Research Article

On Vertex Covering Transversal Domination Number of Regular Graphs

R. Vasanthi¹ and K. Subramanian²

¹Department of Mathematics, Alagappa Chettiar College of Engineering and Technology, Karaikudi, Tamil Nadu 630 004, India

²Department of Mathematics, Alagappa Government Arts College, Karaikudi, Tamil Nadu 630 003, India

Correspondence should be addressed to R. Vasanthi; vasanthi2014accet@gmail.com

Received 19 December 2015; Revised 22 February 2016; Accepted 6 March 2016

Academic Editor: Kinkar Ch Das

Copyright © 2016 R. Vasanthi and K. Subramanian. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

A simple graph $G = (V, E)$ is said to be r -regular if each vertex of G is of degree r . The vertex covering transversal domination number $\gamma_{\text{vct}}(G)$ is the minimum cardinality among all vertex covering transversal dominating sets of G . In this paper, we analyse this parameter on different kinds of regular graphs especially for Q_n and $H_{3,n}$. Also we provide an upper bound for γ_{vct} of a connected cubic graph of order $n \geq 8$. Then we try to provide a more stronger relationship between γ and γ_{vct} .

1. Introduction

Hamid [1] introduced independent transversal domination in graphs. It was defined using maximum independent set in a graph. Vasanthi and Subramanian [2] introduced vertex covering transversal domination in graphs using minimum vertex covering set in a graph. The vertex covering transversal domination number of some standard graphs such as K_n , $K_{m,n}$, P_n , C_n , and W_n and trees is dealt with in paper [2]. Bounds of γ_{vct} are also established through various parameters in [2]. Lam et al. [3] worked on independent domination number of regular graphs. In this paper, we investigate our parameter γ_{vct} for regular graphs. Also we try to provide a more stronger relationship between γ and γ_{vct} .

A simple graph $G = (V, E)$ is said to be r -regular if each vertex of G is of degree r . A set $I \subseteq V$ of vertices in G is called an *independent set* if no two vertices in I are adjacent. Also I is said to be a *maximum independent set* if there is no other independent set I' such that $|I'| > |I|$. The cardinality of a maximum independent set is called the *independence number* and is denoted by $\beta_0(G)$. A set $C \subseteq V$ of vertices in G is called a *vertex covering set* (or simply *covering set*) if every edge of G is incident to at least one vertex in C . Also C is

said to be a *minimum vertex covering set* if there is no other vertex covering set C' such that $|C'| < |C|$. The cardinality of a minimum vertex covering set is called the *vertex covering number* and is denoted by $\alpha_0(G)$. A set $D \subseteq V$ of vertices in the graph G is called a *dominating set* if every vertex in $V - D$ is adjacent to a vertex in D . A dominating set which intersects every minimum vertex covering set in G is called a *vertex covering transversal dominating set*. The minimum cardinality of a vertex covering transversal dominating set is called *vertex covering transversal domination number* of G and is denoted by $\gamma_{\text{vct}}(G)$.

The parameter *independent domination number* $i(G)$ was introduced by Cockayne and Hedetniemi in [4]. The *independent domination number* $i(G)$ is the minimum cardinality among all independent dominating sets of G . An independent set is dominating if and only if it is maximal. So $i(G)$ is the minimum cardinality of a maximal independent set in G . In paper [3], the following theorem which gives the upper bound for independent domination number of a connected cubic graph has been proved.

Theorem 1. *If G is a connected cubic graph of order n where $n \geq 8$, then $i(G) \leq 2n/5$.*

2. Notations

We use the following notations throughout the paper:

- α_0 -set to denote minimum vertex covering set,
- β_0 -set to denote maximum independent set,
- γ -set to denote a dominating set of minimum cardinality,
- γ_{vct} -set to denote a vertex covering transversal dominating set of minimum cardinality,
- $\gamma(G)$ to denote domination number of G ,
- $\gamma_{\text{vct}}(G)$ to denote vertex covering transversal domination number of G ,
- $i(G)$ to denote independent domination number of G ,
- $O(G)$ to denote the order of G ,
- $\deg_G(u)$ to denote the degree of a vertex u in G .

3. γ_{vct} for Regular Graphs

Here, we provide the vertex covering transversal domination number of some standard regular graphs such as complete graphs, complete bipartite regular graphs, cycles, and hypercube Q_n . We also establish γ_{vct} for certain family of regular graphs defined in [3].

Example 2. K_n is a $(n-1)$ -regular graph and $\gamma_{\text{vct}}(K_n) = 2$ for $n \geq 2$.

Example 3. C_n is a 2-regular graph of order $n \geq 3$ and

$$\gamma_{\text{vct}}(C_n) = \begin{cases} 2 & \text{if } n = 3, 4 \\ 3 & \text{if } n = 5 \\ \left\lceil \frac{n}{3} \right\rceil & \text{otherwise.} \end{cases} \quad (1)$$

Example 4. $K_{n,n}$ is a complete bipartite n -regular graph and $\gamma_{\text{vct}}(K_{n,n}) = 2$.

The following theorem provides the vertex covering transversal domination number of n -dimensional cube or hypercube Q_n defined in [5].

Theorem 5. If Q_n is a hypercube containing 2^n vertices which is n -regular, then

$$\gamma_{\text{vct}}(Q_n) = \begin{cases} 2 & \text{if } n = 2 \\ 2^{n-2} & \text{if } n \geq 3. \end{cases} \quad (2)$$

Proof. The n -dimensional cube or hypercube Q_n contains 2^n vertices and is n -regular. Each vertex in Q_n is represented by a n -tuple with 0's and 1's. Two vertices in Q_n are adjacent if and only if the n -tuples differ in exactly one position. Also any $v \in Q_n$ is the n -tuple binary number and its complement v^c is also an n -tuple binary number obtained by replacing 0 by 1 and 1 by 0 in v . The weight of a 0, 1 vertex is the number of 1's occurring in it. There are exactly 2^{n-1} vertices of odd weight

FIGURE 1

FIGURE 2

and 2^{n-1} vertices of even weight. Each edge of Q_n consists of a vertex of even weight and a vertex of odd weight. The vertices of even weight form an independent set and so do the vertices of odd weight. Therefore Q_n is bipartite with bipartitions S_1 and S_2 where S_1 is the set of all n -tuples of even weight and S_2 is the set of all n -tuples of odd weight with $|S_1| = |S_2| = 2^{n-1}$.

Also S_1 and S_2 are the only β_0 -sets of Q_n . Since they are complements of each other, S_1 and S_2 are the only α_0 -sets of Q_n .

For $n = 2$, Q_2 is as shown in Figure 1. Obviously, $S_1 = \{00, 11\}$ and $S_2 = \{10, 01\}$ are α_0 -sets of Q_2 . Then $D = \{00, 10\}$ is a γ_{vct} -set of Q_2 and so $\gamma_{\text{vct}}(Q_2) = 2$.

Now suppose $n \geq 3$.

For $n = 3$, Q_3 is the hypercube on 8 vertices which is 3-regular and is represented as in Figure 1. The only two α_0 -sets of Q_3 are $S_1 = \{000, 011, 110, 101\}$ and $S_2 = \{001, 010, 100, 111\}$. Then $\gamma_{\text{vct}}(Q_3) = 2$ since every two-element set of the form $\{v, v^c\}$ where $v \in S_1$ and $v^c \in S_2$ is a γ_{vct} -set.

If $n = 4$, the hypercube Q_4 contains 2^4 vertices and is 4-regular as shown in Figure 2. Q_4 is bipartite with bipartitions $S_1 = \{0000, 0011, 0110, 1100, 0101, 1010, 1001, 1111\}$ and $S_2 = \{0001, 0010, 0100, 1000, 0111, 1110, 1011, 1101\}$. Also S_1 and S_2 are the only α_0 -sets of Q_4 .

Let $D = \{0000, 1111, 0001, 1110\}$. Then D is a dominating set of Q_4 . Clearly it intersects both S_1 and S_2 . Therefore D is a vertex covering transversal dominating set and so $\gamma_{\text{vct}}(Q_4) = 4$. Hence it remains to show that D is of minimum cardinality.

FIGURE 3

Suppose there exists a vertex covering transversal dominating set D' of cardinality less than 4. It must have at least 2 vertices as it intersects both S_1 and S_2 . Suppose $|D'| = 3$. Since each vertex is of degree 4, all the three vertices in D' may dominate at most 12 vertices. But there are 16 vertices in Q_4 and so D' do not dominate at least 1 vertex. This is a contradiction to the assumption that D' is a vertex covering transversal dominating set.

Also any set containing two mutually complementary vertices from S_1 , say, 1100, 0011, and the other two mutually complementary vertices from S_2 , say, 1000, 0111, form a γ_{vct} -set. Thus $S = \{v_1, v_1^c, v_2, v_2^c\}$ where $v_1, v_1^c \in S_1$ and $v_2, v_2^c \in S_2$ is a γ_{vct} -set of Q_4 .

If $n = 5$, the hypercube Q_5 contains $2^5 = 32$ vertices and the bipartition S_1 contains 2^4 vertices and S_2 contains 2^4 vertices. Let $S = \{v_1, v_1^c, v_2, v_2^c, v_3, v_3^c, v_4, v_4^c\}$ where $v_1, v_2, v_3, v_4 \in S_1$ and $v_1^c, v_2^c, v_3^c, v_4^c \in S_2$. Then S is a γ -set which intersects both S_1 and S_2 . Hence $\gamma_{\text{vct}}(Q_5) = 2^3$.

Thus in general, $S = \{v_1, v_1^c, v_2, v_2^c, \dots, v_{2^{n-3}-1}, v_{2^{n-3}-1}^c, v_{2^{n-3}}, v_{2^{n-3}}^c\}$ is a γ_{vct} -set of Q_n . In particular, if n is odd, $v_1, v_2, \dots, v_{2^{n-3}} \in S_1$ and $v_1^c, v_2^c, \dots, v_{2^{n-3}}^c \in S_2$. If n is even, $v_1, v_1^c, v_2, v_2^c, \dots, v_{2^{n-4}}, v_{2^{n-4}}^c \in S_1$ and $v_{2^{n-4}+1}, v_{2^{n-4}+1}^c, \dots, v_{2^{n-3}}, v_{2^{n-3}}^c \in S_2$. Hence $\gamma_{\text{vct}}(Q_n) = 2^{n-2}$. $\square$

Theorem 6. If G is a connected regular graph of degree $n - 2$ and $O(G) = n$, then $\gamma_{\text{vct}}(G) = 2$.

Proof. Choose any vertex $u \in V(G)$. Then $\deg_G(u) = n - 2$; that is, u is adjacent to $n - 2$ vertices in G . Then there remains exactly one vertex, say, v , which is not adjacent to u . Therefore $S = \{u, v\}$ is an independent set of G . Also v is adjacent to $n - 2$ vertices in G except u . Hence no other vertex may be included in S . Therefore S is a maximum independent set of G .

Now let $w \in V - S$. Then w is adjacent to both u and v . Since u dominates every vertex in G except v , and w dominates $n - 2$ vertices including v , it is obvious that $D = \{u, w\}$ is a dominating set which intersects every minimum vertex covering set of G . Also D is of minimum cardinality in G . Hence $\gamma_{\text{vct}}(G) = 2$.

It is noted that $\{v, w\}$ is also a γ_{vct} -set. $\square$

Remark 7. In the above theorem, n should be even. For otherwise, if n is odd, then $n - 2$ is odd which is impossible as the number of vertices of odd degree in a graph is even.

Lemma 8. Given positive integers $p \geq 2$ and $q \geq 3$, let $G(p, q)$ be the family of graphs such that $V = \bigcup_{i=1}^p (X_i \cup Y_i \cup Z_i)$ and $E = E_1 \cup E_2 \cup E_3 \cup E_4$ with

$$X_i = \{x_{i1}, x_{i2}, \dots, x_{i(q-1)}\}, Y_i = \{y_{i1}, y_{i2}, \dots, y_{iq}\}, Z_i = \{z_{i1}, z_{i2}, \dots, z_{iq}\},$$

$$E_1 = \bigcup_{i=1}^p \{x_{ik}y_{il}; 1 \leq k \leq q-1, 1 \leq l \leq q\},$$

$$E_2 = \bigcup_{i=1}^p \{y_{ik}z_{ik}; 1 \leq k \leq q\},$$

$$E_3 = \bigcup_{i=1}^p \{z_{ik}z_{il}; 1 \leq k, l \leq q, k \neq l\} - \{z_{i1}z_{iq}\},$$

$$E_4 = \{z_{iq}z_{(i+1)1}; 1 \leq i \leq p-1\} \cup \{z_{pq}z_{11}\}.$$

Then

- (i) $|V(G)| = p(3q - 1)$,
- (ii) $G(p, q)$ is connected and q -regular,
- (iii) $\gamma_{\text{vct}}(G(p, q)) = 4p$.

Proof. $G(p, q)$ contains p subgraphs which we shall call blocks each containing $3q - 1$ vertices and isomorphic to each other. By the edge set E_4 , we observe that they are connected to each other.

Thus (i) and (ii) are obvious.

For $q = 4$, two connected blocks of $G(p, 4)$ each consisting of 11 vertices are as shown in Figure 3.

Now $I = \bigcup_{i=1}^p Y_i$ is a maximum independent set of $G(p, q)$. Then its complement $S = V - I = \bigcup_{i=1}^p (X_i \cup Z_i)$ is a minimum vertex covering set of $G(p, q)$.

Also $I_j = \bigcup_{i=1}^p (Y_i - \{y_{ij}\}) \cup \{z_{ij}\}$ and $J_j = \bigcup_{i=1}^p (X_i \cup \{z_{ij}\})$, $1 \leq j \leq q$ are maximum independent sets in $G(p, q)$. Let S_j and T_j be the complement of each I_j and J_j . Then $S_j = \bigcup_{i=1}^p ((X_i \cup Z_i) - \{z_{ij}\} \cup \{y_{ij}\})$ and $T_j = \bigcup_{i=1}^p ((Y_i \cup Z_i) - \{z_{ij}\})$, $1 \leq j \leq q$ are minimum vertex covering sets in $G(p, q)$. Now the subgraph G_i induced by $X_i \cup Y_i$ in each block is a complete bipartite graph $K_{q-1, q}$.

Since $\gamma_{\text{vct}}(K_{m, n}) = 2$ if $m, n > 1$, we have $\gamma_{\text{vct}}(K_{q-1, q}) = 2$. Also each $\{x_{ij}, y_{ik}\}$ for $1 \leq j \leq q - 1$ and $1 \leq k \leq q$ is a vertex covering transversal dominating set of G_i . Then $\{x_{ij}, y_{ik}, z_{i1}, z_{iq}\}$, $1 \leq j \leq q - 1$ and $1 \leq k \leq q$, is a dominating set for each block. Therefore $D_{jk} = \bigcup_{i=1}^p \{x_{ij}, y_{ik}, z_{i1}, z_{iq}\}$,

FIGURE 4

$1 \leq j \leq q-1$ and $1 \leq k \leq q$, is a γ -set which intersects the only α_0 -sets S , S_j and T_j of $G(p, q)$ for each j .

Hence $\gamma_{\text{vct}}(G(p, q)) = 4p$. $\square$

Theorem 9. If $q \geq 3$, then there exists a connected q -regular graph with $\gamma_{\text{vct}}(G) \geq \lceil 4n/3q \rceil$ where n is the order of G .

Proof. Let $G = G(p, q)$ be defined as in Lemma 8. Then $\gamma_{\text{vct}}(G)/n = 4/(3q-1) \geq 4/3q$. Thus $\gamma_{\text{vct}}(G) \geq 4n/3q$. $\square$

Lemma 10. Given positive integers $p \geq 1$ and $q \geq 2$, let $G^*(p, q)$ be the graph (V, E) with $V = U \cup [\bigcup_{i=1}^{2p+1} (V_i \cup W_i)]$ and $E = E_1 \cup E_2 \cup E_3 \cup E_4$ with

$$\begin{aligned} U &= \{u_1, u_2, \dots, u_{2p+1}\}, V_i = \{v_{i1}, v_{i2}, \dots, v_{i(q+2p)}\}, \\ W_i &= \{w_{i1}, w_{i2}, \dots, w_{i(q+2p-1)}\}, \\ E_1 &= \{u_j u_k; 1 \leq j < k \leq 2p+1\}, \\ E_2 &= \bigcup_{i=1}^{2p+1} \{u_i v_{ik}; 1 \leq k \leq q\}, \\ E_3 &= \bigcup_{i=1}^{2p+1} \{v_{i(q+2k-1)} v_{i(q+2k)}; 1 \leq k \leq p\}, \\ E_4 &= \bigcup_{i=1}^{2p+1} \{v_{ik} w_{il}; 1 \leq k \leq q+2p, 1 \leq l \leq q+2p-1\}. \end{aligned}$$

Then

- (i) $|V(G)| = 2(2p+1)(q+2p)$,
- (ii) $G^*(p, q)$ is connected and $(q+2p)$ -regular,
- (iii) $\gamma_{\text{vct}}(G^*(p, q)) = 2(2p+1)$.

Proof. (i) and (ii) are obvious. If $p = 1$ and $q = 2$, the graph $G^*(1, 2)$ is as shown in Figure 4.

It is clear that each $I_j = [\bigcup_{i=1}^{2p+1} (W_i)] \cup \{u_j\}$, $j = 1$ to $2p+1$, is a maximum independent set in $G^*(p, q)$. Therefore its complement $J_j = V - I_j = [\bigcup_{i=1}^{2p+1} (V_i)] \cup [U - \{u_j\}]$, $j = 1$ to $2p+1$, is an α_0 set in $G^*(p, q)$. Further $J_1, J_2, \dots, J_{2p}$ and J_{2p+1} are the only α_0 -sets of $G^*(p, q)$. Now each $S_{jk} = \bigcup_{i=1}^{2p+1} \{v_{ik}, w_{ij}\}$, $j = 1$ to $q+2p-1$, $k = 1$ to q is a dominating set intersecting $J_1, J_2, \dots, J_{2p}$ and J_{2p+1} and also of minimum cardinality $2(2p+1)$.

Hence $\gamma_{\text{vct}}(G^*(p, q)) = 2(2p+1)$. $\square$

Theorem 11. For every $r \geq 4$, there exists a connected r -regular graph G of order n such that $\gamma_{\text{vct}}(G) = n/r$.

Proof. Let $G = G^*(p, q)$ be defined as in Lemma 10. Then G is a connected r -regular graph with $r = q+2p$. Also $\gamma_{\text{vct}}(G)/n = 1/(q+2p)$.

Thus $\gamma_{\text{vct}}(G) = n/r$. $\square$

Remark 12. Theorems 9 and 11 hold good if γ_{vct} is replaced by γ .

4. γ_{vct} for Regular Cubic Graphs

In this section, we provide the vertex covering transversal domination number of some regular cubic graphs especially Harary graph $H_{3,n}$ defined in [6]. We also obtain an upper bound for the vertex covering transversal domination number of a connected cubic graph.

Example 13. Consider the triangular prism graph Y_3 shown in Figure 5. It is a regular cubic graph.

Y_3 has 6 vertices and 9 edges. Assume that the graph Y_3 is labelled as shown in the diagram. It is clear that $\{u_{i \bmod 3}, v_{(i+1) \bmod 3}\}$ and $\{u_{i \bmod 3}, v_{(i+2) \bmod 3}\}$ for $i = 0, 1, 2$ are β_0 -sets of Y_3 . Then their complements $C_i = \{u_{(i+1) \bmod 3}, u_{(i+2) \bmod 3}, v_{i \bmod 3}, v_{(i+2) \bmod 3}\}$ and $S_i = \{u_{(i+1) \bmod 3}, u_{(i+2) \bmod 3}, v_{i \bmod 3}, v_{(i+1) \bmod 3}\}$ for $i = 0, 1, 2$ are α_0 -sets of Y_3 . Now each $D_i = \{u_{i \bmod 3}, v_{i \bmod 3}\}$, $i = 0, 1, 2$ is a γ -set for Y_3 . Clearly it intersects each C_i and S_i . Therefore $\gamma_{\text{vct}}(Y_3) = 2$.

Example 14. Consider Peterson graph which is cubic regular shown in Figure 6.

Assuming that the graph G is labelled as shown in Figure 6, it is obvious that $I_i = \{v_{i \bmod 5}, v_{(i+3) \bmod 5}, u_{(i+1) \bmod 5}, u_{(i+2) \bmod 5}\}$, $i = 0, 1, 2, 3, 4$ are β_0 -sets of G . Then their complements $C_i = \{v_{(i+1) \bmod 5}, v_{(i+2) \bmod 5}, v_{(i+4) \bmod 5}, u_{i \bmod 5}, u_{(i+3) \bmod 5}, u_{(i+4) \bmod 5}\}$, $i = 0, 1, 2, 3, 4$ are α_0 -sets

FIGURE 5

FIGURE 6

of G . Now $S_i = \{v_i \bmod 5, u_{(i+2) \bmod 5}, u_{(i+3) \bmod 5}\}$ are γ -sets intersecting each C_i . Hence $\gamma_{\text{vct}}(G) = 3$.

Note that $S_i = \{v_i \bmod 5, v_{(i+3) \bmod 5}, u_{(i+4) \bmod 5}\}$ are also γ_{vct} -sets in G .

Theorem 15. If $H_{3,n}$ is a Harary graph with $n \geq 6$, then $\gamma_{\text{vct}}(H_{3,n}) = \lfloor (n+1)/3 \rfloor$.

Proof. $H_{3,n}$ is a 3-regular graph and so n is even. By the definition of $H_{3,n}$, every vertex $v_i \in H_{3,n}$ is adjacent to the vertices v_{i+1} , v_{i-1} , and v_{i+k} where $n = 2k$.

Let $V(H_{3,n}) = \{v_0, v_1, v_2, \dots, v_{n-1}\}$. The graphs $H_{3,10}$ and $H_{3,12}$ are shown in Figure 7.

Case 1. Suppose $n = 2k$ where k is odd.

Then $C_1 = \{v_0, v_2, v_4, \dots, v_{n-2}\}$ and $C_2 = \{v_1, v_3, v_5, \dots, v_{n-1}\}$ are the only α_0 -sets of $H_{3,n}$.

Subcase 1. Let $n \equiv 0 \pmod{3}$.

Then $S = \{v_0, v_3, v_6, \dots, v_{n-3}\}$ is a γ -set which intersects C_1 and C_2 and $|S| = n/3$.

Subcase 2. Suppose $n \equiv 1 \pmod{3}$.

Then $S = \{v_0, v_3, v_6, \dots, v_{n-4}\}$ is a γ -set which intersects C_1 and C_2 and $|S| = (n-1)/3$.

Subcase 3. Suppose $n \equiv 2 \pmod{3}$.

Then $S = \{v_0, v_3, v_6, \dots, v_{n-2}\}$ is a γ -set which intersects C_1 and C_2 with $|S| = (n+1)/3$.

Thus in all the subcases of Case 1, $\gamma_{\text{vct}}(H_{3,n}) = \lfloor (n+1)/3 \rfloor$.

Case 2. Suppose $n = 2k$ where k is even.

Then $I_i = \{v_{i \bmod n}, v_{(i+2) \bmod n}, \dots, v_{(i+k-2) \bmod n}, v_{(i+k+1) \bmod n}, v_{(i+k+3) \bmod n}, \dots, v_{(i+n-5) \bmod n}, v_{(i+n-3) \bmod n}\}$ is a β_0 -set for each $i = 0, 1, 2, \dots, n-1$.

Therefore $C_i = \{v_{(i+1) \bmod n}, v_{(i+3) \bmod n}, \dots, v_{(i+k-1) \bmod n}, v_{(i+k) \bmod n}, v_{(i+k+2) \bmod n}, \dots, v_{(i+n-4) \bmod n}, v_{(i+n-2) \bmod n}, v_{(i+n-1) \bmod n}\}$ is an α_0 -set for each $i = 0, 1, 2, \dots, n-1$.

Subcase 1. Let $n \equiv 0 \pmod{3}$.

Then $S_i = \{v_{i \bmod n}, v_{(i+3) \bmod n}, v_{(i+6) \bmod n}, \dots, v_{(i+n-3) \bmod n}\}$ is a γ -set which intersects each C_i for $i = 0, 1, 2, \dots, n-1$.

Subcase 2. Let $n \equiv 1 \pmod{3}$.

Then $S_i = \{v_{i \bmod n}, v_{(i+3) \bmod n}, v_{(i+6) \bmod n}, \dots, v_{(i+n-4) \bmod n}\}$ is a γ -set which intersects each C_i for $i = 0, 1, 2, \dots, n-1$.

Subcase 3. Let $n \equiv 2 \pmod{3}$.

Then $S_i = \{v_{i \bmod n}, v_{(i+3) \bmod n}, v_{(i+6) \bmod n}, \dots, v_{(i+n-2) \bmod n}\}$ is a γ -set which intersects each C_i for $i = 0, 1, 2, \dots, n-1$.

The γ_{vct} -sets mentioned in all the subcases of Case 2 are also of cardinality $\lfloor (n+1)/3 \rfloor$.

Thus $\gamma_{\text{vct}}(H_{3,n}) = \lfloor (n+1)/3 \rfloor$. $\square$

Remark 16. In most of the graphs considered by us, it is observed that $\gamma_{\text{vct}} = \gamma$.

Theorem 17. If G is a connected cubic graph of order n with $n \geq 8$, then $\gamma_{\text{vct}}(G) \leq \lceil 2n/5 \rceil$.

Proof. Let I be an independent dominating set of cardinality $i(G)$. Then I is a maximal independent set of minimum cardinality. Since I is independent, no two vertices of I are adjacent in G . Let $J = V - I$. Then the vertices in I are adjacent only to the vertices in J .

Case 1. Suppose I itself is a β_0 -set. Then J is an α_0 -set. Let $S = I \cup \{v\}$ where $v \in J$. Then S is a vertex covering transversal dominating set of G . Therefore $\gamma_{\text{vct}}(G) \leq i(G) + 1$. Hence $\gamma_{\text{vct}}(G) \leq 2n/5 + 1$ (by Theorem 1 proved in [3]).

Case 2. Suppose I is not a β_0 -set. But I is a maximal independent dominating set of minimum cardinality. We claim that I intersects every α_0 -set of G .

Suppose that I does not intersect an α_0 -set C of G . Then $I \subset V - C$ where $V - C$ is a β_0 -set of G . This is a contradiction to the maximality of I .

Hence I itself is a vertex covering transversal dominating set of G . Therefore $\gamma_{\text{vct}}(G) \leq 2n/5$.

Thus Cases 1 and 2 imply that $\gamma_{\text{vct}}(G) \leq \lceil 2n/5 \rceil$. $\square$

FIGURE 7

5. Relation between $\gamma(G)$ and $\gamma_{\text{vct}}(G)$

In this section, we prove a more stronger relationship between γ and γ_{vct} than that proved in [2]. In view of the results and theorems dealt with in the previous sections, we try to characterize graphs for which $\gamma = \gamma_{\text{vct}}$ and $\gamma < \gamma_{\text{vct}}$.

Theorem 18. *If G is a simple connected graph, then $\gamma_{\text{vct}}(G) \leq \gamma(G) + 1$.*

Proof. Let D be a minimum dominating set. If $D = V(G)$, then obviously $\gamma_{\text{vct}}(G) = \gamma(G)$. If not, then $D \subset V(G)$ and $V(G) - D \neq \emptyset$. Let $u \in V(G) - D$. Then u is dominated by some vertex v in D . Let $S = D \cup \{u\}$. Since uv is an edge in G , either u or v is included in every minimum vertex covering set of G . This implies that S intersects every minimum vertex covering set in G . Hence $\gamma_{\text{vct}}(G) \leq \gamma(G) + 1$. $\square$

Theorem 19. *Let G be a simple connected graph. If there exists a γ -set which is not independent, then $\gamma_{\text{vct}}(G) = \gamma(G)$.*

Proof. Let D be a minimum dominating set which is not an independent set of G . Then at least two vertices, say, u, v in D , are adjacent to each other. Therefore uv is an edge in G and hence either u or v lies in every minimum vertex covering set of G . So D intersects every α_0 -set of G . Therefore D itself is a γ_{vct} -set. Hence $\gamma_{\text{vct}}(G) = \gamma(G)$. $\square$

Remark 20. The converse is not true. If $\gamma_{\text{vct}}(G) = \gamma(G)$, then there may exist a γ -set which is independent also. For example, consider C_6 , the cycle on 6 vertices as shown in Figure 8.

Obviously $\{v_1, v_3, v_5\}$ and $\{v_2, v_4, v_6\}$ are the only α_0 -sets of C_6 . Also $\{v_1, v_4\}$ is a γ -set which is independent. Further, it is a γ_{vct} -set as it intersects both the α_0 -sets of C_6 . Thus there exists a γ -set which is independent in C_6 even though $\gamma_{\text{vct}}(C_6) = \gamma(C_6)$.

Remark 21. Now, the obvious question is “If $\gamma_{\text{vct}}(G) = \gamma(G)$, is every γ -set of G a γ_{vct} -set?” The answer is “not always.” The γ -sets and γ_{vct} -sets in the graphs Q_2 and Y_3 discussed in the previous sections are the best examples for it. So it is noted

FIGURE 8

that this happens if there exists a γ -set which is also a β_o -set. It obviously produces the result that “If $\gamma_{\text{vct}}(G) = \gamma(G) = \beta_o(G)$, then there exists at least one γ -set in G which is not a γ_{vct} -set.” The next general question is that “What happens if all the γ -sets of G are β_o -sets?”. The following theorem provides the answer to it.

Theorem 22. *Let G be a simple connected graph. If every γ -set of G is a β_o -set, then $\gamma_{\text{vct}}(G) = \gamma(G) + 1$.*

Proof. Since every γ -set D of G is a β_o -set, choose a vertex v in its complement. This is possible since $D \neq V(G)$ as D is a β_o -set of a connected graph G . Obviously D is not a γ_{vct} -set as it does not intersect the α_0 -set $V(G) - D$. Let $S = D \cup \{v\}$. We claim that S intersects every α_0 -set of G . Suppose that $S \cap C = \emptyset$ for some α_0 -set C in G . Then $S \subseteq I$ where $I = V(G) - C$ is a β_o -set. This implies that $\beta_o(G) + 1 \leq \beta_o(G)$ which is a contradiction. Hence S intersects every α_0 -set of G . Also S is a γ_{vct} -set of G as it contains exactly one vertex more than that of the γ -set D . Thus $\gamma_{\text{vct}}(G) = \gamma(G) + 1$. $\square$

Remark 23. It is easy to conclude that even though $\gamma_{\text{vct}} = \gamma$, there are graphs in which γ -sets do not become γ_{vct} -sets. This implies that the collection of γ_{vct} -sets in such graphs is contained in the collection of γ -sets. So this may lead to consider γ_{vct} -sets in the graphs for which $\gamma_{\text{vct}} = \gamma$ when we are in a situation to select a minimum number of γ -sets

in such graphs. This approach may affect a new variation in domination theory.

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

References

- [1] I. S. Hamid, "Independent transversal domination in graphs," *Discussiones Mathematicae. Graph Theory*, vol. 32, no. 1, pp. 5–17, 2012.
- [2] R. Vasanthi and K. Subramanian, "Vertex covering transversal domination in graphs," *International Journal of Mathematics and Soft Computing*, vol. 5, no. 2, pp. 1–7, 2015.
- [3] P. C. Lam, W. C. Shiu, and L. Sun, "On independent domination number of regular graphs," *Discrete Mathematics*, vol. 202, no. 1–3, pp. 135–144, 1999.
- [4] E. J. Cockayne and S. T. Hedetniemi, "Towards a theory of domination in graphs," *Networks*, vol. 7, no. 3, pp. 247–261, 1977.
- [5] D. B. West, *Introduction to Graph Theory*, Prentice-Hall of India Private Limited, New Delhi, India, 2000.
- [6] G. Chatrand and P. Zhang, *Introduction to Graph Theory*, Tata McGraw-Hill Education Private Limited, New Delhi, India, 8th edition, 2012.