Research Article

On a New Class of p -Valent Meromorphic Functions Defined in Conic Domains

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We define a new class of multivalent meromorphic functions using the generalised hypergeometric function. We derived this class related to conic domain. It is also shown that this new class of functions, under certain conditions, becomes a class of starlike functions. Some results on inclusion and closure properties are also derived.

1. Introduction

Let M_p denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_n z^{n-p}, \quad p \in \mathbb{N} = 1, 2, 3, \dots, \quad (1)$$

which are analytic and p -valent in the punctured unit disc centred at origin $E^* = \{z : 0 < |z| < 1\} = E \setminus \{0\}$. Also by $f(z) \prec g(z)$ we mean $f(z)$ is subordinate to $g(z)$ which implies the existence of an analytic function, called Schwartz function $w(z)$ with $|w(z)| < 1$, for $z \in E^*$ such that $f(z) = g(w(z))$, where $f(z)$ and $g(z)$ are multivalent

meromorphic functions. Note that if g is univalent in E then the above subordination is equivalent to $f(0) = g(0)$ and $f(E) \subset g(E)$.

The set of points, for $0 < \gamma < 1$ and $k \in [0, \infty)$,

$$\Omega_{k,\gamma} = \gamma\Omega_k + (1-\gamma), \quad (2)$$

where

$$\Omega_k = \left\{u + iv : u > k\sqrt{(u-1)^2 + v^2}, u > 0\right\}, \quad (3)$$

[1] showed that the extremal functions $q_{k,\gamma}(z)$ for conic regions are convex univalent and given by

$$q_{k,\gamma}(z) = \begin{cases} \frac{1 + (1-2\gamma)z}{1-z}, & k = 0, \\ 1 + \frac{2\gamma}{1-k^2} \left[\frac{2}{\pi} (\arccos k) \operatorname{arctanh} \sqrt{z} \right], & 0 < k < 1, \\ 1 + \frac{2\gamma}{\pi^2} \left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2, & k = 1, \\ 1 + \frac{\gamma}{k^2-1} \left[\sin \left(\frac{\pi}{2R(t)} \int_0^{u(z)/\sqrt{t}} \frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}} dx \right) - 1 \right], & k > 1, \end{cases} \quad (4)$$

where $R(t)$ is Legendre's complete elliptic integral of the first kind with $R'(t) = \sqrt{1-t^2}$ as its complementary integral,

$u(z) = (z - \sqrt{t})/(1 - \sqrt{t}z)$, $t \in (0, 1)$, and $z \in E$ is chosen in such a way that $k = \cosh(\pi R'(t)/R(t))$.

The generalised hypergeometric function ${}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$ for complex parameters $\alpha_1, \dots, \alpha_q$ and $\beta_1, \dots, \beta_s$ with $\beta_j \neq 0, -1, -2, -3, \dots$, for $j = 1, 2, 3, \dots, s$, is defined as

$${}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n n!} z^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n (n)!} z^n, \quad (5)$$

with $q \leq s+1$, $q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $(\alpha)_n$ is the well-known Pochhammer symbol related to the factorial and the Gamma function by the relation

$$(\alpha)_n = \frac{(\alpha + n - 1)!}{(\alpha - 1)!} = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)}. \quad (6)$$

Also (5) implies

$$h_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = z^{-p} {}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) \in M_p. \quad (7)$$

Liu and Srivastava [2] defined a linear operator for functions belonging to the class of multivalent meromorphic function $H_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s) : M_p \rightarrow M_p$ as follows:

$$H_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s) = h_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) * f(z). \quad (8)$$

If we assume for brevity that $H_{p,q,s}(\alpha_1) = H_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s)$ then the following identity holds for this operator:

$$z [H_{p,q,s}(\alpha_1) f(z)]' = \alpha_1 H_{p,q,s}(\alpha_1 + 1) f(z) - (\alpha_1 + p) H_{p,q,s}(\alpha_1) f(z). \quad (9)$$

Shareef [3] defined and studied subclass $MQ_p(k, \lambda, \alpha_1)$ of meromorphic function associated with conic domain, for $k \geq 0$, $0 \leq \lambda < 1$, and $p \geq 1$, as follows:

$$-\frac{1}{p} \left[\frac{z (H_{p,q,s}(\alpha_1) f(z))' + \lambda z^2 (H_{p,q,s}(\alpha_1) f(z))''}{(1-\lambda) H_{p,q,s}(\alpha_1) f(z) + \lambda z (H_{p,q,s}(\alpha_1) f(z))'} \right] < q_{k,\gamma}(z). \quad (10)$$

We now define a new subclass $MQ_p(b, k, \lambda, \alpha_1)$ of meromorphic function associated with conic domain, for $k \geq 0$, $0 \leq \lambda < 1$, $p \geq 1$, and $b \geq 1$, as follows:

$$-\frac{1}{p} \left[1 + \frac{1}{b} \left(\frac{z (H_{p,q,s}(\alpha_1) f(z))' + \lambda z^2 (H_{p,q,s}(\alpha_1) f(z))''}{(1-\lambda) H_{p,q,s}(\alpha_1) f(z) + \lambda z (H_{p,q,s}(\alpha_1) f(z))'} \right) - b \right] < q_{k,\gamma}(z). \quad (11)$$

Since $q_{k,\gamma}$ is a convex and univalent function, for $h(z) < q_{k,\gamma}(z)$ it means $h(E^*)$ is contained in $q_{k,\gamma}(E^*)$, where

$$h(z) = -\frac{1}{p} \left[1 + \frac{1}{b} \left(\frac{z (H_{p,q,s}(\alpha_1) f(z))' + \lambda z^2 (H_{p,q,s}(\alpha_1) f(z))''}{(1-\lambda) H_{p,q,s}(\alpha_1) f(z) + \lambda z (H_{p,q,s}(\alpha_1) f(z))'} \right) - b \right]. \quad (12)$$

In the next two sections, for brevity, we drop the subscripts of the operator $H_{p,q,s}(\alpha_1)$.

2. Preliminary Results

Lemma 1 (see [4]). Let $h_2(z)$ be convex in E and $\Re(\lambda h_2(z) + \mu) > 0$, where $\mu \in \mathbb{C}$, $\lambda \in \mathbb{C} \setminus \{0\}$, and $z \in E$. If $h_1(z)$ is analytic in E , with $h_1(0) = h_2(0)$, then

$$h_1(z) + \frac{zh_1'(z)}{\lambda h_1(z) + \mu} < h_2(z) \quad (13)$$

implies $h_1(z) < h_2(z)$.

Lemma 2 (see [5]). Let $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ and $H(z) = 1 + \sum_{n=1}^{\infty} d_n z^n$ and $h < H$. If $H(z)$ is univalent and convex in E , then

$$|c_n| \leq |d_n|, \quad (14)$$

for $n \geq 1$.

Lemma 3 (see [6]). If $q_{k,\gamma}(z) = 1 + q_1 z + q_2 z^2 + \dots$ then

$$q_1 = \begin{cases} \frac{8(1-\gamma)(\arccos(k))^2}{\pi^2(1-k^2)}, & 0 < k < 1, \\ \frac{8(1-\gamma)}{\pi^2}, & k = 1, \\ \frac{\pi^2(1-\gamma)}{4\sqrt{t}(k^2-1)k^2(t)(1+t)}, & k > 1. \end{cases} \quad (15)$$

One now states and proves the main results.

3. Main Results

In this section we explore some of the geometric properties exhibited by the class $MQ_p(b, k, \lambda, \alpha_1)$.

We begin by discussing an inclusion property for the class $MQ_p(b, k, \lambda, \alpha_1)$.

Theorem 4. If $\Re(\alpha_1) > p\Re(bq_{k,\lambda}(z) - 1)$ then

$$MQ_p(b, k, \lambda, \alpha_1 + 1) \subset MQ_p(b, k, \lambda, \alpha_1). \quad (16)$$

Proof. Let $f \in MQ_p(b, k, \lambda, \alpha_1 + 1)$ and set

$$-\frac{1}{p} \left[1 + \frac{1}{b} \left(\frac{z(H(\alpha_1)f(z))' + \lambda z^2(H(\alpha_1)f(z))''}{(1-\lambda)H(\alpha_1)f(z) + \lambda z(H(\alpha_1)f(z))'} \right) - b \right] = h(z). \quad (17)$$

But differentiating (9) with respect to z we get

$$\begin{aligned} & [zH(\alpha_1)f(z)]' \\ &= \alpha_1 H((\alpha_1 + 1)f(z))' \end{aligned}$$

$$\begin{aligned} & -(\alpha_1 + p)(H(\alpha_1)f(z))', \\ & z[H(\alpha_1)f(z)]'' + [H(\alpha_1)f(z)]' \\ &= \alpha_1 H((\alpha_1 + 1)f(z))' \\ & -(\alpha_1 + p)(H(\alpha_1)f(z))', \\ & z^2(H(\alpha_1)f(z))'' \\ &= \alpha_1 z(H(\alpha_1 + 1)f(z))' \\ & -(\alpha_1 + p + 1)z(H(\alpha_1)f(z))'. \end{aligned} \quad (18)$$

Putting (18) in (17) we have

$$\begin{aligned} & \frac{\lambda z \alpha_1 (H(\alpha_1 + 1)f(z))' + [1 - \lambda(\alpha_1 + p + 1)]z(H(\alpha_1)f(z))'}{(1-\lambda)H(\alpha_1)f(z) + \lambda z(H(\alpha_1)f(z))'} = -bph(z), \\ & \frac{\lambda \alpha_1 z(H(\alpha_1 + 1)f(z))' + (1-\lambda)\alpha_1(H(\alpha_1 + 1)f(z))}{(1-\lambda)H(\alpha_1)f(z) + \lambda z(H(\alpha_1)f(z))'} = -bph(z) + (\alpha_1 + p). \end{aligned} \quad (19)$$

Taking logarithmic derivative of (19) we have

$$\begin{aligned} & \frac{z(H(\alpha_1 + 1)f(z))' + \lambda z^2(H(\alpha_1 + 1)f(z))''}{(1-\lambda)(H(\alpha_1 + 1)f(z))' + \lambda z(H(\alpha_1 + 1)f(z))'} \\ & + bph(z) + b(b-1) = \frac{-bpzh'(z)}{-bph(z) + \alpha_1 + p}, \\ & \frac{z(H(\alpha_1 + 1)f(z))' + \lambda z^2(H(\alpha_1 + 1)f(z))''}{(1-\lambda)(H(\alpha_1 + 1)f(z))' + \lambda z(H(\alpha_1 + 1)f(z))'} \\ & = -bp \left[h(z) + \frac{zh'(z)}{-bph(z) + \alpha_1 + p} \right] - b(b-1), \\ & -\frac{1}{p} \left[1 + \frac{1}{b} \left(\frac{z(H(\alpha_1 + 1)f(z))' + \lambda z^2(H(\alpha_1 + 1)f(z))''}{(1-\lambda)H(\alpha_1 + 1)f(z) + \lambda z(H(\alpha_1 + 1)f(z))'} \right) - b \right] \\ & = h(z) + \frac{zh'(z)}{-bph(z) + (\alpha_1 + p)}. \end{aligned} \quad (20)$$

Since $f \in MQ_q(b, k, \lambda, \alpha_1 + 1)$, therefore

$$h(z) + \frac{zh'(z)}{-bph(z) + (\alpha_1 + p)} < q_{k,\gamma}(z). \quad (21)$$

Using Lemma 2 we have

$$h(z) < q_{k,\gamma}(z), \quad (22)$$

provided $\Re(-bpq_{k,\lambda}(z) + \alpha_1 + p) > 0$ or equivalently $\Re(\alpha_1) > p\Re(bq_{k,\gamma}(z) - 1)$. Hence, $f \in MQ_p(b, k, \lambda, \alpha_1)$. $\square$

We now show that the class $MQ_p(b, k, \lambda, \alpha_1)$ is closed under a certain integral.

Theorem 5. If $f(z) \in MQ_p(b, k, \lambda, \alpha_1)$, then the integral

$$B_\eta(H(\alpha)f(z)) = \frac{\eta - p}{z^\eta} \int_0^z t^{\eta-1} (H(\alpha)f(z)) dt \quad (23)$$

maps $f(z)$ into $MQ_p(b, k, \lambda, \alpha_1)$.

Proof. From (23) we have

$$z^\eta B_\eta(H(\alpha)f(z)) = (\eta - p) \int_0^z t^{\eta-1} (H(\alpha)f(z)) dt. \quad (24)$$

Note that

$$B_\eta(H(\alpha)f(z)) = (H(\alpha)) B_\eta f(z). \quad (25)$$

Differentiating (24) above we get

$$\begin{aligned} & \eta z^{\eta-1} B_\eta(H(\alpha)f(z)) + z(B_\eta(H(\alpha)f(z)))' \\ &= (\eta - p) z^{\eta-1} (H(\alpha)f(z)), \end{aligned}$$

$$\begin{aligned} & \eta B_{\eta}(H(\alpha) f(z)) + z(B_{\eta}(H(\alpha) f(z)))' \\ &= (\eta - p)(H(\alpha) f(z)). \end{aligned} \quad (26)$$

Differentiate again

$$\begin{aligned} & \eta(B_{\eta}(H(\alpha) f(z)))' + z(B_{\eta}(H(\alpha) f(z)))'' \\ &+ (B_{\eta}(H(\alpha) f(z)))' = (\eta - p)(H(\alpha) f(z))', \\ & z(B_{\eta}(f(z)))'' = (\eta - p)(B_{\eta}(H(\alpha) f(z)))' \\ &- (\eta + 1)(B_{\eta}(f(z)))'. \end{aligned} \quad (27)$$

Now let

$$\begin{aligned} & -\frac{1}{p} \left[1 \right. \\ & \left. + \frac{1}{b} \left(\frac{z(B_{\eta}(H(\alpha_1) f))' + \lambda z^2(B_{\eta}(H(\alpha_1) f))''}{(1 - \lambda) B_{\eta}(H(\alpha_1) f) + \lambda z(B_{\eta}(H(\alpha_1) f))'} \right) \right. \\ & \left. - b \right] = g(z). \end{aligned} \quad (28)$$

Using (26) and (27) in (28) we get

$$\begin{aligned} & \frac{z B_{\eta}(H(\alpha_1) f)' + \lambda [z(\eta - p)(H(\alpha_1) f)' - z(1 + \eta)(B_{\eta}(H(\alpha_1) f))']}{(1 - \lambda) B_{\eta}(H(\alpha_1) f) + \lambda z(B_{\eta}(H(\alpha_1) f))'} = -bpg(z) + b^2 - b, \\ & \frac{\lambda z(\eta - p)(H(\alpha_1) f)' + [1 - \lambda(1 + \eta)][(\eta - p)(H(\alpha_1) f) - \eta(B_{\eta}(H(\alpha_1) f))']}{(1 - \lambda) B_{\eta}(H(\alpha_1) f) + \lambda z(B_{\eta}(H(\alpha_1) f))'} = -bpg(z) + b^2 - b, \\ & \frac{(\eta - p)[\lambda z(H(\alpha_1) f)' - (1 - \lambda)(H(\alpha_1) f)] - \lambda \eta z(B_{\eta}(H(\alpha_1) f))' - (1 - \lambda)\eta(B_{\eta}(H(\alpha_1) f))}{(1 - \lambda) B_{\eta}(H(\alpha_1) f) + \lambda z(B_{\eta}(H(\alpha_1) f))'} = -bpg(z) + b^2 - b, \\ & \frac{(\eta - p)[\lambda z(H(\alpha_1) f)' - (1 - \lambda)(H(\alpha_1) f)]}{(1 - \lambda) B_{\eta}(H(\alpha_1) f) + \lambda z(B_{\eta}(H(\alpha_1) f))'} = -bpg(z) + b^2 - b + \eta. \end{aligned} \quad (29)$$

Now taking logarithmic derivative we have

$$\begin{aligned} & \frac{(\eta - p)[\lambda z(H(\alpha_1) f)'' + \lambda(H(\alpha_1) f)' + (1 - \lambda)(H(\alpha_1) f)']}{(\eta - p)[\lambda z(H(\alpha_1) f)' - (1 - \lambda)(H(\alpha_1) f)]} \\ & - \frac{(1 - \lambda) B_{\eta}(H(\alpha_1) f)' + \lambda z(B_{\eta}(H(\alpha_1) f))'' + \lambda(B_{\eta}(H(\alpha_1) f))'}{(1 - \lambda) B_{\eta}(H(\alpha_1) f) + \lambda z(B_{\eta}(H(\alpha_1) f))'} = \frac{-bpg'(z)}{-bpg(z) + b^2 - b}, \\ & -\frac{1}{p} \left[1 + \frac{1}{b} \left(\frac{z(H(\alpha_1) f)' + \lambda z^2(H(\alpha_1) f)''}{(1 - \lambda)(H(\alpha_1) f) + \lambda z(H(\alpha_1) f)'} \right) - b \right] = g(z) + \frac{zg'(z)}{-bpg(z) + b^2 - b}. \end{aligned} \quad (30)$$

Using Lemma 2 we get $g(z) < q_{k,\gamma}(z)$ which implies

$$B_{\eta}(f(z)) \in MQ_p(b, k, \lambda, \alpha_1). \quad (31)$$

This proves the assertion. $\square$

Now we get coefficient estimates of the class $MQ_p(b, k, \lambda, \alpha_1)$.

Theorem 6. If $f(z) \in MQ_p(b, k, \lambda, \alpha_1)$ and $f(z)$ is given by (1) then

$$|a_n| \leq \left| \frac{bpq_1}{b(1 - b) + bp - p + n} \right| \sum_{k=0}^{n-1} |a_k|, \quad a_0 = 1 \quad (32)$$

provided

$$2(\lambda p - 1) + (3 - n)\lambda < k\lambda \quad \text{for } 1 - \lambda p + (n - 1)\lambda > 0,$$

$$2(\lambda p - 1) + (3 - n)\lambda > k\lambda$$

$$\text{for } 1 - \lambda p + (n - 1)\lambda < 0, \quad (33)$$

for all $k \leq n$.

Proof. Let $f(z) \in MQ_p(b, k, \lambda, \alpha_1)$; then, by definition, we have

$$\begin{aligned} & -\frac{1}{p} \left[1 \right. \\ & + \frac{1}{b} \left(\frac{z(H(\alpha_1)f(z))' + \lambda z^2(H(\alpha_1)f(z))''}{(1-\lambda)H(\alpha_1)f(z) + \lambda z(H(\alpha_1)f(z))'} \right. \\ & \left. \left. - b \right) \right] = h(z) < q_{k,\gamma}(z), \quad (34) \\ & -\frac{z(H(\alpha_1)f(z))' + \lambda z^2(H(\alpha_1)f(z))''}{p(1-\lambda)H(\alpha_1)f(z) + \lambda z(H(\alpha_1)f(z))'} \\ & = bh(z) + \frac{b}{p} - \frac{b^2}{p}, \end{aligned}$$

which gives

$$\begin{aligned} & -\frac{z(H(\alpha_1)f(z))' + \lambda z^2(H(\alpha_1)f(z))''}{p} \\ & = \left[bh(z) + \frac{b(1-b)}{p} \right] \\ & \cdot \left[(1-\lambda)H(\alpha_1)f(z) + \lambda z(H(\alpha_1)f(z))' \right]. \quad (35) \end{aligned}$$

Let us write $H(\alpha_1)f(z) = f(z)$; then

$$\begin{aligned} f(z) &= z^{-p} + \sum_{n=0}^{\infty} a_n z^{n-p}, \\ f'(z) &= -pz^{-p-1} + \sum_{n=0}^{\infty} a_n(n-p)z^{n-p-1}, \\ zf'(z) &= -pz^{-p} + \sum_{n=0}^{\infty} a_n(n-p)z^{n-p}, \\ f''(z) &= -p(-p-1)z^{-p-2} \\ &+ \sum_{n=0}^{\infty} a_n(n-p)(n-p-1)z^{n-p-2}, \\ z^2 f''(z) &= -p(-p-1)z^{-p} \\ &+ \sum_{n=0}^{\infty} a_n(n-p)(n-p-1)z^{n-p}. \quad (36) \end{aligned}$$

Assuming $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$, then (35) becomes

$$\begin{aligned} & z^{-p} - \sum_{n=0}^{\infty} a_n \frac{(n-p)}{p} z^{n-p} + \lambda \left(- (p+1) z^{-p} \right. \\ & \left. + \sum_{n=0}^{\infty} a_n \frac{(n-p)(n-p-1)}{p} z^{n-p} \right), \\ & (1 - \lambda p - \lambda) z^{-p} \\ & - \sum_{n=1}^{\infty} a_n \frac{n-p}{p} [1 + \lambda(n-p-1)] z^{n-p} = \left[b \right. \\ & \left. + \sum_{n=1}^{\infty} bc_n z^n + \frac{b(1-b)}{p} \right] \left[(1 - \lambda p - \lambda) z^{-p} \right. \\ & \left. + \sum_{n=1}^{\infty} a_n [1 + \lambda(n-p-1)] z^{n-p} \right]. \quad (37) \end{aligned}$$

From (37) we have

$$\begin{aligned} & (1 - \lambda p - \lambda) z^{-p} \\ & - \sum_{n=1}^{\infty} a_n \frac{n-p}{p} [1 + \lambda(n-p-1)] z^{n-p} = \left[b + bc_1 z \right. \\ & \left. + bc_2 z^2 + bc_3 z^3 + \dots + bc_n z^n + \dots + \frac{b(1-b)}{p} \right] \\ & \cdot \left[(1 - \lambda p - \lambda) z^{-p} \right. \\ & \left. + \sum_{n=1}^{\infty} a_n [1 + \lambda(n-p-1)] z^{n-p} \right]. \quad (38) \end{aligned}$$

Now comparing coefficients of z^{1-p} we have

$$\begin{aligned} & -a_1 \frac{1-p}{p} (1 - \lambda p) = a_1 b (1 - \lambda p) + a_1 \left(\frac{b(1-b)}{p} \right) \\ & \cdot (1 - \lambda p) + bc_1 (1 - \lambda p - \lambda), \\ & -a_1 (1 - \lambda p) \left[\frac{b(1-b) + bp - p + 1}{p} \right] \\ & = bc_1 (1 - \lambda p - \lambda), \quad (39) \\ & a_1 = -\frac{p}{(1 - \lambda p) [b(1-b) + bp - p + 1]} (1 - \lambda p - \lambda) \\ & \cdot bc_1, \end{aligned}$$

and comparing the coefficients of z^{2-p} gives

$$\begin{aligned}
 -a_2 \left[\frac{2-p}{p} \right] [1 + \lambda(1-p)] &= a_2 [1 + \lambda(1-p)] \left[b \right. \\
 &\quad \left. + \frac{b(1-b)}{p} \right] + a_1 [1 - \lambda p] bc_1 + bc_2 (1 - \lambda p - \lambda), \\
 -a_2 [1 + \lambda(1-p)] \left[\frac{2-p}{p} + \frac{bp + b(1-b)}{p} \right] &= a_1 [1 \\
 &\quad - \lambda p] bc_1 + bc_2 (1 - \lambda p - \lambda), \\
 -a_2 [1 + \lambda(1-p)] \left[\frac{b(1-b) + bp - p + 2}{p} \right] &= a_1 [1 \\
 &\quad - \lambda p] bc_1 + bc_2 (1 - \lambda p - \lambda), \\
 a_2 &= -\frac{p}{(1 + \lambda(1-p))(b(1-b) + bp - p + 2)} ((1 \\
 &\quad - \lambda p - \lambda) bc_2 + [1 - \lambda p] ba_1 c_1),
 \end{aligned} \tag{40}$$

and for the coefficient of z^{3-p} we have

$$\begin{aligned}
 -a_3 [1 + \lambda(2-p)] \left[\frac{b(1-b) + bp - p + 3}{p} \right] \\
 &= bc_1 a_2 [1 + \lambda(1-p)] + bc_2 a_1 [1 - \lambda p] + bc_3 [1 \\
 &\quad - \lambda p - \lambda], \\
 a_3 &= -\frac{p}{(1 + \lambda(2-p))(b(1-b) + bp - p + 3)} ((1 \\
 &\quad - \lambda p - \lambda) bc_3 + [1 - \lambda p] bc_2 a_1 + [1 + \lambda(1-p)] \\
 &\quad \cdot bc_1 a_2),
 \end{aligned} \tag{41}$$

which generalise to

$$\begin{aligned}
 a_n &= -\frac{p}{(1 - \lambda p + (n-1)\lambda)(b(1-b) + bp - p + n)} [(1 \\
 &\quad - \lambda p - \lambda) bc_n + (1 - \lambda p) bc_{n-1} a_1 + (1 - \lambda p + \lambda) \\
 &\quad \cdot bc_{n-2} a_2 + \dots + (1 - \lambda p + (n-2)\lambda) bc_1 a_{n-1}].
 \end{aligned} \tag{42}$$

The above expression can also be written as

$$\begin{aligned}
 a_n &= -\frac{p}{(1 - \lambda p + (n-1)\lambda)(b(1-b) + bp - p + n)} \\
 &\quad \cdot \sum_{k=1}^n [1 - \lambda p + (k-2)\lambda] c_{n-k-1} a_{k-1},
 \end{aligned} \tag{43}$$

with $a_0 = 1$.

Now taking

$$A_{n,k} = -\frac{1 - \lambda p + (k-2)\lambda}{1 - \lambda p + (n-1)\lambda}, \quad n \geq k, \tag{44}$$

we have

$$\begin{aligned}
 |A_{n,k}| &< 1, \\
 \text{if } \begin{cases} 2(\lambda p - 1) + (3-n)\lambda < k\lambda & \text{for } 1 - \lambda p + (n-1)\lambda > 0, \\ 2(\lambda p - 1) + (3-n)\lambda > k\lambda & \text{for } 1 - \lambda p + (n-1)\lambda < 0, \end{cases}
 \end{aligned} \tag{45}$$

for all $n \geq k$. Since $q_{k,y}(z)$ is univalent and $q_{k,y}(E)$ is convex, applying Rogosinski's theorem we have

$$|c_n| \leq |q_1|, \tag{46}$$

where q_1 is given in (15). Under the conditions given in (45), expressions (39)–(42) give

$$\begin{aligned}
 |a_1| &\leq \left| \frac{bpq_1}{b(1-b) + bp - p + 1} \right|, \\
 |a_2| &\leq \left| \frac{bpq_1}{b(1-b) + bp - p + 2} \right| (1 + |a_1|), \\
 |a_3| &\leq \left| \frac{bpq_1}{b(1-b) + bp - p + 3} \right| (1 + |a_1| + |a_2|), \\
 &\vdots \\
 |a_n| &\leq \left| \frac{bpq_1}{b(1-b) + bp - p + n} \right| \\
 &\quad \cdot (1 + |a_1| + |a_2| + \dots + |a_{n-1}|).
 \end{aligned} \tag{47}$$

This can also be written as

$$|a_n| \leq \left| \frac{bpq_1}{b(1-b) + bp - p + n} \right| \sum_{k=0}^{n-1} |a_k|. \tag{48}$$

This concludes the proof. $\square$

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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