we obtain

$$(f \circ g)(\theta) = f([0]_\theta) = \theta \quad \forall \theta \in C_N(\mathcal{A}). \quad (15)$$

We deduce from (13) and (15) that $g \circ f = \text{id}_{\text{Id}_N(\mathcal{A})}$ and $f \circ g = \text{id}_{C_N(\mathcal{A})}$. Thus, f and g are inverse bijections between $\text{Id}_N(\mathcal{A})$ and $C_N(\mathcal{A})$. $\square$

Let $\mathcal{A} = (A, *, 0_A)$ and $\mathcal{B} = (B, *, 0_B)$ be BN-algebras. A mapping $f : A \rightarrow B$ is called a *homomorphism* from $\mathcal{A}$ into $\mathcal{B}$ if $f(x * y) = f(x) * f(y)$ for any $x, y \in A$.

Observe that $f(0_A) = 0_B$. Indeed, $f(0_A) = f(0_A * 0_A) = f(0_A) * f(0_A) = 0_B$. We denote by $\ker f$ the subset $\{x \in A : f(x) = 0_B\}$ of A (it is the *kernel* of the homomorphism f).

Lemma 30. Let $f : A \rightarrow B$ be a homomorphism from $\mathcal{A}$ into $\mathcal{B}$. Then $\ker f$ is an ideal of $\mathcal{A}$.

Proof. Obviously, $0_A \in \ker f$; that is, (I_1) holds. Let $x * y \in \ker f$ and $y \in \ker f$. Then $0_B = f(x * y) = f(x) * f(y) = f(x) * 0_B = f(x)$. Consequently, $x \in \ker f$. Therefore, (I_2) is satisfied. Thus I is an ideal of $\mathcal{A}$. $\square$

Remark 31. The kernel of a homomorphism is not always a normal ideal. Let $(A; *, 0)$ be the algebra given in Example 16. Clearly, $\text{id}_A : A \rightarrow A$ is a homomorphism and the ideal $\ker(\text{id}_A) = \{0\}$ is not normal.

Proposition 32. Let $\mathcal{A} = (A, *, 0_A)$ be a BN-algebra and let $\mathcal{B} = (B, *, 0_B)$ be a BM-algebra. Let $f : A \rightarrow B$ be a homomorphism from $\mathcal{A}$ into $\mathcal{B}$. Then $\ker f$ is a normal ideal.

Proof. By Lemma 30, $\ker f$ is an ideal of $\mathcal{A}$. Let $x, y, a, b \in A$ and $x * y, a * b \in \ker f$. Then $0_B = f(x * y) = f(x) * f(y)$. From Proposition 3(ii) it follows that $f(x) = f(y)$. Similarly, $f(a) = f(b)$. Consequently, $f((x * a) * (y * b)) = (f(x) * f(a)) * (f(y) * f(b)) = (f(x) * f(a)) * (f(x) * f(a)) = 0_B$, and hence, $(x * a) * (y * b) \in \ker f$. $\square$

Let I be a normal ideal of $\mathcal{A}$. For $x \in A$, we write $x/I = \{y \in A : x \sim_I y\}$; that is, $x/I = [x]_{\sim_I}$. We note that

$$x \sim_I y \iff \frac{x}{I} = \frac{y}{I}. \quad (16)$$

Denote $A/I = \{x/I : x \in A\}$ and set $x/I *' y/I = x * y/I$. The operation $*$ ' is well-defined, since $\sim_I$ is a congruence of $\mathcal{A}$. It is easy to see that $\mathcal{A}/I = (A/I, *, 0/I)$ is a BN-algebra. The algebra $\mathcal{A}/I$ is called the *quotient BN-algebra of $\mathcal{A}$ modulo I* .

Example 33. Let $\mathcal{A}$ and θ be given as in Example 24. We know that $\mathcal{A}$ is a BN-algebra and θ is a normal congruence of $\mathcal{A}$. Since $[0]_\theta = \{0, 1\}$, from Proposition 28 we see that $I = \{0, 1\}$ is a normal ideal of $\mathcal{A}$. We have $0/I = 1/I = \{0, 1\}$ and $2/I = 3/I = \{2, 3\}$. Then $A/I = \{0/I, 2/I\}$. Clearly, $0/I *' 0/I = 0/I = 2/I *' 2/I$ and $0/I *' 2/I = 2/I = 2/I *' 0/I$.

Theorem 34. Let $\mathcal{A}$ be a BN-algebra and let $\mathcal{B}$ be a BM-algebra. Let $f : A \rightarrow B$ be a homomorphism from $\mathcal{A}$ onto $\mathcal{B}$. Then $\mathcal{A}/\ker f$ is isomorphic to $\mathcal{B}$.

Proof. By Proposition 32, $I = \ker f$ is a normal ideal of $\mathcal{A}$. Define a mapping $g : A/I \rightarrow B$ by $g(x/I) = f(x)$ for all $x \in I$. Let $x/I = y/I$. Then, $x \sim_I y$; that is, $x * y \in I$. Hence, $f(x) * f(y) = 0_B$. By Proposition 3(ii) we have $f(x) = f(y)$. Consequently, $g(x/I) = g(y/I)$. This means that g is well-defined. It is easy to check that g is a homomorphism from $\mathcal{A}/I$ onto $\mathcal{B}$. Observe that g is one-to-one. Let $g(x/I) = g(y/I)$. Then $f(x) = f(y)$ and hence $f(x * y) = 0_B$; that is, $x * y \in I$. Therefore, $x \sim_I y$ and consequently, $x/I = y/I$. Thus g is an isomorphism from $\mathcal{A}/I$ onto $\mathcal{B}$. $\square$

3. Fuzzy Ideals

We now review some fuzzy logic concepts. First, for $\Gamma \subseteq [0; 1]$ we define $\bigwedge \Gamma = \inf \Gamma$ and $\bigvee \Gamma = \sup \Gamma$. Obviously, if $\Gamma = \{\alpha, \beta\}$, then $\alpha \wedge \beta = \min\{\alpha, \beta\}$ and $\alpha \vee \beta = \max\{\alpha, \beta\}$. Recall that a *fuzzy set* in A is a function $\mu : A \rightarrow [0; 1]$.

For any fuzzy sets μ and ν in A , we define

$$\mu \leq \nu \iff \mu(x) \leq \nu(x) \quad \forall x \in A. \quad (17)$$

A trivial verification shows that this relation is an order relation in the set of fuzzy sets in A .

Let A and B be any two sets, μ any fuzzy set in A , and $f : A \rightarrow B$ any function. Set $f^-(y) = \{x \in A : f(x) = y\}$ for $y \in B$. The fuzzy set ν in B defined by

$$\nu(y) = \begin{cases} \bigvee \{\mu(x) : x \in f^-(y)\}, & \text{if } f^-(y) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases} \quad (18)$$

for all $y \in B$, is called the *image* of μ under f and is denoted by $f(\mu)$.

Let A and B be any two sets, $f : A \rightarrow B$ any function, and ν any fuzzy set in $f(A)$. The fuzzy set μ in A defined by

$$\mu(x) = \nu(f(x)), \quad \forall x \in A \quad (19)$$

is called the *preimage* of ν under f and is denoted by $f^-(\nu)$.

Now, we give the definition of a fuzzy ideal in a BN-algebra.

Definition 35. A fuzzy set μ in A is called a *fuzzy ideal* of a BN-algebra $\mathcal{A}$ if it satisfies, for all $x, y \in A$,

- (d1) $\mu(0) \geq \mu(x)$,
- (d2) $\mu(x) \geq \mu(x * y) \wedge \mu(y)$.

Proposition 36. Let μ be a fuzzy ideal of $\mathcal{A}$. Then, for any $x, y \in A$, if $x \leq y$, then $\mu(x) \leq \mu(y)$.

Proof. If $x \leq y$, then $x * y = 0$. From Proposition 8(iv) we obtain $y * x = 0$. Hence, by (d2) and (d1), we have $\mu(y) \geq \mu(y * x) \wedge \mu(x) = \mu(0) \wedge \mu(x) = \mu(x)$. $\square$

Denote by $\mathcal{FId}(\mathcal{A})$ the set of all fuzzy ideals of a BN-algebra $\mathcal{A}$.

Example 37. Let $\mathcal{A} = (A, *, 0)$ be the BN-algebra given in Example 6. Let $0 \leq \alpha_3 < \alpha_2 < \alpha_1 \leq 1$. Define a fuzzy set μ in A by

$$\mu(x) = \begin{cases} \alpha_1, & \text{if } x = 0, \\ \alpha_2, & \text{if } x = 2, \\ \alpha_3, & \text{if } x \in \{1, 3\}. \end{cases} \quad (20)$$

It is easily checked that μ satisfies (d1) and (d2). Thus $\mu \in \mathcal{FId}(\mathcal{A})$.

Example 38. Let I be an ideal of a BN-algebra $\mathcal{A}$ and let $\alpha, \beta \in [0; 1]$ with $\alpha \geq \beta$. Define $\mu_I^{\alpha, \beta}$ as follows:

$$\mu_I^{\alpha, \beta}(x) = \begin{cases} \alpha, & \text{if } x \in I, \\ \beta, & \text{otherwise.} \end{cases} \quad (21)$$

We denote $\mu_I^{\alpha, \beta} = \mu$. Since $0 \in I$, $\mu(0) = \alpha \geq \mu(x)$ for all $x \in A$. To prove (d2), let $x, y \in A$. If $x \in I$, then $\mu(x) = \alpha \geq \mu(x * y) \wedge \mu(y)$. Suppose now that $x \notin I$. By the definition of an ideal, $x * y \notin I$ or $y \notin I$. Therefore, $\mu(x * y) \wedge \mu(y) = \beta = \mu(x)$. Thus μ is a fuzzy ideal of $\mathcal{A}$.

In particular, the characteristic function χ_I of I :

$$\chi_I(x) = \begin{cases} 1, & \text{if } x \in I, \\ 0, & \text{otherwise,} \end{cases} \quad (22)$$

is a fuzzy ideal of $\mathcal{A}$.

Proposition 39. A fuzzy set μ in A is a fuzzy ideal of $\mathcal{A}$ if and only if it satisfies (d1) and

- (d3) for all $x, y, z \in A$, if $(z * y) * x = 0$, then $\mu(z) \geq \mu(x) \wedge \mu(y)$.

Proof. Let $\mu \in \mathcal{FId}(\mathcal{A})$ and let $x, y, z \in A$. Suppose that $(z * y) * x = 0$. Since μ is a fuzzy ideal, we have $\mu(z * y) \geq \mu((z * y) * x) \wedge \mu(x) = \mu(0) \wedge \mu(x) = \mu(x)$ and $\mu(z) \geq \mu(z * y) \wedge \mu(y)$. Therefore, $\mu(z) \geq \mu(x) \wedge \mu(y)$.

Conversely, let μ satisfy (d3). From (B1) we have $(x * y) * (x * y) = 0$. By (d3), $\mu(x) \geq \mu(x * y) \wedge \mu(y)$. Then μ satisfies (d2) and hence $\mu \in \mathcal{FId}(\mathcal{A})$. $\square$

Theorem 40. Let μ be a fuzzy set in A . Then $\mu \in \mathcal{FId}(\mathcal{A})$ if and only if its nonempty level subset

$$U(\mu; \alpha) = \{x \in A : \mu(x) \geq \alpha\} \quad (23)$$

is an ideal of $\mathcal{A}$ for all $\alpha \in [0; 1]$.

Proof. Assume that $\mu \in \mathcal{FId}(\mathcal{A})$. Let $\alpha \in [0; 1]$ and $U(\mu; \alpha) \neq \emptyset$. Then $\mu(x_0) \geq \alpha$ for some $x_0 \in A$. Since $\mu(0) \geq \mu(x_0)$, we have $0 \in U(\mu; \alpha)$. Let $x, y \in A$ such that $x * y, y \in U(\mu; \alpha)$. Then $\mu(x * y) \geq \alpha$ and $\mu(y) \geq \alpha$. It follows from (d2) that

$$\mu(x) \geq \mu(x * y) \wedge \mu(y) \geq \alpha \quad (24)$$

so that $x \in U(\mu; \alpha)$. Therefore $U(\mu; \alpha)$ is an ideal of $\mathcal{A}$.

Conversely, suppose that, for each $\alpha \in [0; 1]$, $U(\mu; \alpha) = \emptyset$ or $U(\mu; \alpha)$ is an ideal of $\mathcal{A}$. If (d1) is not valid, then there exists $x_0 \in A$ such that $\mu(0) < \mu(x_0) = \beta$. Then $U(\mu; \beta) \neq \emptyset$ and, by assumption, $U(\mu; \beta)$ is an ideal of $\mathcal{A}$. Hence $0 \in U(\mu; \beta)$ and consequently $\mu(0) \geq \beta$. This is a contradiction and (d1) is valid. Now assume that (d2) does not hold. Then there are $a, b \in A$ such that $\mu(a) < \mu(a * b) \wedge \mu(b)$. Taking

$$\gamma = \frac{1}{2} (\mu(a) + \mu(a * b) \wedge \mu(b)), \quad (25)$$

we get $\mu(a) < \gamma < \mu(a * b) \wedge \mu(b) \leq \mu(a * b)$ and $\gamma < \mu(b)$. Therefore $a * b, b \in U(\mu; \gamma)$ but $a \notin U(\mu; \gamma)$. This is impossible, and μ is a fuzzy ideal of $\mathcal{A}$. $\square$

By Theorem 40, we have the following.

Corollary 41. If μ is a fuzzy ideal of a BN-algebra $\mathcal{A}$, then the set

$$A_\mu = \{x \in A : \mu(x) = \mu(0)\} \quad (26)$$

is an ideal of $\mathcal{A}$.

The following example shows that the converse of Corollary 41 does not hold.

Example 42. Let $\mathcal{A}$ be a BN-algebra. Define a fuzzy set μ in A by

$$\mu(x) = \begin{cases} 0.4, & \text{if } x = 0, \\ 0.6, & \text{if } x \neq 0. \end{cases} \quad (27)$$

Then $A_\mu = \{0\}$ is the ideal of $\mathcal{A}$ but $\mu \notin \mathcal{FId}(\mathcal{A})$ (because μ does not satisfy (d1)).

Example 43. Consider the BN-algebra $\mathcal{A} = (A, *, 0)$ given in Example 6. Let μ be defined as in Example 37. It is easy to check that for all $\alpha \in [0; 1]$ we have

$$U(\mu; \alpha) = \begin{cases} \emptyset, & \text{if } \alpha > \alpha_1, \\ \{0\}, & \text{if } \alpha_2 < \alpha \leq \alpha_1, \\ \{0, 2\}, & \text{if } \alpha_3 < \alpha \leq \alpha_2, \\ A, & \text{if } \alpha \leq \alpha_3. \end{cases} \quad (28)$$

Since $\{0\}, \{0, 2\}$ and A are ideals of $\mathcal{A}$, this is another proof (by Theorem 40) that μ is a fuzzy ideal of $\mathcal{A}$.

Lemma 44. Let $I_1 \subset I_2 \subset \dots \subset I_n \subset \dots$ be a strictly ascending sequence of ideals in a BN-algebra $\mathcal{A}$ and let (t_n) be a strictly decreasing sequence in $(0; 1)$. Let μ be the fuzzy set in A defined by

$$\mu(x) = \begin{cases} 0, & \text{if } x \notin I_n \text{ for each } n \in \mathbb{N}, \\ t_n, & \text{if } x \in I_n - I_{n-1} \text{ for some } n \in \mathbb{N}, \end{cases} \quad (29)$$

where $I_0 = \emptyset$. Then μ is a fuzzy ideal of $\mathcal{A}$.

Proof. Let $I = \bigcup_{n \in \mathbb{N}} I_n$. By Remark 14, I is an ideal of $\mathcal{A}$. Obviously, $\mu(0) = t_1 \geq \mu(x)$ for all $x \in A$; that is, (d1) holds. Now we show that μ satisfies (d2). Let $x, y \in A$. We have two cases.

Case 1 ($x \notin I$). Then $x * y \notin I$ or $y \notin I$. Therefore $\mu(x * y) \wedge \mu(y) = 0 = \mu(x)$.

Case 2 ($x \in I_n - I_{n-1}$ for some $n \in \mathbb{N}$). Then $x * y \notin I_{n-1}$ or $y \notin I_{n-1}$. Hence $\mu(x * y) \leq t_n$ or $\mu(y) \leq t_n$. Therefore $\mu(x * y) \wedge \mu(y) \leq t_n = \mu(x)$.

Thus (d2) is also satisfied and consequently μ is a fuzzy ideal of $\mathcal{A}$. $\square$

Let T be a nonempty set of indexes. Let $\mu_t \in \mathcal{FId}(\mathcal{A})$ for $t \in T$. The meet $\bigwedge_{t \in T} \mu_t$ of fuzzy ideals μ_t of $\mathcal{A}$ is defined as follows:

$$\left(\bigwedge_{t \in T} \mu_t \right)(x) = \bigwedge \{ \mu_t(x) : t \in T \}. \quad (30)$$

Theorem 45. Let $\mu_t \in \mathcal{FId}(\mathcal{A})$ for $t \in T$. Then $\bigwedge_{t \in T} \mu_t \in \mathcal{FId}(\mathcal{A})$.

Proof. Let $\mu = \bigwedge_{t \in T} \mu_t$. Then, by (d1),

$$\begin{aligned} \mu(0) &= \bigwedge \{ \mu_t(0) : t \in T \} \geq \bigwedge \{ \mu_t(x) : t \in T \} \\ &= \mu(x) \end{aligned} \quad (31)$$

for all $x \in A$. Let $x, y \in A$. Since $\mu_t \in \mathcal{FId}(\mathcal{A})$, we have $\mu_t(x) \geq \mu_t(x * y) \wedge \mu_t(y)$. Hence

$$\begin{aligned} \bigwedge \{ \mu_t(x) : t \in T \} &\geq \bigwedge \{ \mu_t(x * y) \wedge \mu_t(y) : t \in T \} \\ &= \bigwedge \{ \mu_t(x * y) : t \in T \} \\ &\quad \wedge \bigwedge \{ \mu_t(y) : t \in T \}. \end{aligned} \quad (32)$$

Consequently, $\mu(x) \geq \mu(x * y) \wedge \mu(y)$ and therefore $\mu \in \mathcal{FId}(\mathcal{A})$. $\square$

Let ν be a fuzzy set in A . A fuzzy ideal μ of $\mathcal{A}$ is said to be generated by ν if $\nu \leq \mu$ and, for any fuzzy ideal ρ of $\mathcal{A}$, $\nu \leq \rho$ implies $\mu \leq \rho$. The fuzzy ideal generated by ν will be denoted by $[\nu]$. The fuzzy ideal $[\nu]$ can be defined equivalently as follows:

$$[\nu] = \bigwedge \{ \rho \in \mathcal{FId}(\mathcal{A}) : \rho \geq \nu \}. \quad (33)$$

For $\mu, \nu \in \mathcal{FId}(\mathcal{A})$ let $\mu \vee \nu$ denote the join of μ and ν ; that is, $\mu \vee \nu = (\rho]$, where ρ is the fuzzy set in A defined by $\rho(x) = \mu(x) \vee \nu(x)$ for all $x \in A$.

From Theorem 45 we obtain the following theorem.

Theorem 46. Let $\mathcal{A}$ be a BN-algebra. Then $(\mathcal{FId}(\mathcal{A}); \wedge, \vee)$ is a complete lattice.

The following two theorems give the homomorphic properties of fuzzy ideals.

Theorem 47. Let $\mathcal{A}$ and $\mathcal{B}$ be BN-algebras and let $f : A \rightarrow B$ be a homomorphism and $\nu \in \mathcal{FId}(\mathcal{B})$. Then $f^{\leftarrow}(\nu) \in \mathcal{FId}(\mathcal{A})$.

Proof. Let $x \in A$. Since $f(x) \in B$ and $\nu \in \mathcal{FId}(\mathcal{B})$, we have $\nu(0) \geq \nu(f(x)) = (f^{\leftarrow}(\nu))(x)$, but $\nu(0) = \nu(f(0)) = (f^{\leftarrow}(\nu))(0)$. Thus we get $(f^{\leftarrow}(\nu))(0) \geq (f^{\leftarrow}(\nu))(x)$ for any $x \in A$; that is, $f^{\leftarrow}(\nu)$ satisfies (d1).

Now let $x, y \in A$. Since $\nu \in \mathcal{FId}(\mathcal{B})$, we obtain

$$\begin{aligned} \nu(f(x)) &\geq \nu(f(x) * f(y)) \wedge \nu(f(y)) \\ &= \nu(f(x * y)) \wedge \nu(f(y)) \end{aligned} \quad (34)$$

and hence $(f^{\leftarrow}(\nu))(x) \geq (f^{\leftarrow}(\nu))(x * y) \wedge (f^{\leftarrow}(\nu))(y)$. Consequently, $f^{\leftarrow}(\nu) \in \mathcal{FId}(\mathcal{A})$. $\square$

Lemma 48. Let $\mathcal{A}$ and $\mathcal{B}$ be BN-algebras and let $f : A \rightarrow B$ be a homomorphism and $\mu \in \mathcal{FId}(\mathcal{A})$. Then, if μ is constant on $\ker f = f^{\leftarrow}(0)$, then $f^{\leftarrow}(f(\mu)) = \mu$.

Proof. Let $x \in A$ and $f(x) = y$. Hence

$$\begin{aligned} (f^{\leftarrow}(f(\mu)))(x) &= (f(\mu))(f(x)) = (f(\mu))(y) \\ &= \bigvee \{\mu(a) : a \in f^{\leftarrow}(y)\}. \end{aligned} \quad (35)$$

For all $a \in f^{\leftarrow}(y)$, we have $f(x) = f(a)$. Hence $f(a * x) = 0$; that is, $a * x \in \ker f$. Thus $\mu(a * x) = \mu(0)$. Therefore, $\mu(a) \geq \mu(a * x) \wedge \mu(x) = \mu(0) \wedge \mu(x) = \mu(x)$. Similarly, $\mu(x) \geq \mu(a)$. Hence $\mu(x) = \mu(a)$. Thus

$$(f^{\leftarrow}(f(\mu)))(x) = \bigvee \{\mu(a) : a \in f^{\leftarrow}(y)\} = \mu(x); \quad (36)$$

that is, $f^{\leftarrow}(f(\mu)) = \mu$. $\square$

Theorem 49. Let $\mathcal{A}$ and $\mathcal{B}$ be BN-algebras and let $f : A \rightarrow B$ be a surjective homomorphism and $\mu \in \mathcal{FId}(\mathcal{A})$ such that $A_\mu \supseteq \ker f$. Then $f(\mu) \in \mathcal{FId}(\mathcal{B})$.

Proof. Since μ is a fuzzy ideal of $\mathcal{A}$ and $0 \in f^{\leftarrow}(0)$, we have

$$(f(\mu))(0) = \bigvee \{\mu(a) : a \in f^{\leftarrow}(0)\} = \mu(0) \geq \mu(x) \quad (37)$$

for any $x \in A$. Hence

$$(f(\mu))(0) \geq \bigvee \{\mu(x) : x \in f^{\leftarrow}(y)\} = (f(\mu))(y) \quad (38)$$

for any $y \in B$. Thus $f(\mu)$ satisfies (d1). Suppose that

$$(f(\mu))(x_B) < (f(\mu))(x_B * y_B) \wedge (f(\mu))(y_B) \quad (39)$$

for some $x_B, y_B \in B$. Since f is surjective, there are $x_A, y_A \in A$ such that $f(x_A) = x_B$ and $f(y_A) = y_B$. Hence

$$\begin{aligned} (f(\mu))(f(x_A)) &< (f(\mu))(f(x_A * y_A)) \\ &\quad \wedge (f(\mu))(f(y_A)). \end{aligned} \quad (40)$$

Therefore

$$\begin{aligned} (f^{\leftarrow}(f(\mu)))(x_A) &< (f^{\leftarrow}(f(\mu)))(x_A * y_A) \\ &\quad \wedge (f^{\leftarrow}(f(\mu)))(y_A). \end{aligned} \quad (41)$$

Since $A_\mu \supseteq \ker f$, μ is constant on $\ker f$. Then, by Lemma 48, we get

$$\mu(x_A) < \mu(x_A * y_A) \wedge \mu(y_A), \quad (42)$$

which is a contradiction with the fact that μ is a fuzzy ideal. Thus, we obtain $f(\mu) \in \mathcal{FId}(\mathcal{B})$. $\square$

4. Fuzzy Characterizations of Noetherian and Artinian BN-Algebras

In this section we characterize Noetherian BN-algebras and Artinian BN-algebras using some fuzzy concepts, in particular, fuzzy ideals.

A BN-algebra $\mathcal{A}$ is called *Noetherian* if for every ascending sequence $I_1 \subseteq I_2 \subseteq \dots$ of ideals of $\mathcal{A}$ there exists $k \in \mathbb{N}$ such that $I_n = I_k$ for all $n \geq k$. A BN-algebra $\mathcal{A}$ is called *Artinian* if for every descending sequence $I_1 \supseteq I_2 \supseteq \dots$ of ideals of $\mathcal{A}$ there exists $k \in \mathbb{N}$ such that $I_n = I_k$ for all $n \geq k$.

Theorem 50. Let $\mathcal{A}$ be a BN-algebra. The following statements are equivalent:

- (a) $\mathcal{A}$ is Noetherian,
- (b) for each fuzzy ideal μ of $\mathcal{A}$, $\text{Im}(\mu) = \{\mu(x) : x \in A\}$ is a well-ordered set.

Proof. (a) $\Rightarrow$ (b): Assume that $\mathcal{A}$ is Noetherian and μ is a fuzzy ideal of $\mathcal{A}$ such that $\text{Im}(\mu)$ is not a well-ordered subset of $[0; 1]$. Then there exists a strictly decreasing sequence $(\mu(x_n))$, where $x_n \in A$. Let $t_n = \mu(x_n)$ and $U_n = U(\mu; t_n) = \{x \in A : \mu(x) \geq t_n\}$. Then, by Theorem 40, U_n is an ideal of $\mathcal{A}$ for every $n \in \mathbb{N}$. So $U_1 \subset U_2 \subset \dots$ is a strictly ascending sequence of ideals of $\mathcal{A}$. This is a contradiction with the assumption that $\mathcal{A}$ is Noetherian. Therefore $\text{Im}(\mu)$ is a well-ordered set for each fuzzy ideal μ of $\mathcal{A}$.

(b) $\Rightarrow$ (a): Assume that (b) is true. Suppose that $\mathcal{A}$ is not Noetherian. Then there exists a strictly ascending sequence $I_1 \subset I_2 \subset \dots \subset I_n \subset \dots$ of ideals of $\mathcal{A}$. Let μ be a fuzzy set in A such that

$$\mu(x) = \begin{cases} 0, & \text{if } x \notin I_n \text{ for each } n \in \mathbb{N}, \\ \frac{1}{n}, & \text{if } x \in I_n - I_{n-1} \text{ for some } n \in \mathbb{N}, \end{cases} \quad (43)$$

where $I_0 = \emptyset$. By Lemma 44, $\mu \in \mathcal{FId}(\mathcal{A})$, but $\text{Im}(\mu)$ is not a well-ordered set, which is impossible. Therefore $\mathcal{A}$ is Noetherian. $\square$

Corollary 51. Let $\mathcal{A}$ be a BN-algebra. If, for every fuzzy ideal μ of $\mathcal{A}$, $\text{Im}(\mu)$ is a finite set, then $\mathcal{A}$ is Noetherian.

Theorem 52. Let $\mathcal{A}$ be a BN-algebra and let $T = \{t_1, t_2, \dots\} \cup \{0\}$, where (t_n) is a strictly decreasing sequence in $(0; 1)$. Then the following conditions are equivalent:

- (a) $\mathcal{A}$ is Noetherian,
- (b) for each fuzzy ideal μ of $\mathcal{A}$, if $\text{Im}(\mu) \subseteq T$, then there exists $k \in \mathbb{N}$ such that $\text{Im}(\mu) \subseteq \{t_1, t_2, \dots, t_k\} \cup \{0\}$.

Proof. (a) $\Rightarrow$ (b): Assume that $\mathcal{A}$ is Noetherian. Let μ be a fuzzy ideal of $\mathcal{A}$ such that $\text{Im}(\mu) \subseteq T$. From Theorem 50 we know that $\text{Im}(\mu)$ is a well-ordered subset of $[0; 1]$. Then, since $1 > t_1 > t_2 > \dots > t_n > \dots > 0$ and $\text{Im}(\mu) \subseteq \{t_1, t_2, \dots\} \cup \{0\}$, there exists $k \in \mathbb{N}$ such that $\text{Im}(\mu) \subseteq \{t_1, t_2, \dots, t_k\} \cup \{0\}$.

(b) $\Rightarrow$ (a): Assume that (b) is true. Suppose that $\mathcal{A}$ is not Noetherian. Then there exists a strictly ascending sequence $I_1 \subset I_2 \subset \dots$ of ideals of $\mathcal{A}$. Define a fuzzy set μ in A by

$$\mu(x) = \begin{cases} 0, & \text{if } x \notin I_n \text{ for each } n \in \mathbb{N}, \\ t_n, & \text{if } x \in I_n - I_{n-1} \text{ for some } n \in \mathbb{N}, \end{cases} \quad (44)$$

where $I_0 = \emptyset$. By Lemma 44, μ is a fuzzy ideal of $\mathcal{A}$. This is a contradiction with our assumption. Thus $\mathcal{A}$ is Noetherian. $\square$

Theorem 53. Let $\mathcal{A}$ be a BN-algebra and let $T = \{t_1, t_2, \dots\} \cup \{0, 1\}$, where (t_n) is a strictly increasing sequence in $(0; 1)$. Then the following conditions are equivalent:

- (a) $\mathcal{A}$ is Artinian,
- (b) for each fuzzy ideal μ of $\mathcal{A}$, if $\text{Im}(\mu) \subseteq T$, then there exists $k \in \mathbb{N}$ such that $\text{Im}(\mu) \subseteq \{t_1, t_2, \dots, t_k\} \cup \{0, 1\}$.

Proof. (a) $\Rightarrow$ (b): Suppose that $t_{i_1} < t_{i_2} < \dots < t_{i_m} < \dots$ is a strictly increasing sequence of elements of $\text{Im}(\mu)$. Let $U_m = U(\mu; t_{i_m})$ for $m \in \mathbb{N}$. It is immediately seen that $U_1 \supset U_2 \supset \dots \supset U_m \supset \dots$ is a strictly descending sequence of ideals of $\mathcal{A}$. This contradicts the assumption that $\mathcal{A}$ is Artinian.

(b) $\Rightarrow$ (a): Assume that (b) is true. Suppose that $\mathcal{A}$ is not Artinian. Then there exists a strictly descending sequence $I_1 \supset I_2 \supset \dots \supset I_n \supset \dots$ of ideals of $\mathcal{A}$. Define a fuzzy set μ in A by

$$\mu(x) = \begin{cases} 0, & \text{if } x \notin I_1, \\ t_n, & \text{if } x \in I_n - I_{n+1} \text{ for } n = 1, 2, \dots, \\ 1, & \text{if } x \in \bigcap \{I_n : n \in \mathbb{N}\}. \end{cases} \quad (45)$$

Obviously, $\mu(0) = 1 \geq \mu(x)$ for all $x \in A$; that is, (d1) holds. Now we show that μ satisfies (d2). Let $x, y \in A$. We have three cases.

Case 1 ($x \notin I_1$). Then $x * y \notin I_1$ or $y \notin I_1$. Therefore $\mu(x * y) \wedge \mu(y) = 0 = \mu(x)$.

Case 2 ($x \in I_n - I_{n+1}$ for some $n \in \mathbb{N}$). Then $x * y \notin I_{n+1}$ or $y \notin I_{n+1}$. Hence $\mu(x * y) \leq t_n$ or $\mu(y) \leq t_n$. Therefore $\mu(x * y) \wedge \mu(y) \leq t_n = \mu(x)$.

Case 3 ($x \in \bigcap \{I_n : n \in \mathbb{N}\}$). It is obvious.

Thus μ is a fuzzy ideal of $\mathcal{A}$. This contradicts our assumption. Thus $\mathcal{A}$ is Artinian. $\square$

Corollary 54. Let $\mathcal{A}$ be a BN-algebra. If, for every fuzzy ideal μ of $\mathcal{A}$, $\text{Im}(\mu)$ is a finite set, then $\mathcal{A}$ is Artinian.

The following example shows that the converse of Corollary 54 does not hold.

Example 55. Let p be a prime number. Set $A = \{z \in \mathbb{C} : z^{p^n} = 1 \text{ for some } n \geq 0\}$. It is known that $(A; \cdot, 1)$ is the p -quasicyclic group. Define $x * y = xy^{-1}$ for all $x, y \in A$. By Example 7, $\mathcal{A} = (A; *, 1)$ is a BN-algebra. Let $I_n = \{z \in \mathbb{C} : z^{p^n} = 1\}$ for $n \in \mathbb{N} \cup \{0\}$. It follows easily that I is an ideal of $\mathcal{A}$ if and only if $I = A$ or $I = I_n$ for some $n \geq 0$. We have $I_0 = \{1\} \subset I_1 \subset \dots \subset I_n \subset \dots \subset A$ and hence $\mathcal{A}$ is Artinian. Define μ by

$$\mu(x) = \frac{1}{n+1} \quad \text{if } x \in I_n - I_{n-1} \text{ for some } n \in \mathbb{N} \cup \{0\}, \quad (46)$$

where $I_{-1} = \emptyset$. Since $A = \bigcup_{n \in \mathbb{N}} I_n$, μ is a fuzzy set in A . By the proof of Lemma 44, $\mu \in \mathcal{FId}(\mathcal{A})$. However, $\text{Im}(\mu) = \{1/n : n \in \mathbb{N}\}$ is not a finite set.

5. Conclusions

This paper begins by considering the notion of ideals in BN-algebras. We give its characterizations and introduce the concept of normal ideals investigating its properties. We also define the notion of normal congruences proving that there is a one-to-one correspondence between normal ideals and normal congruences of a BN-algebra. Moreover we obtain the isomorphism theorem for BN-algebras. Next, we define the notion of fuzzy ideals of BN-algebras giving its characterizations and providing conditions for a fuzzy set to be a fuzzy ideal. We give the relationships between ideals and fuzzy ideals of a BN-algebra and also provide the homomorphic properties of fuzzy ideals. Finally, we display characterizations of Noetherian BN-algebras and Artinian BN-algebras via fuzzy ideals.

The next step in studying fuzzy ideals in BN-algebras may be introducing and investigating the notions of fuzzy maximal ideals and fuzzy prime ideals of BN-algebras.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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