Research Article

Fuzzy C_e - I (ec, eo) and Fuzzy Completely C_e - I (rc, eo) Functions via Fuzzy e-Open Sets

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We introduced the notions of fuzzy C_e - I (ec, eo) functions and fuzzy completely C_e - I (rc, eo) functions via fuzzy e-open sets. Some properties and several characterization of these types of functions are investigated.

1. Introduction

With the introduction of fuzzy sets by Zadeh [1] and fuzzy topology by Chang [2], the theory of fuzzy topological spaces was subsequently developed by several fuzzy topologist based on the concepts of general topology. In 2014, the concept of fuzzy e-open sets and fuzzy e-continuity and separations axioms and their properties were defined by Seenivasan and Kamala [3]. In this paper, we introduce the notion of fuzzy C_e - I (ec, eo) functions, fuzzy C_e -continuous, fuzzy completely C_e - I (rc, eo) functions, and fuzzy e-kernel via fuzzy e-open sets and studied their properties and several characterizations of these types of functions are investigated. In this paper, we denote fuzzy e-open, fuzzy e-closed, and fuzzy regular closed as, eo, ec, and rc, respectively.

2. Preliminaries

Throughout this paper, (X, τ) and (Y, σ) (or simply X and Y) represent nonempty fuzzy topological spaces on which no separation axioms are assumed, unless otherwise mentioned.

Let μ be any fuzzy set of X . The fuzzy closure of μ , fuzzy interior of μ , fuzzy δ -closure of μ , and the fuzzy δ -interior of μ are denoted by $\text{cl}(\mu)$, $\text{int}(\mu)$, $\text{cl}_\delta(\mu)$, and $\text{int}_\delta(\mu)$, respectively. A fuzzy set μ of X is called fuzzy regular open [4] (resp., fuzzy regular closed) if $\mu = \text{int}(\text{cl}(\mu))$ (resp., $\mu = \text{cl}(\text{int}(\mu))$).

The fuzzy δ -interior of fuzzy set μ of X is the union of all fuzzy regular open sets contained in μ . A fuzzy set μ is called fuzzy δ -open [5] if $\mu = \text{int}_\delta(\mu)$. The complement of fuzzy δ -open set is called fuzzy δ -closed (i.e., $\mu = \text{cl}_\delta(\mu)$). A

fuzzy set μ of X is called fuzzy δ -preopen [6] (resp., fuzzy δ -semi open [7]) if $\mu \leq \text{int}(\text{cl}_\delta(\mu))$ (resp., $\mu \leq \text{cl}(\text{int}_\delta(\mu))$). The complement of a fuzzy δ -preopen set (resp., fuzzy δ -semiopen set) is called fuzzy δ -preclosed (resp., fuzzy δ -semiclosed).

Definition 1. A fuzzy set μ of a fuzzy topological space X is called fuzzy e-open [3] if $\mu \leq \text{cl}(\text{int}_\delta \mu) \vee \text{int}(\text{cl}_\delta \mu)$. Fuzzy e-closed if $\mu \geq \text{cl}(\text{int}_\delta \mu) \wedge \text{int}(\text{cl}_\delta \mu)$.

The intersection of all fuzzy e-closed sets containing μ is called fuzzy e-closure of μ and is denoted by $\text{fe-cl}(\mu)$ and the union of all fuzzy e-open sets contained in μ is called fuzzy e-interior of μ and is denoted by $\text{fe-int}(\mu)$.

Definition 2. A mapping $f : X \rightarrow Y$ is said to be fuzzy e^* -open [8] if the image of every fuzzy e-open set in X is fuzzy e-open set in Y .

Definition 3. A function $f : X \rightarrow Y$ is called fuzzy e-irresolute [3]. $f^{-1}(\lambda)$ is fuzzy e-open in X for every fuzzy e-open set λ of Y .

Definition 4. A fuzzy set μ is quasicoincident [9] with a fuzzy set λ denoted by $\mu q \lambda$ iff there exist $x \in X$ such that $\mu(x) + \lambda(x) > 1$. If μ and λ are not quasicoincident, then we write $\mu \bar{q} \lambda$ and $\mu \leq \lambda \Leftrightarrow \mu \bar{q} 1 - \lambda$.

Definition 5. A fuzzy point x_p is quasicoincident [9] with a fuzzy set λ denoted by $x_p q \lambda$ iff there exist $x \in X$ such that $p + \lambda(x) > 1$.

Definition 6. A fuzzy topological space (X, τ) is said to be fuzzy $e-T_1$ [3] if for each pair of distinct points x and y of X there exist fuzzy e -open sets μ_1 and μ_2 such that $x \in \mu_1$ and $y \in \mu_2$ and $x \notin \mu_2$ and $y \notin \mu_1$.

Definition 7. A fuzzy topological space (X, τ) is said to be fuzzy $e-T_2$ [3] if for each pair of distinct points x and y of X there exists disjoint fuzzy e -open sets η and ρ such that $x \in \eta$ and $y \in \rho$.

Definition 8. A fuzzy topological space X is said to be fuzzy weakly Hausdorff [10] if each element of X is an intersection of fuzzy regular closed sets.

Definition 9. A fuzzy topological space X is said to be fuzzy e -normal [3] if for every two disjoint fuzzy closed sets η and ρ of X there exist two disjoint fuzzy e -open sets μ and λ such that $\eta \leq \mu$ and $\rho \leq \lambda$ and $\mu \wedge \lambda = 0$.

Definition 10. A fuzzy topological space X is said to be fuzzy strongly normal [10] if for every two disjoint fuzzy closed sets η and ρ of X there exist two disjoint fuzzy open sets μ and λ such that $\eta \leq \mu$ and $\rho \leq \lambda$.

Definition 11. A fuzzy topological space X is said to be fuzzy Urysohn [11] if for every distinct points x and y in X there exist fuzzy open sets μ and λ in X such that $x \in \mu$ and $y \in \lambda$ and $\text{cl}(\mu) \wedge \text{cl}(\lambda) = 0$.

Definition 12. A space (X, τ) is called fuzzy S -closed [2] (resp., fuzzy e -compact [3]) if every fuzzy regular closed (resp., fuzzy e -open) cover of X has a finite subcover.

Definition 13. A function $f : X \rightarrow Y$ is called fuzzy completely continuous [12] if $f^{-1}(\lambda)$ is fuzzy regular open in X for every fuzzy open set λ of Y .

Definition 14. A fuzzy filter base ξ is said to be fuzzy rc -convergent [10] to a fuzzy point x_e in X if for any fuzzy regular closed set η in X containing x_e there exists a fuzzy set $\rho \in \xi$ such that $\rho \leq \eta$.

Definition 15. A collection of fuzzy subsets Δ of a fuzzy topological spaces X is said to form fuzzy filterbases [13] iff for every finite collection $\{\lambda_\alpha : \alpha = 1, 2, \dots, n\}$, $\bigwedge_{\alpha=1}^n \lambda_\alpha \neq 0_X$.

3. Fuzzy $C_e-I(ec, eo)$ Functions

In this section, the notion of fuzzy $C_e-I(ec, eo)$ functions is introduced and some characteristics and properties are studied.

Definition 16. A mapping $\varphi : (X, \tau) \rightarrow (Y, \sigma)$ is called fuzzy $C_e-I(ec, eo)$ if the inverse image of every fuzzy e -open set of Y is fuzzy e -closed in X .

Remark 17. The concepts of fuzzy $C_e-I(ec, eo)$ and fuzzy e -irresolute are independent notions as illustrated in the following example.

Example 18. Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$ and the fuzzy sets μ_1, μ_2, v be defined as follows:

$$\begin{aligned}\mu_1(a) &= 0.1, \\ \mu_2(a) &= 0.7, \\ v(x) &= 0.2, \\ \mu_1(b) &= 0.6, \\ \mu_2(b) &= 0.5, \\ v(y) &= 0.2, \\ \mu_1(c) &= 0.4, \\ \mu_2(c) &= 0.6, \\ v(z) &= 0.4.\end{aligned}\tag{1}$$

Let $\tau = \{0, 1, \mu_1, \mu_2, \mu_1 \vee \mu_2, \mu_1 \wedge \mu_2\}$ and $\sigma = \{0, v, 1\}$. Then, the mapping $\varphi : (X, \tau) \rightarrow (Y, \sigma)$ is defined by $\varphi(a) = x$, $\varphi(b) = y$, $\varphi(c) = z$. Then, φ is fuzzy $C_e-I(ec, eo)$ but not fuzzy e -irresolute.

Example 19. Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$ and the fuzzy sets η_1, η_2, ρ are defined as follows:

$$\begin{aligned}\eta_1(a) &= 0.4, \\ \eta_2(a) &= 0.3, \\ \rho(x) &= 0.4, \\ \eta_1(b) &= 0.7, \\ \eta_2(b) &= 0.5, \\ \rho(y) &= 0.5, \\ \eta_1(c) &= 0.8, \\ \eta_2(c) &= 0.2, \\ \rho(z) &= 1.\end{aligned}\tag{2}$$

Let $\tau = \{0, 1, \eta_1, \eta_2\}$ and $\sigma = \{0, \rho, 1\}$. Then, the mapping $\varphi : (X, \tau) \rightarrow (Y, \sigma)$ is defined by $\varphi(a) = x$, $\varphi(b) = y$, $\varphi(c) = z$. Then, φ is fuzzy e -irresolute but not fuzzy $C_e-I(ec, eo)$.

Definition 20. A mapping $\varphi : (X, \tau) \rightarrow (Y, \sigma)$ is called fuzzy C_e -continuous if the inverse image of every fuzzy open set of Y is fuzzy e -closed in X .

Remark 21. Every fuzzy $C_e-I(ec, eo)$ function is fuzzy C_e -continuous, but not conversely from the following example.

Example 22. Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$ and the fuzzy sets μ_1, μ_2, ν, v are defined as follows:

$$\begin{aligned}
 \mu_1(a) &= 0.6, \\
 \mu_2(a) &= 0.4, \\
 \nu(a) &= 0.8, \\
 \nu(x) &= 0.3, \\
 \mu_1(b) &= 0.5, \\
 \mu_2(b) &= 0.7, \\
 v(b) &= 0.5, \\
 \nu(y) &= 0.4, \\
 \mu_1(c) &= 0.3, \\
 \mu_2(c) &= 0.5, \\
 v(c) &= 0.3, \\
 \nu(z) &= 0.6.
 \end{aligned} \tag{3}$$

Let $\tau = \{0, 1, \mu_1, \mu_2, \mu_1 \vee \mu_2, \mu_1 \wedge \mu_2\}$ and $\sigma = \{0, \nu, 1\}$. Then, the mapping $\varphi : (X, \tau) \rightarrow (Y, \sigma)$ is defined by $\varphi(a) = x$, $\varphi(b) = y$, and $\varphi(c) = z$. Then, φ is fuzzy C_e -continuous but not fuzzy C_e -I(ec, eo) as the fuzzy set v is fuzzy e-open in Y but $\varphi^{-1}(v)$ is not fuzzy e-closed set in X .

Theorem 23. For a fuzzy function $\varphi : X \rightarrow Y$, if $\varphi(x_\epsilon)q\mu$, the inverse image of for every fuzzy e-closed set of Y is fuzzy e-open in X iff for any $x_\epsilon \in X$; if $\varphi(x_\epsilon)q\mu$, then $x_\epsilon qfe\text{-int}(\varphi^{-1}(\mu))$.

Proof. Let $\mu \leq Y$ be a fuzzy e-closed set and $\varphi(x_\epsilon)q\mu$. Then, $x_\epsilon q\varphi^{-1}(\mu)$ and, by hypothesis, $\varphi^{-1}(\mu) = fe\text{-int}(\varphi^{-1}(\mu))$. We obtain, $x_\epsilon qfe\text{-int}(\varphi^{-1}(\mu))$. Converse can be shown easily. $\square$

Theorem 24. For a fuzzy function $\varphi : X \rightarrow Y$, if $\varphi(x_\epsilon)q\mu$, for any fuzzy e-closed set $\mu \leq Y$ and for any $x_\epsilon \in X$, $x_\epsilon qfe\text{-int}(\varphi^{-1}(\mu))$ iff there exists a fuzzy e-open set δ such that $x_\epsilon q\delta$ and $\varphi(\delta) \leq \mu$.

Proof. Let $\mu \leq Y$ be any fuzzy e-closed set and let $\varphi(x_\epsilon)q\mu$. Then, $x_\epsilon qfe\text{-int}(\varphi^{-1}(\mu))$. Take $\delta = fe\text{-int}(\varphi^{-1}(\mu))$ then $\varphi(\delta) = \varphi(fe\text{-int}(\varphi^{-1}(\mu))) \leq \varphi(\varphi^{-1}(\mu)) \leq \mu$, and δ is fuzzy e-open in X and $x_\epsilon q\delta$.

Conversely, let $\mu \leq Y$ be any fuzzy e-closed set and let $\varphi(x_\epsilon)q\mu$. By hypothesis, there exists fuzzy e-open set δ such that $x_\epsilon q\delta$ and $\varphi(\delta) \leq \mu$. This implies, $\delta \leq \varphi^{-1}(\mu)$ and then $x_\epsilon qfe\text{-int}(\varphi^{-1}(\mu))$. $\square$

Theorem 25. For a fuzzy function $\varphi : X \rightarrow Y$, the following statements are equivalent:

- (1) f is fuzzy C_e -I(ec, eo).
- (2) For every fuzzy e-closed set μ in Y , $\varphi^{-1}(\mu)$ is fuzzy e-open in X .
- (3) For every fuzzy open set μ , $\varphi^{-1}(fe\text{-int}(\mu))$ is fuzzy e-closed.

(4) For every fuzzy closed set η , $\varphi^{-1}(fe\text{-cl}(\eta))$ is fuzzy e-open.

(5) For each $x_\epsilon \in X$ and each fuzzy e-closed set μ in Y containing $\varphi(x_\epsilon)$, there exists a fuzzy e-open set ρ in X containing x_ϵ such that $\varphi(\rho) \leq \mu$.

(6) For each $x_\epsilon \in X$ and each fuzzy e-open set μ in Y noncontaining $\varphi(x_\epsilon)$, there exists a fuzzy e-closed set ν in X noncontaining x_ϵ such that $\varphi^{-1}(\mu) \leq \nu$.

Proof. (1) $\Leftrightarrow$ (2): let ρ be a fuzzy e-open set in Y . Then, $1_Y - \rho$ is fuzzy e-closed. By (2), $\varphi^{-1}(1_Y - \rho) = 1_X - \varphi^{-1}(\rho)$ is fuzzy e-open. Thus, $\varphi^{-1}(\rho)$ is fuzzy e-closed. Converse can be shown easily.

(1) $\Leftrightarrow$ (3): let μ be a fuzzy open set. Since $fe\text{-int}(\mu)$ is fuzzy e-open, then by (1) it follows that $\varphi^{-1}(fe\text{-int}(\mu))$ is fuzzy e-closed. The converse is easy to prove.

(2) $\Leftrightarrow$ (4): let η be a fuzzy closed set. Since $fe\text{-cl}(\eta)$ is fuzzy e-closed set, then by (2) it follows that $\varphi^{-1}(fe\text{-cl}(\eta))$ is fuzzy e-open. The converse is easy to prove.

(2) $\Leftrightarrow$ (5): let μ be any fuzzy e-closed set in Y containing $\varphi(x_\epsilon)$. By (2), $\varphi^{-1}(\mu)$ is fuzzy e-open set in X and $x_\epsilon \in \varphi^{-1}(\mu)$. Take $\rho = \varphi^{-1}(\mu)$. Then, $\varphi(\rho) \leq \mu$. The converse can be shown easily.

(2) $\Leftrightarrow$ (6): let μ be any fuzzy e-open set in Y noncontaining $\varphi(x_\epsilon)$. Then, $1 - \mu$ is a fuzzy e-closed set containing $\varphi(x_\epsilon)$. By (5), there exists a fuzzy e-open set ρ in X containing x_ϵ such that $\varphi(\rho) \leq 1 - \mu$. Hence, $\rho \leq \varphi^{-1}(1 - \mu) = 1 - \varphi^{-1}(\mu)$ and $\varphi^{-1}(\mu) \leq 1 - \rho$. Take $\nu = 1 - \rho$. We obtain that ν is a fuzzy e-closed set in X noncontaining x_ϵ . The converse can be shown easily. $\square$

Theorem 26. Let $\phi : X \rightarrow Y$ be a function and let $\varphi : X \rightarrow X \times Y$ be the fuzzy graph function of ϕ , defined by $\varphi(x_\epsilon) = (x_\epsilon, \phi(x_\epsilon))$ for every $x_\epsilon \in X$. If ϕ is fuzzy C_e -I(ec, eo), then ϕ is fuzzy C_e -I(ec, eo).

Proof. Let μ be a fuzzy e-closed set in Y ; then, $1_X \times \mu$ is a fuzzy e-closed set in $X \times Y$. Since φ is fuzzy C_e -I(ec, eo), then $\varphi^{-1}(\mu) = \varphi^{-1}(1_X \times \mu)$ is fuzzy e-open in X . Thus, ϕ is fuzzy C_e -I(ec, eo). $\square$

Theorem 27. Let $\{Y_\lambda : \lambda \in \Lambda\}$ be a family of product spaces. If a function $\varphi : X \rightarrow \prod Y_\lambda$ is fuzzy C_e -I(ec, eo), then $P_\lambda \circ \varphi : X \rightarrow Y_\lambda$ is fuzzy C_e -I(ec, eo) for each $\lambda \in \Lambda$ where P_λ is the projection of $\prod Y_\lambda$ onto Y_λ .

Proof. Let δ be any fuzzy e-open set in Y_λ . Since P_λ is a fuzzy continuous and fuzzy open set, it is a fuzzy e-open set. Now $P_\lambda : \prod Y_\lambda \rightarrow Y_\lambda$, $P_\lambda^{-1}(\delta)$ is a fuzzy e-open in $\prod Y_\lambda$. Therefore, P_λ is a fuzzy e-irresolute function. Now $(P_\lambda \circ \varphi)^{-1}(\delta) = \varphi^{-1}(P_\lambda^{-1}(\delta))$, since φ is fuzzy C_e -I(ec, eo). Hence $\varphi^{-1}(P_\lambda^{-1}(\delta))$ is a fuzzy e-closed set, since $P_\lambda^{-1}(\delta)$ is a fuzzy e-open set. Hence, $P_\lambda \circ \varphi$ is fuzzy C_e -I(ec, eo). $\square$

Theorem 28. If the function $\varphi : \prod X_\lambda \rightarrow \prod Y_\lambda$ is fuzzy C_e -I(ec, eo), then $\varphi_\lambda : X_\lambda \rightarrow Y_\lambda$ is fuzzy C_e -I(ec, eo) for each $\lambda \in \Lambda$.

Proof. Let $\lambda_0 \in \Lambda$ be an arbitrary fixed index and let v_{λ_0} be any fuzzy e-open set of Y_{λ_0} ; then, $\prod Y_{\mu} \times v_{\lambda_0}$ is fuzzy e-open in $\prod Y_{\lambda}$, where $\lambda_0 \neq \mu \in \Lambda$. Since φ is fuzzy C_e -I(ec, eo) function, then $\varphi^{-1}(\prod Y_{\mu} \times v_{\lambda_0}) = \prod X_{\mu} \times \varphi_{\lambda_0}^{-1}(v_{\lambda_0})$ is fuzzy e-closed in $\prod X_{\lambda}$ and hence $\varphi_{\lambda_0}^{-1}(v_{\lambda_0})$ is fuzzy e-closed in X_{λ_0} . This implies φ_{λ_0} is fuzzy C_e -I(ec, eo). $\square$

Theorem 29. If $\varphi : X \rightarrow Y$ is fuzzy C_e -I(ec, eo) and δ is fuzzy closed set of X , then $\varphi|_{\delta} : \delta \rightarrow Y$ is fuzzy C_e -I(ec, eo).

Proof. Let λ be a fuzzy e-open set of Y ; then, $(\varphi|_{\delta})^{-1}(\lambda) = \varphi^{-1}(\lambda) \cap \delta$. Since $\varphi^{-1}(\lambda)$ and δ are fuzzy closed, hence $(\varphi|_{\delta})^{-1}(\lambda)$ is fuzzy e-closed in the relative topology of δ . $\square$

Definition 30. The intersection of all fuzzy e-open set η of a fuzzy topological space (X, τ) containing μ is called the fuzzy e-kernel of μ (briefly, $fe-K_{\mu}$), $fe-K_{\mu} = \bigwedge \{\eta : \mu \leq \eta, \eta \text{ is fuzzy e-open set of } X\}$.

The following properties hold for fuzzy sets μ, λ of X :

- (1) $x \in fe-K_{\mu}$ iff $\mu \wedge \gamma \neq 0$ for any fuzzy e-closed set γ containing x .
- (2) $\mu \leq fe-K_{\mu}$ and $\mu = fe-K_{\mu}$ if μ is fuzzy e-open in X .
- (3) $\mu \leq \lambda$; then, $fe-K_{\mu} \leq fe-K_{\lambda}$.

Theorem 31. For a fuzzy function $\varphi : X \rightarrow Y$, the following statements are equivalent:

- (1) φ is fuzzy C_e -I(ec, eo).
- (2) $\varphi(fe-cl(\mu)) \leq fe-K_{\varphi(\mu)}$ for every fuzzy set μ of X .
- (3) $fe-cl(\varphi^{-1}(\eta)) \leq \varphi^{-1}(fe-K_{\eta})$ for every fuzzy set η of Y .

Proof. (1) $\Rightarrow$ (2): let $\mu \leq X$ and $\gamma \notin fe-K_{\varphi(\mu)}$. There exists a fuzzy e-closed set γ in Y , such that $\gamma \in \gamma$ and $\varphi(\mu) \wedge \gamma = 0$. Therefore, $\varphi^{-1}(\varphi(\mu) \wedge \gamma) = 0$. This implies that $\mu \wedge \varphi^{-1}(\gamma) = 0$ and $fe-cl(\mu) \wedge \varphi^{-1}(\gamma) = 0$. Thus, $\varphi(fe-cl(\mu)) \wedge \gamma = 0$ and $\gamma \notin \varphi(fe-cl(\mu))$. Hence, $\varphi(fe-cl(\mu)) \leq fe-K_{\varphi(\mu)}$.

(2) $\Rightarrow$ (3): let $\eta \leq Y$; then, $\varphi^{-1}(\eta) \leq X$. By hypothesis, $\varphi(fe-cl(\varphi^{-1}(\eta))) \leq fe-K_{\varphi(\varphi^{-1}(\eta))} \leq fe-K_{\eta}$. Hence, $fe-cl(\varphi^{-1}(\eta)) \leq \varphi^{-1}(fe-K_{\eta})$.

(3) $\Rightarrow$ (1): let η be any fuzzy e-open set of Y ; we have $fe-cl(\varphi^{-1}(\eta)) \leq \varphi^{-1}(fe-K_{\eta}) = \varphi^{-1}(\eta)$, since η is fuzzy e-open and $fe-cl(\varphi^{-1}(\eta)) = \varphi^{-1}(\eta)$. This implies that $\varphi^{-1}(\eta)$ is fuzzy e-closed in X . $\square$

Definition 32. The fuzzy e-Frontier of a fuzzy set γ of a fuzzy topological space X is given by $fe-Fr(\gamma) = fe-cl(\gamma) \wedge fe-cl(1_X - \gamma)$.

Theorem 33. The fuzzy point $x_{\varepsilon} \in X$ such that $\varphi : X \rightarrow Y$ is not fuzzy C_e -I(ec, eo) is exactly the union of fuzzy e-Frontier if the inverse image of the fuzzy e-closed set in Y contains $\varphi(x_{\varepsilon})$.

Proof. Suppose that φ is not fuzzy C_e -I(ec, eo) at the point $x_{\varepsilon} \in X$; then there exists a fuzzy e-closed set γ such that $\varphi(x_{\varepsilon}) \in \gamma$ and $\varphi(\mu) \wedge (1_Y - \gamma) \neq 0$ for all fuzzy e-open set μ such that $x_{\varepsilon} \in \mu$. It follows that $\mu \wedge \varphi^{-1}(1_Y - \gamma) \neq 0$ and hence $x_{\varepsilon} \in fe-cl(\varphi^{-1}(1_Y - \gamma)) = fe-cl(1_X - \varphi^{-1}(\gamma))$. Thus, $x_{\varepsilon} \in \varphi^{-1}(\gamma) \leq fe-cl(\varphi^{-1}(\gamma))$ and hence $x_{\varepsilon} \in fe-Fr(\varphi^{-1}(\gamma))$.

Conversely, suppose that $x_{\varepsilon} \in fe-Fr(\varphi^{-1}(\gamma))$, γ is fuzzy e-closed set of Y containing $\varphi(x_{\varepsilon})$, and φ is fuzzy C_e -I(ec, eo) at $x_{\varepsilon} \in X$. There exists fuzzy e-open set μ such that $x_{\varepsilon} \in \mu$ and $\mu \leq \varphi^{-1}(\gamma)$. Thus, $x_{\varepsilon} \in fe-int\varphi^{-1}(\gamma)$ and hence $x_{\varepsilon} \notin fe-Fr(\varphi^{-1}(\gamma))$ for each fuzzy e-closed set γ of Y containing $\varphi(x_{\varepsilon})$, a contradiction. Therefore, φ is not fuzzy C_e -I(ec, eo). $\square$

Theorem 34. The following hold for functions $\phi : X \rightarrow Y$ and $\varphi : Y \rightarrow Z$:

- (a) If $\phi : X \rightarrow Y$ is fuzzy C_e -I(ec, eo) and $\varphi : Y \rightarrow Z$ is fuzzy C_e -continuous then $\varphi \circ \phi : X \rightarrow Z$ is fuzzy C_e -continuous.
- (b) If $\phi : X \rightarrow Y$ is fuzzy C_e -I(ec, eo) and $\varphi : Y \rightarrow Z$ is fuzzy e-irresolute then $\varphi \circ \phi : X \rightarrow Z$ is fuzzy C_e -I(ec, eo).

Theorem 35. If $\phi : X \rightarrow Y$ is a fuzzy e-irresolute surjective function and $\varphi : Y \rightarrow Z$ is a fuzzy function such that $\varphi \circ \phi : X \rightarrow Z$ is fuzzy C_e -I(ec, eo), then φ is fuzzy C_e -I(ec, eo).

Proof. Let η be any fuzzy e-closed set in Z . Since $\varphi \circ \phi$ is fuzzy C_e -I(ec, eo), $(\varphi \circ \phi)^{-1}(\eta)$ is fuzzy e-open in X . Therefore, $\phi^{-1}(\varphi^{-1}(\eta)) = (\varphi \circ \phi)^{-1}(\eta)$ is fuzzy e-open in X . Since ϕ is fuzzy e-irresolute, surjection implies $\phi(\phi^{-1}(\varphi^{-1}(\eta))) = \varphi^{-1}(\eta)$ is fuzzy e-open in Y . Thus, φ is fuzzy C_e -I(ec, eo). $\square$

Theorem 36. If $\phi : X \rightarrow Y$ is a fuzzy e^* -open surjective function and $\varphi : Y \rightarrow Z$ is a fuzzy function such that $\varphi \circ \phi : X \rightarrow Z$ is fuzzy C_e -continuous, then φ is fuzzy C_e -continuous.

Proof. Let η be any fuzzy closed set in Z . Since $\varphi \circ \phi$ is fuzzy C_e -continuous, $(\varphi \circ \phi)^{-1}(\eta)$ is fuzzy e-open in X . Therefore, $\phi^{-1}(\varphi^{-1}(\eta)) = (\varphi \circ \phi)^{-1}(\eta)$ is fuzzy e-open in X . Since ϕ is fuzzy e^* -open, surjection implies $\phi(\phi^{-1}(\varphi^{-1}(\eta))) = \varphi^{-1}(\eta)$ is fuzzy e-open in Y . Thus, φ is fuzzy C_e -continuous. $\square$

4. Fuzzy Completely C_e -I(rc, eo) Functions

In this section, the notion of fuzzy completely C_e -I(rc, eo) functions is introduced and the relation between other functions is studied and further some structure preservation properties are investigated.

Definition 37. A mapping $\varphi : (X, \tau) \rightarrow (Y, \sigma)$ is called fuzzy completely C_e -I(rc, eo) if inverse image of every fuzzy e-open set in Y is fuzzy regular closed in X .

Example 38. Let $X = \{x, y, z\}$ and the fuzzy sets μ_1, μ_2 are defined as follows:

$$\begin{aligned} \mu_1(x) &= 0.4, \\ \mu_2(x) &= 0.7, \\ \mu_1(y) &= 0.6, \\ \mu_2(y) &= 0.3, \\ \mu_1(z) &= 0.1, \\ \mu_2(z) &= 0.5. \end{aligned} \tag{4}$$

Let $\tau = \{0, 1, \mu_1, \mu_2, \mu_1 \vee \mu_2, \mu_1 \wedge \mu_2\}$ and $\sigma = \{0, \mu_1, \mu_1 \wedge \mu_2, 1\}$. Then, the mapping $\varphi : (X, \tau) \rightarrow (X, \sigma)$ is defined by $\varphi(x) = 1 - x$. Then, φ is fuzzy completely C_e -I(rc, eo).

Remark 39. Every fuzzy completely C_e -I(rc, eo) function is fuzzy C_e -I(ec, eo) and fuzzy C_e -continuous, but the converse is not true, which can be seen in the following example.

Example 40. Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$ and the fuzzy sets μ_1, μ_2 , and ν are defined as follows:

$$\begin{aligned} \mu_1(a) &= 0.6, \\ \mu_2(a) &= 0.3, \\ \nu(x) &= 0.4, \\ \mu_1(b) &= 0.5, \\ \mu_2(b) &= 0.7, \\ \nu(y) &= 0.3, \\ \mu_1(c) &= 0.2, \\ \mu_2(c) &= 0.8, \\ \nu(z) &= 0.1. \end{aligned} \quad (5)$$

Let $\tau = \{0, 1, \mu_1, \mu_2, \mu_1 \vee \mu_2, \mu_1 \wedge \mu_2\}$ and $\sigma = \{0, \nu, 1\}$. Then, the mapping $\varphi : (X, \tau) \rightarrow (Y, \sigma)$ is defined by $\varphi(a) = x$, $\varphi(b) = y$, $\varphi(c) = z$. Then, φ is fuzzy C_e -continuous and also fuzzy C_e -I(ec, eo) but not fuzzy completely C_e -I(rc, eo) as the fuzzy set ν is fuzzy e-open in Y but $\varphi^{-1}(\nu)$ is not fuzzy regular closed set in X .

From the above examples, we have the following implications.

None of these implications is reversible.

Theorem 41. For a fuzzy function $\varphi : X \rightarrow Y$, if $\varphi(x_e)q\mu$, the inverse image of every fuzzy e-closed set of Y is fuzzy δ -open in X iff for any $x_e \in X$ if $\varphi(x_e)q\mu$, then $x_e q \text{int}_\delta(\varphi^{-1}(\mu))$.

Proof. Let $\mu \leq Y$ be a fuzzy e-closed set and $\varphi(x_e)q\mu$. Then, $x_e q \varphi^{-1}(\mu)$ and, by hypothesis, $\varphi^{-1}(\mu) = \text{int}_\delta(\varphi^{-1}(\mu))$. From here, $x_e q \text{int}_\delta(\varphi^{-1}(\mu))$. The converse can be shown easily. $\square$

Theorem 42. For a fuzzy function $\varphi : X \rightarrow Y$, if $\varphi(x_e)q\mu$, for any fuzzy e-closed set $\mu \leq Y$ and for any $x_e \in X$, $x_e q \text{int}_\delta(\varphi^{-1}(\mu))$ iff there exists a fuzzy δ -open set θ such that $x_e q \theta$ and $\varphi(\theta) \leq \mu$.

Proof. Let $\mu \leq Y$ be any fuzzy e-closed set and let $\varphi(x_e)q\mu$. Then, $x_e q \text{int}_\delta(\varphi^{-1}(\mu))$. Take $\theta = \text{int}_\delta(\varphi^{-1}(\mu))$; then, $\varphi(\theta) = \varphi(\text{int}_\delta(\varphi^{-1}(\mu))) \leq \varphi(\varphi^{-1}(\mu)) \leq \mu$; θ is fuzzy δ -open in X and $x_e q \theta$.

Conversely, let $\mu \leq Y$ be any fuzzy e-closed set and let $\varphi(x_e)q\mu$. By hypothesis, there exists fuzzy δ -open set θ such that $x_e q \theta$ and $\varphi(\theta) \leq \mu$. This implies $\theta \leq \varphi^{-1}(\mu)$ and then $x_e q \text{int}_\delta(\varphi^{-1}(\mu))$. $\square$

Theorem 43. For a fuzzy function $\varphi : X \rightarrow Y$, the following statements are equivalent:

- (1) f is fuzzy completely C_e -I(rc, eo).
- (2) For every fuzzy e-closed set μ in Y , $\varphi^{-1}(\mu)$ is fuzzy regular open in X .
- (3) For every fuzzy open set μ , $\varphi^{-1}(\text{fe-int}(\mu))$ is fuzzy regular closed.
- (4) For every fuzzy closed set η , $\varphi^{-1}(\text{fe-cl}(\eta))$ is fuzzy regular open.
- (5) For each $x_e \in X$ and each fuzzy e-closed set μ in Y containing $\varphi(x_e)$, there exists a fuzzy regular open set ρ in X containing x_e such that $\varphi(\rho) \leq \mu$.
- (6) For each $x_e \in X$ and each fuzzy e-open set μ in Y non containing $\varphi(x_e)$, there exists a fuzzy regular closed set ν in X non containing x_e such that $\varphi^{-1}(\mu) \leq \nu$.

Proof. (1) $\Leftrightarrow$ (2): let ρ be a fuzzy e-open set in Y . Then, $1_Y - \rho$ is fuzzy e-closed. By (2), $\varphi^{-1}(1_Y - \rho) = 1_X - \varphi^{-1}(\rho)$ is fuzzy regular open. Thus, $\varphi^{-1}(\rho)$ is fuzzy regular closed. Thus, φ is fuzzy completely C_e -I(rc, eo). The converse can be shown easily.

(1) $\Leftrightarrow$ (3): let μ be a fuzzy open set. Since $\text{fe-int}(\mu)$ is fuzzy e-open, then by (1) it follows that $\varphi^{-1}(\text{fe-int}(\mu))$ is fuzzy regular closed. The converse is easy to prove.

(2) $\Leftrightarrow$ (4): let η be a fuzzy closed set. Since $\text{fe-cl}(\eta)$ is fuzzy e-closed set, then by (2) it follows that $\varphi^{-1}(\text{fe-cl}(\eta))$ is fuzzy regular open. The converse is easy to prove.

(2) $\Leftrightarrow$ (5): let μ be any fuzzy e-closed set in Y containing $\varphi(x_e)$. By (2), $\varphi^{-1}(\mu)$ is fuzzy regular open set in X and $x_e \in \varphi^{-1}(\mu)$. Take $\rho = \varphi^{-1}(\mu)$. Then, $\varphi(\rho) \leq \mu$. The converse can be shown easily.

(5) $\Leftrightarrow$ (6): let μ be any fuzzy e-open set in Y non containing $\varphi(x_e)$. Then, $1 - \mu$ is a fuzzy e-closed set containing $\varphi(x_e)$. By (5), there exists a fuzzy regular open set ρ in X containing x_e such that $\varphi(\rho) \leq 1 - \mu$. Hence, $\rho \leq \varphi^{-1}(1 - \mu) = 1 - \varphi^{-1}(\mu)$ and $\varphi^{-1}(\mu) \leq 1 - \rho$. Take $\nu = 1 - \rho$. We obtain that ν is a fuzzy regular closed set in X non containing x_e . The converse can be shown easily. $\square$

Theorem 44. Let $\varphi_1 : X \rightarrow Y$ be a function and let $\varphi_2 : X \rightarrow X \times Y$ be the fuzzy graph function of φ_1 , defined by $\varphi_2(x_e) = (x_e, \varphi_1(x_e))$ for every $x_e \in X$. If φ_2 is fuzzy completely C_e -I(rc, eo), then φ_1 is fuzzy completely C_e -I(rc, eo).

Proof. Let μ be a fuzzy e-closed set in Y ; then, $1_X \times \mu$ is a fuzzy e-closed set in $X \times Y$. Since φ_2 is fuzzy completely C_e -I(rc, eo), then $\varphi_1^{-1}(\mu) = \varphi_2^{-1}(1_X \times \mu)$ is fuzzy regular open in X . Thus, φ_1 is fuzzy completely C_e -I(rc, eo). $\square$

Theorem 45. *The following holds for functions $\varphi_1 : X \rightarrow Y$ and $\varphi_2 : Y \rightarrow Z$:*

- (a) *If $\varphi_1 : X \rightarrow Y$ is fuzzy C_e -I(ec, eo) and $\varphi_2 : Y \rightarrow Z$ is fuzzy completely C_e -I(rc, eo), then $\varphi_2 \circ \varphi_1 : X \rightarrow Z$ is fuzzy e-irresolute.*
- (b) *If $\varphi_1 : X \rightarrow Y$ is fuzzy completely C_e -I(rc, eo) and $\varphi_2 : Y \rightarrow Z$ is fuzzy C_e -continuous, then $\varphi_2 \circ \varphi_1 : X \rightarrow Z$ is fuzzy completely continuous.*

Definition 46. A fuzzy filter base ξ is said to be fuzzy e-convergent to a fuzzy point x_ϵ in X if for any fuzzy e-open set η in X containing x_ϵ there exists a fuzzy set $\rho \in \xi$ such that $\rho \leq \eta$.

Theorem 47. *If a fuzzy function $\varphi : X \rightarrow Y$ is fuzzy completely C_e -I(rc, eo) for each fuzzy point $x_\epsilon \in X$ and each fuzzy filter base ξ in X is fuzzy rc-convergent to x_ϵ , then the fuzzy filter base $\varphi(\xi)$ is fuzzy e-convergent to $\varphi(x_\epsilon)$.*

Proof. Let $x_\epsilon \in X$ and let ξ be any fuzzy filter base in X which is fuzzy rc-converging to x_ϵ . Since φ is fuzzy completely C_e -I(rc, eo), then for any fuzzy e-open set η in Y containing $\varphi(x_\epsilon)$, there exists a fuzzy regular closed set ρ in X containing x_ϵ such that $\varphi(\rho) \leq \eta$. Since ξ is fuzzy rc-converging to x_ϵ , there exists a $\delta \in \xi$ such that $\delta \leq \rho$. This means that $\varphi(\delta) \leq \rho$ and therefore the fuzzy filter base $\varphi(\xi)$ is fuzzy e-convergent to $\varphi(x_\epsilon)$. $\square$

Theorem 48. *If $\varphi : X \rightarrow Y$ is a fuzzy completely C_e -I(rc, eo) surjection and X is fuzzy S-closed, then Y is fuzzy e-compact.*

Proof. Suppose that $\varphi : X \rightarrow Y$ is a fuzzy completely C_e -I(rc, eo) surjection and X is fuzzy S-closed. Let $\{\nu_i\}_{i \in I}$ be a fuzzy e-open cover of Y . Since φ is a fuzzy completely C_e -I(rc, eo), then $\{\varphi^{-1}(\nu_i)\}_{i \in I}$ is fuzzy regular closed cover of X and hence there exists finite set I_0 of I such that $X = \bigvee \{\varphi^{-1}(\nu_i); i \in I_0\}$. Therefore, we have $Y = \bigvee \{\nu_i; i \in I_0\}$ and Y is fuzzy e-compact. $\square$

Theorem 49. *If $\varphi : X \rightarrow Y$ is a fuzzy completely C_e -I(rc, eo) injection and Y is fuzzy e- T_1 , then X is fuzzy weakly Hausdorff.*

Proof. Suppose Y is fuzzy e- T_1 . For any distinct fuzzy points x_ϵ and y_ν in X , there exist fuzzy e-open sets η and ρ in Y . Since φ is injective, $\varphi(x_\epsilon) \in \eta$, $\varphi(y_\nu) \notin \eta$, $\varphi(x_\epsilon) \notin \rho$, and $\varphi(y_\nu) \in \rho$. Since φ is fuzzy completely C_e -I(rc, eo), $\varphi^{-1}(\eta)$ and $\varphi^{-1}(\rho)$ are fuzzy regular closed sets of X such that $x_\epsilon \in \varphi^{-1}(\eta)$, $y_\nu \notin \varphi^{-1}(\eta)$, $x_\epsilon \notin \varphi^{-1}(\rho)$, and $y_\nu \in \varphi^{-1}(\rho)$. This shows that X is fuzzy weakly Hausdorff. $\square$

Theorem 50. *If $\varphi : X \rightarrow Y$ is a fuzzy completely C_e -I(rc, eo) injection and Y is fuzzy e-normal, then X is fuzzy strongly normal.*

Proof. Let η and ρ be disjoint nonempty fuzzy closed sets of X . Since φ is injective, $\varphi(\eta)$ and $\varphi(\rho)$ are disjoint fuzzy closed sets. Since Y is fuzzy e-normal, there exist fuzzy e-open sets μ and λ such that $\varphi(\eta) \leq \mu$ and $\varphi(\rho) \leq \lambda$ and $\mu \wedge \lambda = 0$. This implies that $\text{fe-cl}(\mu)$ and $\text{fe-cl}(\lambda)$ are fuzzy e-closed sets

in Y . Then, since φ is fuzzy completely C_e -I(rc, eo), $\varphi^{-1}(\text{fe-cl}(\mu))$ and $\varphi^{-1}(\text{fe-cl}(\lambda))$ are fuzzy regular open sets. Then, $\eta \leq \varphi^{-1}(\text{fe-cl}(\mu))$ and $\rho \leq \varphi^{-1}(\text{fe-cl}(\lambda))$ and $\varphi^{-1}(\text{fe-cl}(\mu))$ and $\varphi^{-1}(\text{fe-cl}(\lambda))$ are disjoint; by definition X is fuzzy strongly normal. $\square$

Definition 51. A fuzzy topological space (X, τ) is said to be fuzzy e- T_0 (r- T_0 [14]) if for every fuzzy set λ of X can be written in the form $\lambda = \bigvee_{i \in I} \bigwedge_{j \in J} \lambda_{ij}$, where λ_{ij} are fuzzy e-open (fuzzy regular open) or fuzzy e-closed (fuzzy regular closed) sets of Y .

Theorem 52. *If $\varphi : X \rightarrow Y$ is a fuzzy completely C_e -I(rc, eo) injection and Y is fuzzy e- T_0 , then X is fuzzy r- T_0 .*

Proof. Let η be a any fuzzy set of X . Since Y is fuzzy e- T_0 , $\varphi(\eta)$ is fuzzy e-open set of Y . Then, $\varphi(\eta) = \bigvee_{i \in I} \bigwedge_{j \in J} \lambda_{ij}$, where λ_{ij} are fuzzy e-open set or fuzzy e-closed sets of Y . Since φ is completely C_e -I(rc, eo) injection we have $\eta = \varphi^{-1}(\varphi(\eta)) = \varphi^{-1}(\bigvee_{i \in I} \bigwedge_{j \in J} \lambda_{ij}) = \bigvee_{i \in I} \bigwedge_{j \in J} \varphi^{-1}(\lambda_{ij})$, where $\varphi^{-1}(\lambda_{ij})$ are fuzzy regular open sets or fuzzy regular closed sets of X . Thus, X is fuzzy r- T_0 . $\square$

Theorem 53. *If $\varphi : X \rightarrow Y$ is a fuzzy completely C_e -I(rc, eo) injection and Y is fuzzy e- T_2 , then X is fuzzy Urysohn.*

Proof. Let x_ϵ and y_ν be any two distinct fuzzy points in X . Since φ is injective, $\varphi(x_\epsilon) \neq \varphi(y_\nu)$ in Y . Since Y is fuzzy e- T_2 , there exist fuzzy e-open sets η and ρ in Y such that $\varphi(x_\epsilon) \in \eta$ and $\varphi(y_\nu) \in \rho$ and $\eta \wedge \rho = 0$. This implies that $\text{fe-cl}(\eta)$ and $\text{fe-cl}(\rho)$ are fuzzy e-closed sets in Y . Then, since φ is fuzzy completely C_e -I(rc, eo), there exists fuzzy regular open sets δ and γ in X containing x_ϵ and y_ν , respectively, such that $\varphi(\delta) \leq \text{fe-cl}(\eta)$ and $\varphi(\gamma) \leq \text{fe-cl}(\rho)$. This implies that $\delta \leq \varphi^{-1}(\text{fe-cl}(\eta))$ and $\gamma \leq \varphi^{-1}(\text{fe-cl}(\rho))$; we have that $\varphi^{-1}(\text{fe-cl}(\eta))$ and $\varphi^{-1}(\text{fe-cl}(\rho))$ are disjoint and hence $\text{cl}(\delta) \wedge \text{cl}(\gamma) = 0$; by definition, X is fuzzy Urysohn. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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