Hindawi Publishing Corporation
The Scientific World Journal
Volume 2016, Article ID 1871602, 9 pages
http://dx.doi.org/10.1155/2016/1871602

Research Article

The Existence of Weak $\mathscr{D}$-Pullback Exponential Attractor for Nonautonomous Dynamical System

Yongjun Li, Xiaona Wei, and Yanhong Zhang

School of Mathematics, Lanzhou City University, Lanzhou 730070, China

Correspondence should be addressed to Yongjun Li; li_liyong120@163.com

Received 5 January 2016; Accepted 22 February 2016

Academic Editor: Tuncay Candan

First, for a process $\{U(t,\tau) \mid t \geq \tau\}$, we introduce a new concept, called the weak $\mathscr{D}$-pullback exponential attractor, which is a family of sets $\{\mathscr{M}(t) \mid t \leq T\}$, for any $T \in \mathbb{R}$, satisfying the following: (i) $\mathscr{M}(t)$ is compact, (ii) $\mathscr{M}(t)$ is positively invariant, that is, $U(t,\tau)\mathscr{M}(\tau) \subset \mathscr{M}(t)$, and (iii) there exist $k, l > 0$ such that $\operatorname{dist}(U(t,\tau)B(\tau), \mathscr{M}(t)) \leq ke^{-(t-\tau)}$; that is, $\mathscr{M}(t)$ pullback exponential attracts $B(\tau)$. Then we give a method to obtain the existence of weak $\mathscr{D}$-pullback exponential attractors for a process. As an application, we obtain the existence of weak $\mathscr{D}$-pullback exponential attractor for reaction diffusion equation in H_0^1 with exponential growth of the external force.

1. Introduction

Pullback attractor is a suitable concept to describe the long time behavior of infinite dimensional nonautonomous dynamical systems or process generated by nonautonomous partial differential equations. There are many references concerned with the existence of pullback attractors for nonautonomous PDEs (see [1–5]). In [2], Caraballo introduced the notion of $\mathscr{D}$-pullback attractor for nonautonomous dynamical systems and gave a general method to prove the existence of $\mathscr{D}$-pullback attractor. However, pullback attractors or $\mathscr{D}$-pullback attractors attract any bounded set of phase space, but the attraction to it may be arbitrarily slow. In order to describe the attracting speed, the concept of pullback exponential attractor is put forward (see [6]), which is a positively invariant family of compact subsets with finite fractal dimension (see [7, 8]) and exponentially attracts each bounded subset. In [6], a new method is given to prove the existence of pullback exponential attractor and it is applied to reaction diffusion equation when the external force is normal; in [9], the same result is obtained when the nonlinear term $f(t,u)$ satisfies $|f(t,u) - f(t,v)| \leq \xi(t)|u - v|$. In fact, these conditions are relatively strict; for general conditions, we can not get the result.

Motivated by these problems and some ideas in [3–6], we introduce a new attractor, called the weak $\mathscr{D}$-pullback exponential attractors that is for a process $\{U(t,\tau)\}$, for any $T \in \mathbb{R}$, there exists a family of sets $\{\mathscr{M}(t) \mid t \leq T\}$ satisfying the following:

(i) $\mathscr{M}(t)$ is compact.

(ii) $\mathscr{M}(t)$ is positively invariant; that is, $U(t,\tau)\mathscr{M}(\tau) \subset \mathscr{M}(t)$.

(iii) $\forall t \in (-\infty, T]$, there exist $k, l > 0$ such that $\operatorname{dist}(U(t,\tau)B(\tau), \mathscr{M}(t)) \leq ke^{-(t-\tau)}$, that is, $\mathscr{M}(t)$ pullback exponential attracts $B(\tau)$ for all $\{B(t)\} \in \mathscr{D}$.

Compared with the pullback exponential attractor, the fractal dimension of the weak $\mathscr{D}$-pullback exponential attractor is not necessarily uniformly bounded or even unbounded, and the positively invariant only holds for any $t \in (-\infty, T]$, compared with the $\mathscr{D}$-pullback attractor, which pullback attracts bounded set with exponential speed and contains $\mathscr{D}$-pullback attractor.

The paper is organized as follows. In Section 2, we recall some basic concepts about pullback attractor. In Section 3, we construct a weak $\mathscr{D}$-pullback exponential attractor for nonautonomous dynamical systems and we provided a method

to verify the existence of weak $\mathscr{D}$-pullback exponential attractor. In Section 4, we apply our result to prove the existence of weak $\mathscr{D}$-pullback exponential attractor for nonautonomous reaction diffusion system in H_0^1 with exponential growth of the external force.

2. Preliminaries

Let X be a complete metric space; let $B(X)$ be the set of all bounded subsets of X; $\mathscr{D}$ is a nonempty class of parameterised sets $\widehat{\mathscr{D}} = \{D(t) \mid t \in \mathbb{R}, D(t) \in B(X)\}$ or $\widehat{\mathscr{D}} = \{D(n) \mid n \in \mathbb{Z}, D(n) \in B(X)\}$; and a two-parameter family of mappings $\{U(t,\tau) \mid t \geq \tau\} = \{U(t,\tau) \mid t \geq \tau, t, \tau \in \mathbb{R}\}$ act on X, that is, $U(t,\tau) : X \to X, \forall t \geq \tau$.

Definition 1. A two-parameter family of mappings $\{U(t,\tau)\}$ is said to be a process in X, if

(1) $U(t,s)U(s,\tau) = U(t,\tau), \forall t \geq s \geq \tau$,

(2) $U(\tau,\tau) = \mathrm{Id}$ is the identity operator, $\tau \in \mathbb{R}$.

The pair $(U(t,\tau), X)$ is generally referred to as a nonautonomous dynamical system, and $(U(n,m), X)$ $(n, m \in \mathbb{Z})$ is called a nonautonomous discrete dynamical system generated by $(U(t,\tau), X)$. If $x \to U(t,\tau)x$ is continuous in X, we say that the process is continuous process; if $U(t,\tau)x_n \rightharpoonup U(t,\tau)x$ as $x_n \to x$, we say that the process is the norm-to-weak continuous process. Obviously, continuous process is also a norm-to-weak continuous process.

Definition 2. A family of sets $\{B(t) \mid t \in \mathbb{R}\} \in \mathscr{D}$ is called $\mathscr{D}$-pullback bounded absorbing sets for the process $\{U(t,\tau)\}$ if, for any $t \in \mathbb{R}$ and any bounded sets $\{D(t) \mid t \in \mathbb{R}\} \in \mathscr{D}$, there exists $\tau_0(t, D(t)) \leq t$ such that $U(t,\tau)D(\tau) \subset B(t)$ for all $\tau \leq \tau_0$.

Definition 3. The family $\mathscr{A} = \{\mathscr{A}(t) \mid t \in \mathbb{R}\} \subset B(X)$ is said to be a $\mathscr{D}$-pullback attractor for $U(t,\tau)$ if the following hold:

(1) $\mathscr{A}(t)$ is compact for all $t \in \mathbb{R}$;

(2) $\mathscr{A}$ is invariant; that is, $U(t,\tau)\mathscr{A}(\tau) = \mathscr{A}(t)$ $\forall t \geq \tau$;

(3) $\mathscr{A}$ is $\mathscr{D}$-pullback attracting; that is, $\lim_{\tau\to-\infty} \mathrm{dist}\,(U(t,\tau)B(\tau), \mathscr{A}(t)) = 0$, $\forall \{B(t)\} \in \mathscr{D}$, and $t \in \mathbb{R}$;

(4) if $\{C(t)\}_{t\in\mathbb{R}}$ is another family of closed attracting sets, then $\mathscr{A}(t) \subset C(t)$ $\forall t \in \mathbb{R}$.

Here $\mathrm{dist}(\cdot,\cdot)$ denotes the nonsymmetric Hausdorff distance between sets in X; that is, $\mathrm{dist}(A,B) = \sup_{a\in A}\inf_{b\in B}\|a-b\|$.

Definition 4. The Kuratowski measure of noncompactness $\alpha(B)$ of $B \subset X$ is defined by

$$\alpha(B) = \inf\{\delta > 0 \mid B \text{ admits a finite cover by sets of diameter} \leq \delta\}. \tag{1}$$

The following summarizes some of the basic properties of the measure of noncompactness.

Lemma 5 (see [10]). *Let $B, B_1, B_2 \subset X$. Then*

(1) *$\alpha(B) = 0$ if, and only if, $\overline{B}$ is compact;*

(2) *$\alpha(B_1 + B_2) \leq \alpha(B_1) + \alpha(B_2)$;*

(3) *$\alpha(B_1) \leq \alpha(B_2)$ for $B_1 \subset B_2$;*

(4) *$\alpha(B_1 \cup B_2) \leq \max\{\alpha(B_1), \alpha(B_2)\}$;*

(5) *if $F_1 \supset F_2 \ldots$ are nonempty closed sets in X such that $\alpha(F_n) \to 0$ as $n \to \infty$, then $F = \bigcap_{n=1}^{\infty} F_n$ is nonempty and compact.*

In addition, let X be an infinite dimensional Banach space with a decomposition $X = X_1 \oplus X_2$ and let $P : X \to X_1$, $Q : X \to X_2$ be projectors with $\dim X_1 < \infty$. Then

(6) *$\alpha(B(\varepsilon)) = 2\varepsilon$, where $B(\varepsilon)$ is a ball of radius ε;*

(7) *$\alpha(B) < \varepsilon$ for any bounded subset B of X for which the diameter of QB is less than ε.*

Definition 6 (see [3–5]). A process $\{U(t,\tau)\}$ is called $\mathscr{D}$-pullback ω-limit compact for $\{B(t) \mid t \in \mathbb{R}\}$ if, for any $\varepsilon > 0$, there exists a $\tau_0(t, B(t), \varepsilon) \leq t$ such that $\alpha(\bigcup_{\tau\leq\tau_0} U(t,\tau)B(\tau)) \leq \varepsilon$.

Lemma 7 (see [3–5]). *Assume that the process $\{U(t,\tau) \mid t \geq \tau\}$ is $\mathscr{D}$-pullback ω-limit compact for $\widehat{B} = \{B(t) \mid t \in \mathbb{R}\}$; then, for any sequence $\{\tau_n\} \subset (-\infty, t]$, $\tau_n \to -\infty$ as $n \to +\infty$ and for any sequence $x_n \in B(\tau_n)$, there exists a convergence subsequence of $\{U(t,\tau_n)x_n\}$ whose limit lies in $\omega(\widehat{B}, t)$; here $\omega(\widehat{B}, t)$ is defined by*

$$\omega\left(\widehat{B}, t\right) = \bigcap_{s\leq t} \overline{\bigcup_{\tau\leq s} U(t,\tau)B(\tau)}. \tag{2}$$

Theorem 8 (see [3, 5]). *Let $\{U(t,\tau) \mid t \geq \tau\}$ be a continuous or norm-to-weak continuous process and $\{U(t,\tau) \mid t \geq \tau\}$ is $\mathscr{D}$-pullback ω-limit compact; let $\{B(t) \mid t \in \mathbb{R}\} \subset B(X)$ be a family of $\mathscr{D}$-pullback bounded absorbing sets for the process. Then the process $\{U(t,\tau) \mid t \geq \tau\}$ has a $\mathscr{D}$-pullback attractor $\mathscr{A} = \{\mathscr{A}(t) \mid t \in \mathbb{R}\}$, and*

$$\begin{aligned}\mathscr{A}(t) &= \bigcap_{s\leq t} \overline{\bigcup_{\tau\leq s} U(t,\tau)B(\tau)} \\ &= \bigcap_{s\leq t} \overline{\bigcup_{\tau\leq s} \{U(t,\tau)B \mid B \in B(X)\}}.\end{aligned} \tag{3}$$

For a discrete process $\{U(n,m) \mid n, m \in \mathbb{Z}, n \geq m\}$, the above conclusions also hold true.

3. The Existence of Weak $\mathscr{D}$-Pullback Exponential Attractor

Let X be a Banach space; $\|\cdot\|$ denotes the norm of X, $\mathscr{D}$ is a nonempty class of parameterised sets $\widehat{\mathscr{D}} = \{D(t) \mid t \in \mathbb{R}\} \subset B(X)$ or $\widehat{\mathscr{D}} = \{D(n) \mid n \in \mathbb{Z}\} \subset B(X)$, and $\{U(t,\tau)\}$ is a continuous process on X.

Now, we give our main theorems which describe the relationship between the measure of noncompactness and the weak $\mathscr{D}$-pullback exponential attractor.

Theorem 9. *Assume that $\{B(n)\} \in \mathscr{D}$ is positively invariant $\mathscr{D}$-pullback bounded absorbing sets of $\{U(n,m)\}$; that is, for any $\{D(n)\} \in \mathscr{D}$, $N \in \mathbb{Z}$, there exists $T \in \mathbb{N}$, such that $U(n,m)D(m) \subset B(n)$ for any $n - m \geq T$, and $U(n,m)B(m) \subset B(n)$ for any $m \leq n \leq N$; then the following are equivalent:*

(I) *The measure of noncompactness $\mathscr{D}$-pullback decays exponentially for the discrete process $\{U(n,m)\}$; that is, there exist $k, l > 0$ such that*

$$\alpha\left(\bigcup_{k\leq m} U(n,k)\,B(k)\right) \leq ke^{-l(n-m)}, \quad \text{for any } m \leq n \leq N. \tag{4}$$

(II) *The process $\{U(n,m)\}$ has a weak $\mathscr{D}$-pullback exponential attractor; that is, there exists a family of sets $\{\mathscr{M}(n) \mid n \leq N\}$ satisfying the following:*

(1) *$\mathscr{M}(n)$ is compact;*

(2) *$\mathscr{M}(n)$ is positively invariant; that is, $U(n,m)\mathscr{M}(m) \subset \mathscr{M}(n)$;*

(3) *$\{\mathscr{M}(n) \mid n \in \mathbb{Z}\}$ attracts $\{D(n)\}$ exponentially in a $\mathscr{D}$-pullback sense; more precisely,*

$$\operatorname{dist}(U(n,m)\,D(m), \mathscr{M}(n)) \leq ke^{-l(n-m)}, \quad \text{for any } \{D(n)\} \in \mathscr{D}. \tag{5}$$

Proof. ((I) $\Rightarrow$ (II)) Since the measure of noncompactness $\mathscr{D}$-pullback decays exponentially for $\{U(n,m)\}$, from Definition 6, we find that $\{U(n,m)\}$ is $\mathscr{D}$-pullback ω-limit compact. By Theorem 8, we get that

$$\mathscr{A}(n) = \bigcap_{k\leq n} \overline{\bigcup_{m\leq k} U(n,m)\,B(m)} \tag{6}$$

is a $\mathscr{D}$-pullback attractor of $\{U(n,m)\}$. Using (3) of Lemma 5, we find that

$$\begin{aligned}\alpha(U(n,m)\,B(m)) &\leq \alpha\left(\bigcup_{k\leq m} U(n,k)\,B(k)\right)\\ &\leq ke^{-l(n-m)},\end{aligned} \tag{7}$$

and by the definition of the measure of noncompactness, for any $n \geq m$, there exist finite points $x_{n,i}^m \in B(n)$ such that $U(n,m)B(m) \subset \bigcup_{i=1}^{n_m} B(x_{n,i}^m, ke^{-l(n-m)})$. Letting $W_n^m = \{x_{n,i}^m \mid i = 1,2,\ldots,n_m\}$ and $M(k) = \bigcup_{n=0}^{+\infty}\bigcup_{i=0}^{n} U(k,k-i)W_{k-i}^{n-i}$, we get

$$\begin{aligned}M(k+1) &= \bigcup_{n=0}^{+\infty}\bigcup_{i=0}^{n} U(k+1,k+1-i)\,W_{k+1-i}^{n-i}\\ &= \bigcup_{n=0}^{+\infty}\bigcup_{i=0}^{n} U(k+1,k)\,U(k,k+1-i)\,W_{k+1-i}^{n-i}\\ &\supset \bigcup_{n=1}^{+\infty}\bigcup_{i=0}^{n} U(k+1,k)\,U(k,k-(i-1))\,W_{k-(i-1)}^{n-1-(i-1)}\\ &\supset U(k+1,k)\bigcup_{n=1}^{+\infty}\bigcup_{i=1}^{n} U(k,k-(i-1))\,W_{k-(i-1)}^{n-1-(i-1)}\\ &= U(k+1,k)\bigcup_{n=1}^{+\infty}\bigcup_{i=0}^{n-1} U(k,k-i)\,W_{k-i}^{n-1-i}\\ &= U(k+1,k)\bigcup_{n=0}^{+\infty}\bigcup_{i=0}^{n} U(k,k-i)\,W_{k-i}^{n-i}\\ &= U(k+1,k)\,M(k).\end{aligned} \tag{8}$$

Consequently, for all $n \in \mathbb{Z}$, the family $\{M(n) \mid n \leq N\}$ is positively invariant.

Let $\mathscr{M}(n) = M(n) \cup \mathscr{A}(n)$; we claim that $\{\mathscr{M}(n) \mid n \leq N\}$ satisfies (II).

(Compactness) for any sequence $x_k \in \mathscr{M}(n)$, there exist m_k and y_k such that $x_k = U(n,m_k)y_k$. By (I), we get that the process $\{U(n,m)\}$ is pullback $\mathscr{D}$-ω-limit compact; we deduce from Lemma 7 that x_k has subsequence convergent in $\mathscr{M}(n)$. We get that $\mathscr{M}(n)$ is compact.

(Positively invariant) since $U(n+1,n)M(n) \subset M(n+1)$, $U(n+1,n)\mathscr{A}(n) = \mathscr{A}(n+1)$, we get

$$\begin{aligned}U(n+1,n)\,\mathscr{M}(n) &= U(n+1,n)\,(M(n) \cup \mathscr{A}(n))\\ &\subset \mathscr{M}(n+1).\end{aligned} \tag{9}$$

(Exponential attracting) for any $\{D(m)\} \in \mathscr{D}$, there exists $T \in \mathbb{N}$, such that

$$U(m,m-T)\,D(m-T) \subset B(m). \tag{10}$$

Since $\{B(n)\}$ is positively invariant, we get

$$\begin{aligned} &U(n,m-T)D(m-T) \\ &\quad = U(n,m)U(m,m-T)D(m-T) \\ &\quad \subset U(n,m)B(m) \subset \bigcup_{i=1}^{n_m} B\left(x_{n,i}^m, ke^{-l(n-m)}\right), \end{aligned} \tag{11}$$

so we obtain

$$\begin{aligned} &\operatorname{dist}(U(n,m-T)D(m-T), \mathscr{M}(n)) \\ &\quad \le \operatorname{dist}(U(n,m)B(m), \mathscr{M}(n)). \end{aligned} \tag{12}$$

Since $W_n^m \subset M(n)$, we get

$$\begin{aligned} &\operatorname{dist}(U(n,m-T)D(m-T), \mathscr{M}(n)) \\ &\quad \le \operatorname{dist}\left(U(n,m)B(m), W_n^m\right) \le ke^{-l(n-m)} \end{aligned} \tag{13}$$

for any $n \ge m$.

((II) $\Rightarrow$ (I)) By the definition of $\operatorname{dist}(\cdot,\cdot)$, we get

$$\begin{aligned} &\operatorname{dist}(U(n,m)B(m), \mathscr{M}(n)) \\ &\quad = \sup_{x\in B(m)} \inf_{y\in\mathscr{M}(n)} d\left(U(n,m)x, y\right) \le ke^{-l(n-m)}, \end{aligned} \tag{14}$$

and, for any $x \in B(m)$, we have

$$\inf_{y\in\mathscr{M}(n)} d\left(U(n,m)x, y\right) \le ke^{-l(n-m)}. \tag{15}$$

Therefore, for any $x \in B(m)$, there exists $y_x \in \mathscr{M}(n)$, such that

$$d\left(U(n,m)x, y_x\right) < 2ke^{-l(n-m)}. \tag{16}$$

We get

$$U(n,m)x \in B\left(y_x, 2ke^{-l(n-m)}\right). \tag{17}$$

Since $\mathscr{M}(n)$ is a compact set, we get that there exist $y_1, y_2, \ldots, y_l \in \mathscr{M}(n)$ such that

$$\mathscr{M}(n) \subset \bigcup_{i=1}^{l} B\left(y_i, 2ke^{-l(n-m)}\right). \tag{18}$$

Therefore, for any y_x, there exists $y_{i_x} \in \{y_1, y_2, \ldots, y_l\}$ such that

$$\begin{aligned} &d\left(y_x, y_{i_x}\right) \le 2ke^{-l(n-m)}, \\ &d\left(U(n,m)x, y_{i_x}\right) \le d\left(U(n,m)x, y_x\right) + d\left(y_x, y_{i_x}\right) \\ &\quad \le 4ke^{-l(n-m)}. \end{aligned} \tag{19}$$

We get

$$U(n,m)B(m) \subset \bigcup_{i=1}^{l} B\left(y_i, 4ke^{-l(n-m)}\right), \tag{20}$$

and, by Definition 4, we obtain

$$\alpha(U(n,m)B(m)) \le Ke^{-l(n-m)} \quad (K = 4k), \tag{21}$$

and, by (4) of Lemma 5, we get

$$\alpha\left(\bigcup_{k\le m} U(n,m)B(m)\right) \le Ke^{-l(n-m)}, \tag{22}$$

which say that the measure of noncompactness $\mathscr{D}$-pullback decays exponentially. □

Theorem 10. *Assume that $\{B(t)\} \in \mathscr{D}$ is positively invariant $\mathscr{D}$-pullback bounded absorbing sets of $\{U(t,\tau)\}$; that is, for any $\{D(t)\} \in \mathscr{D}$, $R \in \mathbb{R}$, there exists $T \ge 0$, such that $U(t,\tau)D(\tau) \subset B(t)$ for any $t-\tau \ge T$, and $U(t,\tau)B(\tau) \subset B(t)$ for any $t \le R$, and there exists a continuous function $r(t)$ that satisfies $\|U(t,\tau)x - U(t,\tau)y\| \le r(t)\|x-y\|$ for any $x, y \in B(\tau)$, $t-\tau \le 1$; then the following are equivalent:*

(I) *The measure of noncompactness $\mathscr{D}$-pullback decays exponentially for the process $\{U(t,\tau)\}$; that is, there exist $k, l > 0$ such that*

$$\alpha\left(\bigcup_{s\le\tau} U(t,s)B(s)\right) \le ke^{-l(t-\tau)}, \quad \text{for any } \tau \le t \le R. \tag{23}$$

(II) *The process $\{U(t,\tau)\}$ has a weak $\mathscr{D}$-pullback attractor; that is, there exists a family of sets $\{\mathscr{M}(t) \mid t \le R\}$ satisfying the following:*

(1) *$\mathscr{M}(t)$ is compact;*

(2) *$\mathscr{M}(t)$ is positively invariant; that is, $U(t,\tau)\mathscr{M}(\tau) \subset \mathscr{A}(t)$;*

(3) *$\{\mathscr{M}(t) \mid t \le R\}$ attracts $\{D(t)\}$ exponentially in $\mathscr{D}$-pullback sense; more precisely,*

$$\operatorname{dist}(U(t,\tau)D(\tau), \mathscr{M}(t)) \le k(t)e^{-l(t-\tau)}, \quad \text{for any } \{D(t)\} \in \mathscr{D}. \tag{24}$$

Proof. ((I) $\Rightarrow$ (II)) By Theorem 9, we know that the discrete process $\{U(n,m)\}$ generated by $\{U(t,\tau)\}$ has a weak $\mathscr{D}$-pullback exponential attractor $\{\mathscr{M}(n)\}$, that is, $\mathscr{M}(n)$ is compact and positively invariant and $\mathscr{D}$-pullback exponentially attracts $\{D(n)\} \in \mathscr{D}$. We set $\mathscr{M}(t) = U(t,k)\mathscr{M}(k)$, $t \in [k, k+1)$, for all $k \le R$. As proof of Theorem 9, it is easy to prove that $\mathscr{M}(t)$ is compact and positively invariant. Next, we will prove that $\mathscr{M}(t)$ attracts $\{D(n)\} \in \mathscr{D}$ exponentially in $\mathscr{D}$-pullback sense.

For any $\{D(t)\} \in \mathscr{D}$, there exists $T \in \mathbb{N}$ such that $U(t,\tau)D(\tau) \subset B(t)$ for any $t-\tau \ge T$. For discrete process $\{U(n,m)\}$, by Theorem 9, there exist $k_0, l_0 > 0$ such that

$$\operatorname{dist}(U(n,m)B(m), \mathscr{M}(n)) \le k_0 e^{-l_0(n-m)}. \tag{25}$$

For any $t, \tau \in \mathbb{R}$, there exist $t_0, \tau_0 \in [0, 1)$ such that $t = n + t_0$, $\tau = m + \tau_0$; therefore

$$\begin{aligned}
\operatorname{dist}(U(t,\tau)D(\tau), \mathscr{M}(t)) &= \operatorname{dist}(U(n+t_0, m+\tau_0)D(m+\tau_0), \mathscr{M}(n+t_0)) \\
&\le \operatorname{dist}(U(n+t_0, n)U(n, m+\tau_0)D(m+\tau_0), U(n+t_0)\mathscr{M}(n)) \\
&\le r(t)\operatorname{dist}(U(n, m+\tau_0)D(m+\tau_0), \mathscr{M}(n)) \\
&\le r(t)\operatorname{dist}(U(n, m+T+1)U(m+T+1, m+\tau_0)D(m+\tau_0), \mathscr{M}(n)) \\
&\le r(t)\operatorname{dist}(U(n, m+T+1)B(m+T+1), \mathscr{M}(n)) \le r(t)k_0 e^{-l_0(n-m-T-1)},
\end{aligned} \tag{26}$$

$$\operatorname{dist}(U(t,\tau)D(\tau), \mathscr{M}(t)) \le r(t)k_0 e^{-l_0(t-t_0-\tau+\tau_0-T-1)} \le r(t)k_0 e^{-l_0(-t_0+\tau_0-T-1)} e^{-l_0(t-\tau)} \le r(t)k_0 e^{l_0(T+2)} e^{-l_0(t-\tau)}.$$

We obtain that $\{\mathscr{M}(t) \mid t \le R\}$ attracts $\{D(t)\}$ exponentially in a $\mathscr{D}$-pullback sense.

((II) $\Rightarrow$ (I)) The proof is the same as that of Theorem 9, so we omit it. □

We now present a method to verify that the measure of noncompactness $\mathscr{D}$-pullback decays exponentially for the process $\{U(t,\tau)\}$.

Let X be a uniformly convex Banach space; that is, for all $\varepsilon > 0$, there exists $\delta > 0$ such that, given $x, y \in X$, $\|x\| \le 1$, $\|y\| \le 1$, $\|x - y\| > \varepsilon$; then $\|x + y\|/2 < 1 - \delta$. Requiring a space to be uniformly convex is not a severe restriction in application, since this property is satisfied by all Hilbert spaces, the L^p space with $1 < p < \infty$, and most Sobolev spaces $W^{k,p}$ with $1 < p < \infty$.

Definition 11 (enhanced flattening property). Let X be a uniformly convex Banach space; for a family of bounded sets $\{B(t)\} \subset X$, there exist $k, l, T > 0$, and for any finite dimension subspace X_1 of X, such that

(i) $P_m(\bigcup_{t-\tau\ge s} U(t,\tau)B(\tau))$ is bounded;

(ii) $\|(I - P_m)\bigcup_{t-\tau\ge s} U(t,\tau)x\| \le ke^{-l(t-\tau)} + k(t,m)$, $\forall x \in B(\tau)$,

for all $s \ge T$. Here $\|\cdot\|$ denote the norm in X and $k(t,s)$ is real-valued function satisfying

$$\lim_{s\to+\infty} k(t,s) = 0. \tag{27}$$

Theorem 12. *Assume that the process $\{U(t,\tau)\}$ satisfies the enhanced flattening property; then the measure of noncompactness $\mathscr{D}$-pullback decays exponentially for $\{U(t,\tau)\}$.*

Proof. For any $\{B(t)\} \in \mathscr{D}$, from (2) and (7) of Lemma 5, and the enhanced flattening property, we get

$$\begin{aligned}
&\alpha\left(\bigcup_{t-\tau\ge T} U(t,\tau)B(\tau)\right) \\
&\quad\le \alpha\left(P_m\left(\bigcup_{t-\tau\ge T} U(t,\tau)B(\tau)\right)\right) \\
&\qquad + \alpha\left((I-P_m)\left(\bigcup_{t-\tau\ge T} U(t,\tau)B(\tau)\right)\right) \\
&\quad= \alpha\left((I-P_m)\left(\bigcup_{t-\tau\ge T} U(t,\tau)B(\tau)\right)\right) \\
&\quad\le ke^{-l(t-\tau)} + k(t,m).
\end{aligned} \tag{28}$$

Since $k(t,m) \to 0$, for $\varepsilon_0 = ke^{-l(t-\tau)}$, there exists $M > 0$, for any $m > M$; we have

$$|k(t,m)| < ke^{-l(t-\tau)}. \tag{29}$$

Hence, $\alpha(\bigcup_{t-\tau\ge T} U(t,\tau)B(t)) \le 2ke^{-l(t-\tau)}$; that is, the measure of noncompactness of $\{U(t,\tau)\}\mathscr{D}$-pullback decays exponentially.

Let $\mathscr{R}$ be the set of all functions $r(t) : R \to (0, +\infty)$ such that $\lim_{t\to-\infty} t^\beta e^{\lambda t} r^2(t) = 0$ for some $\beta \ge 0$, $\lambda > 0$, and denote by $\mathscr{D}$ the class of all families $\mathscr{D} = \{D(t) \mid t \in \mathbb{R}\} \subset B(X)$ such that $D(t) \subset \overline{B}(r(t))$ for some $r(t) \in \mathscr{R}$, $\overline{B}(r(t))$ denote the closed ball in X with radius $r(t)$. □

Theorem 13. *Assume that the process $\{U(t,\tau)\}$ satisfies*

$$\begin{aligned}
\|U(t,\tau)u_\tau\|^2 &\le K_0(t-\tau)^\beta e^{-\lambda(t-\tau)}\|u_\tau\|^2 + K_1 \\
&\quad + K_2 e^{\alpha|t|}
\end{aligned} \tag{30}$$

for some $\beta \geq 0$, $0 < \alpha < \lambda$, and $t - \tau \geq T'$ and for any $t \leq R$; then the process $\{U(t,\tau)\}$ has a family of positively invariant $\mathscr{D}$-pullback bounded absorbing sets $\{B(t) \mid t \leq R\}$; that is, for any $D \in \mathscr{D}$, there exists $T > 0$ such that $U(t,\tau)D(\tau) \subset B(t)$ for any $t - \tau \geq T$ and $U(t,\tau)B(\tau) \subset B(t)$.

Proof. Let us define

$$D(t) = \left\{x \in X \mid \|x\| \leq 2K_1 + K_2 e^{\alpha|t|}\right\}, \quad t \leq R. \tag{31}$$

For every $\{D_0(t)\} \in \mathscr{D}$, there exists $T_0 > 0$ such that

$$U(t, t-s) D_0(t-s) \subset D(t), \quad s \geq T_0, \ t \leq R. \tag{32}$$

Obviously, $\{D(t)\}$ is a family of $\mathscr{D}$-pullback bounded absorbing sets. Moreover, there exists $T > 0$ such that

$$U(t, t-s) D(t-s) \subset D(t), \quad s \geq T, \ t \leq R. \tag{33}$$

Note that these can not hold for any $t \in \mathbb{R}$. Let

$$B(t) = \overline{\bigcup_{s \geq T} U(t, t-s) D(t-s)}, \quad t \leq R. \tag{34}$$

We know that $B(t) \subset D(t)$ and $\{B(t)\}$ is also a family of $\mathscr{D}$-pullback bounded absorbing sets. We also have

$$\begin{aligned} &U(t,\tau) B(\tau) \\ &\quad = \overline{\bigcup_{s \geq T} U(t, t-(t-\tau+s)) D(t-(t-\tau+s))} \\ &\quad \subset \overline{\bigcup_{s \geq T} U(t, t-s) D(t-s)} = B(t). \end{aligned} \tag{35}$$

□

By Theorems 10–13, we get the following theorem.

Theorem 14. *Let X be a uniformly convex Banach space; $\{U(t,\tau)\}$ is a process on X, and the process $\{U(t,\tau)\}$ satisfies the following:*

(I) *$\|U(t,\tau)u_\tau\|^2 \leq K_0(t-\tau)^\beta e^{-\lambda(t-\tau)}\|u_\tau\|^2 + K_1 + K_2 e^{\alpha|t|}$ for some $\beta \geq 0$, $0 < \alpha < \lambda$, and $t - \tau \geq T$ and any $t \leq R$.*

(II) *$\|(I - P_m)(\bigcup_{t-\tau \geq s} U(t,\tau)x)\| \leq k e^{-l(t-\tau)} + k(t,m)$, $\forall x \in B(\tau) = \{x : \|x\| \leq 2K_1 + K_2 e^{\alpha|\tau|}\}$, for all $s \geq T$. Here m is the dimension of subspace X_1 of X, and $k(t,s)$ is real-valued function that satisfies*

$$\lim_{s \to +\infty} k(t,s) = 0. \tag{36}$$

(III) *$\|U(t,\tau)x - U(t,\tau)y\| \leq r(t)\|x - y\|$, for any $t - \tau < 1$, $x, y \in B(\tau)$; then the process $\{U(t,\tau)\}$ has a weak $\mathscr{D}$-pullback exponential attractor; that is, for any $R \in \mathbb{R}$, there exists a family of sets $\{\mathscr{M}(t) \mid t \leq R\}$ satisfying the following:*

(1) *$\mathscr{M}(t)$ is compact.*

(2) *$\mathscr{M}(t)$ is positively invariant; that is, $U(t,\tau)\mathscr{M}(\tau) \subset M(t)$.*

(3) *$\mathscr{M}(t)$ attracts $\{D(t)\}$ exponentially in a $\mathscr{D}$-pullback sense; more precisely,*

$$\begin{aligned} &\operatorname{dist}(U(t,\tau) D(\tau), \mathscr{M}(t)) \leq \beta(t) e^{-l(t-\tau)}, \\ &\qquad \text{for any } \{D(t)\} \in \mathscr{D}. \end{aligned} \tag{37}$$

4. Application to Nonautonomous Reaction Diffusion Equation

As an application of Theorem 14, we prove the existence of the weak $\mathscr{D}$-pullback exponential attractor in $H_0^1(\Omega)$ for the process generated by the solution of the following nonautonomous reaction diffusion equation:

$$\begin{aligned} u_t - \triangle u + f(u) &= g(t), \quad x \in \Omega, \\ u|_{\partial\Omega} &= 0, \\ u(\tau) &= u_\tau, \end{aligned} \tag{38}$$

where $f \in C^1(\mathbb{R}, \mathbb{R})$, $g(\cdot) \in L^2_{\mathrm{loc}}(\mathbb{R}, L^2(\Omega))$, Ω is a bounded open subset of $\mathbb{R}^n$, and there exist $p \geq 2$, $c_i > 0$, $i = 1, \ldots, 5$, $l > 0$ such that

$$c_1 |u|^p - c_2 \leq f(u) u \leq c_3 |u|^p + c_4, \tag{39}$$

$$\begin{aligned} &f'(u) \geq -l, \\ &\left|f'(u)\right| \leq c_5 \left(1 + |u|^{p-2}\right) \end{aligned} \tag{40}$$

for all $u \in \mathbb{R}$.

We set $A := -\triangle$, naming λ the first eigenvalue of A, and denote $H = L^2(\Omega)$ by scalar product $(\cdot,\cdot)$ and norm $|\cdot|$; let $((\cdot,\cdot))$ and $\|\cdot\|$ denote the scalar product and norm of $H_0^1(\Omega)$ and $((u,v)) = \int_\Omega \nabla u \nabla v \, dx$ for all $u, v \in H_0^1(\Omega)$. Moreover, we suppose for any $t \in \mathbb{R}$ that there exist $M \geq 0$ and $0 \leq \alpha < \lambda$ such that

$$\left|g(t)\right|^2 \leq M e^{\alpha|t|}. \tag{41}$$

For this initial boundary value problem, we know from [7, 8] that, for any $\tau, T \in \mathbb{R}$, $T > \tau$, there exists a unique solution $u(\cdot) \in C([\tau, T]; H) \cap L^2(\tau, T; H_0^1(\Omega)) \cap L^p(\tau, T; L^p(\Omega))$.

Thanks to the existence theorem, the initial boundary value problem is equivalent to a process $\{U(t,\tau)\}_{t \geq \tau}$ defined by

$$\begin{aligned} U(t,\tau) : H \times [\tau, +\infty) &\longrightarrow H_0^1 \\ u(t) &= U(t,\tau) u_\tau, \end{aligned} \tag{42}$$

where $u(t)$ is the solution of (38)–(40) with u_τ as initial data at time τ.

Theorem 15 (see [3]). *Assume that f and g satisfy (39)–(41) and $u(t)$ is a weak solution associated with (38). Then the following inequality holds for $t > \tau$:*

$$\begin{aligned}\|u(t)\|^2 \le c\Bigg(&\left(1+(t-\tau)+\frac{1}{t-\tau}\right)e^{-\lambda(t-\tau)}\left|u_\tau\right|^2\\&+\left(1+\frac{1}{t-\tau}\right)\\&+\left(1+\frac{1}{t-\tau}\right)e^{-\lambda t}\int_{-\infty}^{t}e^{\lambda s}\left|g(s)\right|^2 ds\\&+\left(1+\frac{1}{t-\tau}\right)e^{-\lambda t}\int_{-\infty}^{t}\int_{-\infty}^{s}e^{\lambda r}\left|g(r)\right|^2 dr\,ds\Bigg).\end{aligned} \tag{43}$$

Theorem 16. *Assume that f and g satisfy (39)–(41), where $2 \le p < +\infty (n \le 2)$, $2 \le p \le (2n-2)/(n-2)$ $(n \ge 3)$. Then the process defined by (42) has a weak $\mathscr{D}$-pullback exponential attractor in H_0^1.*

Next, we will prove that the process defined by (42) satisfy (I)–(III) of Theorem 14.

Proof. By (42), for $t \le 0$,

$$\begin{aligned}e^{-\lambda t}\int_{-\infty}^{t}e^{\lambda s}\left|g(s)\right|^2 ds &\le Me^{-\lambda t}\int_{-\infty}^{t}e^{\lambda s}e^{-\alpha s}ds\\&\le \frac{Me^{-\alpha t}}{\lambda-\alpha},\end{aligned} \tag{44}$$

and, for $t > 0$,

$$\begin{aligned}&e^{-\lambda t}\int_{-\infty}^{t}e^{\lambda s}\left|g(s)\right|^2 ds\\&\quad= e^{-\lambda t}\left(\int_{-\infty}^{0}e^{\lambda s}\left|g(s)\right|^2 ds+\int_{0}^{t}e^{\lambda s}\left|g(s)\right|^2 ds\right)\\&\quad\le M\left(\frac{1}{\lambda-\alpha}+\frac{e^{\alpha t}}{\lambda+\alpha}\right).\end{aligned} \tag{45}$$

Therefore, for any $t \in \mathbb{R}$, we have

$$e^{-\lambda t}\int_{-\infty}^{t}e^{\lambda s}\left|g(s)\right|^2 ds \le \frac{M}{\lambda-\alpha}\left(1+e^{\alpha|t|}\right). \tag{46}$$

Using the same proof, we can get

$$e^{-\lambda t}\int_{-\infty}^{t}\int_{-\infty}^{s}e^{\lambda r}\left|g(r)\right|^2 dr\,ds \le \frac{M}{(\lambda-\alpha)^2}\left(1+e^{\alpha|t|}\right). \tag{47}$$

By (43) and using (46) and (47) we find that there exists $T > 0$, for any $t-\tau \ge T$; we have

$$\begin{aligned}\left\|U(t,\tau)u_\tau\right\|^2 \le K_0(t-\tau)e^{-\lambda(t-\tau)}\left|u_\tau\right|^2 + K_1\\+K_2e^{\alpha|t|}.\end{aligned} \tag{48}$$

By Theorem 13, for any fixed $R \in \mathbb{R}$, the process $\{U(t,\tau)\}$ generated by (38) is a family of positively invariant $\mathscr{D}$-pullback bounded absorbing sets $\{B(t) \mid t \le R\}$ and for any $x \in B(t)$, $\|x\| \le 2K_1 + K_2e^{\alpha|t|}$.

Let $\mathscr{R}$ be the set of all functions $r : R \to (0,+\infty)$ such that $\lim_{t\to-\infty}te^{\lambda t}r^2(t) = 0$ and denote by $\mathscr{D}$ the class of all families $\mathscr{D} = \{D(t) \mid t \in R\} \subset B(H)$ such that $D(t) \subset \overline{B}(r(t))$ for some $r(t) \in \mathscr{R}$, $\overline{B}(r(t))$ denotes the closed ball in H with radius $r(t)$.

Since A^{-1} is a continuous compact operator in H, by the classical spectral theorem, there exist a sequence $\{\lambda_j\}_{j=1}^{\infty}$,

$$\begin{aligned}&0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \cdots,\\&\qquad\qquad \lambda_j \longrightarrow +\infty, \text{ as } j \longrightarrow \infty,\end{aligned} \tag{49}$$

and a family of elements $\{e_j\}_{j=1}^{\infty}$ of $H_0^1(\Omega)$ which are orthogonal in H such that

$$Ae_j = \lambda_j e_j, \quad \forall j \in N. \tag{50}$$

Let $H_m = \operatorname{span}\{e_1, e_2, \ldots, e_m\}$ in H and $P : H \to H_m$ is an orthogonal projector. For any $u \in H$ we write

$$u = Pu + (I-P)u \triangleq u_1 + u_2. \tag{51}$$

We set $u_1(t) = U(t,\tau)u_{1\tau}$ and $u_2(t) = U(t,\tau)u_{2\tau}$ to be solutions associated with (38) with initial data $u_{1\tau}, u_{2\tau} \in B(\tau)$. Let $w(t) = u_1(t) - u_2(t)$; by (38) we get

$$w_t - \triangle w + f(u_1(t)) - f(u_2(t)) = 0. \tag{52}$$

Taking inner product of (52) with $-\triangle w$ in H, we have

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 + |\triangle w|^2 + (f(u_1) - f(u_2), -\triangle w) = 0. \tag{53}$$

Taking into account (40) and Hölder inequality, it is immediate to see that

$$\begin{aligned}&\left|(f(u_1)-f(u_2),-\triangle w)\right|\\&\quad\le \int_\Omega\left|f(u_1)-f(u_2)\right||\triangle w|\,dx \le \frac{1}{2}|\triangle w|^2+\frac{1}{2}\\&\quad\cdot\int_\Omega\left|f(u_1)-f(u_2)\right|^2dx,\\&\int_\Omega\left|f(u_1)-f(u_2)\right|^2dx\\&\quad=\int_\Omega\left|f'(u_1+\theta(u_2-u_1))\right|^2\left|u_1-u_2\right|^2dx\\&\quad\le c\int_\Omega\left(1+\left|u_1\right|^{p-2}+\left|u_2\right|^{p-2}\right)^2\left|u_1-u_2\right|^2dx\\&\quad\le c\left(\int_\Omega\left(1+\left|u_1\right|^{2(p-1)}+\left|u_2\right|^{2(p-1)}dx\right)^{(p-2)/(p-1)}\right)\\&\quad\cdot\left(\int_\Omega\left|u_1-u_2\right|^{2(p-1)}\right)^{1/(p-1)}\\&\quad\le c\left(1+\left|u_1\right|_{2(p-1)}^{2(p-2)}+\left|u_2\right|_{2(p-1)}^{2(p-2)}\right)|w|_{2(p-1)}^2.\end{aligned} \tag{54}$$

Since $2 \le p < \infty (n \le 2)$, $2 \le p \le n/(n-2)+1 (n \ge 3)$, using Sobolev embedding theorem, we obtain

$$\begin{aligned}\int_\Omega |f(u_1) - f(u_2)|^2 dx \\ \le c\left(1 + \|u_1\|^{2(p-2)} + \|u_2\|^{2(p-2)}\right)\|w\|^2 \le c\|w\|^2.\end{aligned} \tag{55}$$

Since $u_1(t), u_2(t) \in B(t)$, we get

$$\|u_i(t)\|^2 \le 2K_1 + K_2 e^{\alpha|t|}, \quad i = 1, 2. \tag{56}$$

From (53)–(56), we have

$$\frac{d}{dt}\|w\|^2 \le c\left(1 + e^{\alpha|t|}\right)\|w\|^2. \tag{57}$$

Therefore

$$\|w(t)\|^2 \le e^{c(1+\int_{t-1}^{t} e^{\alpha|s|}ds)}\|w(\tau)\|^2, \quad t - \tau < 1. \tag{58}$$

We obtain

$$\begin{aligned}\|U(t,\tau)u_{1\tau} - U(t,\tau)u_{2\tau}\|^2 \\ \le e^{c(1+\int_{t-1}^{t} e^{\alpha|s|}ds)}\|u_{1\tau} - u_{2\tau}\|^2.\end{aligned} \tag{59}$$

For any $u \in H$ we write

$$u = P_m u + (I - P_m)u \triangleq u_1 + u_2. \tag{60}$$

Taking the inner product of (38) with $-\triangle u_2$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_2\|^2 + |\triangle u_2|^2 + (f(u), -\triangle u_2) \\ = (g(t), -\triangle u_2).\end{aligned} \tag{61}$$

Applying the Poincaré inequality and Hölder inequality, we get

$$\frac{d}{dt}\|u_2\|^2 + \lambda_m\|u_2\|^2 \le 2\left(\int_\Omega |f(u)|^2 dx + |g(t)|^2\right). \tag{62}$$

By (40), we find

$$|f(u)| \le c\left(|u|^{p-1} + 1\right). \tag{63}$$

Hence,

$$\begin{aligned}\frac{d}{dt}\|u_2\|^2 + \lambda_m\|u_2\|^2 \\ \le c\left(1 + \int_\Omega |u(s)|^{2p-2}dx + |g(t)|^2\right).\end{aligned} \tag{64}$$

Thanks to Sobolev embedding theorem, we obtain

$$\frac{d}{dt}\|u_2\|^2 + \lambda_m\|u_2\|^2 \le c\left(1 + \|u\|^{2p-2} + |g(t)|^2\right). \tag{65}$$

Since $u(t) \in B(t)$, hence

$$\|u\|^{2p-2} \le c\left(1 + e^{\alpha(p-1)|t|}\right). \tag{66}$$

By (41), we get

$$\frac{d}{dt}\|u_2\|^2 + \lambda_m\|u_2\|^2 \le c\left(1 + e^{\alpha(p-1)|t|} + e^{\alpha|t|}\right). \tag{67}$$

Using Gronwall lemma, we have

$$\begin{aligned}\|u_2(t)\|^2 &\le e^{-\lambda_m(t-\tau)}\|u_{2\tau}\|^2 \\ &+ ce^{-\lambda_m t}\int_{-\infty}^{t} e^{\lambda_m s}\left(1 + e^{\alpha(p-1)|s|} + e^{\alpha|s|}\right)ds \\ &\le c\left(e^{-(\lambda_m-\lambda)(t-\tau)}e^{-\lambda(t-\tau)}\|u_\tau\|^2 + \frac{1}{\lambda_m}\right. \\ &+ \frac{1}{\lambda_m - \alpha(p-1)} + \frac{1}{\lambda_m - \alpha} + \frac{e^{\alpha(p-1)|t|}}{\lambda_m - \alpha(p-1)} \\ &\left.+ \frac{e^{\alpha|t|}}{\lambda_m - \alpha}\right).\end{aligned} \tag{68}$$

Since $e^{-\lambda(t-\tau)}\|u_\tau\|^2 \to 0$ for any $u_\tau \in B(\tau)$ and $\lambda_m \to +\infty$, which imply that there exists $T > 0$, for any $t - \tau \ge T$, we have

$$\|u_2(t)\|^2 \le c\left(e^{-(\lambda_m-\lambda)(t-\tau)} + k(t,m)\right). \tag{69}$$

Here $k(t,m) = 1/\lambda_m + 1/(\lambda_m - \alpha(p-1)) + 1/(\lambda_m - \alpha) + e^{\alpha(p-1)|t|}/(\lambda_m - \alpha(p-1)) + e^{\alpha|t|}/(\lambda_m - \alpha)$; obviously $k(t,m) \to 0$ as $m \to +\infty$.

Therefore, by (48), (59), and (69), the process $\{U(t,\tau)\}_{t\ge\tau}$ generated by (38) satisfies all the conditions of Theorem 14. □

Competing Interests

The authors declare that they have no competing interests.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (Grant no. 11261027), Longyuan Youth Innovative Talents Support Programs of 2014, and the Innovation Funds of Principal (LZCU-XZ2014-05).

References

[1] D. N. Cheban, *Global Attractors of Non-Autonomous Dissipative Dynamical Systems*, World Scientific Publishing, 2004.

[2] T. Caraballo, G. Łukaszewicz, and J. Real, "Pullback attractors for asymptotically compact non-autonomous dynamical systems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 3, pp. 484–498, 2006.

[3] Y. Li and C. Zhong, "Pullback attractors for the norm-to-weak continuous process and application to the nonautonomous reaction-diffusion equations," *Applied Mathematics and Computation*, vol. 190, no. 2, pp. 1020–1029, 2007.

[4] H. Song, "Pullback attractors of non-autonomous reaction-diffusion equations in H_0^1," *Journal of Differential Equations*, vol. 249, no. 10, pp. 2357–2376, 2010.

[5] Y. Wang, C. Zhong, and S. Zhou, "Pullback attractors of nonautonomous dynamical systems," *Discrete and Continuous Dynamical Systems. Series A*, vol. 16, no. 3, pp. 587–614, 2006.

[6] Y. Li, S. Wang, and T. Zhao, "The existence of pullback exponential attractors for nonautonomous dynamical system and applications to nonautonomous reaction diffusion equations," *The Journal of Applied Analysis and Computation*, vol. 5, no. 3, pp. 388–405, 2015.

[7] I. Chueshov, *Introduction to the Theory of Infinite-Dimensional Dissipative System*, ACTA Scientific Publishing House, 1999.

[8] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, vol. 68, Springer, New York, NY, USA, 1997.

[9] R. Czaja, "Pullback exponential attractors with admissible exponential growth in the past," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 104, pp. 90–108, 2014.

[10] K. Deimling, *Nonlinear Function Analysis*, Springer, Berlin, Germany, 1985.