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SHORT COMMUNICATION

Evaluation of the Integral $\int_{-1}^{+1} P_l^{m,n}(x)P_l^{m-2,n}(x)\,dx$

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Integrals of the type indicated can be evaluated using the Fourier-series development of the Jacobi-polynomials. An alternative procedure is given yielding an analytical expression in terms of l, m and n.

KEY WORDS: Jacobi-polynomials, definite integral, texture development.

In texture analysis, for instance in the case of the series-development of crystallite rotations used for the prediction of texture-development, one frequently encounters integrals of the type:

$$\int_{-1}^{+1} P_l^{m,n}(x)P_l^{m-2,n}(x)\,dx$$

The expression can be evaluated using the Fourier-series development of the P_l^{mn} given by Bunge (1982, p. 359). However, an analytical expression can be obtained relatively easy.

Assume $l > 0$ odd or even, m even and $\geqq 2$, n even and $\geqq 0$. Using the definition of the P_l^{mn} given by Bunge (1982, p. 351) the factor independent of x in the integral is written:

$$c = \frac{-(l+n)!}{2^{2l}(l-n)!}\{(l-m)!\,(l-m+2)!\,(l+m)!\,(l+m-2)!\}^{-1/2} \quad (1)$$

Writing†

$$P(x) = (1-x)^{-n+m-1}(1+x)^{-n-m+1}D^{(l-n)}\{(1-x)^{l-m}(1+x)^{l+m}\}$$
$$= \sum_{k=0}^{l-n}\binom{l-n}{k}(-1)^{l-n-k}\frac{(l+m)!\,(l-m)!}{(l+m-k)!\,(n-m+k)!}$$
$$\times (1-x)^{k-1}(1+x)^{l-n+1-k} \quad (2)$$

$$Q(x) = (1-x)^{l-m+2}(1+x)^{l+m-2} \quad (3)$$

The integral reads:

$$c\int_{-1}^{+1} P(x)\frac{d}{dx}D^{(l-n-1)}Q(x)\,dx \quad (4)$$

Integrating by parts yields:

$$cP(x)D^{(l-n-1)}Q(x)|_{-1}^{+1} - c\int_{-1}^{+1} D^{(1)}P(x)\frac{d}{dx}D^{(l-n-2)}Q(x)\,dx \quad (5)$$

Evaluation of the stock term requires analyzing whether negative exponents of $(1-x)$ or $(1+x)$ occur.

It can be shown that negative exponents do not arise. Consequently, the stock term equals zero.

It can be shown for the general case $(1 \leqq q \leqq l-n)$ that:

$$c(-1)^{q-1}D^{(q-1)}P(x)D^{(l-n-q)}Q(x)\,|_{-1}^{+1} \equiv 0 \quad (6)$$

Eventually, after repeatedly integrating by parts Eq. (5) yields:

$$c(-1)^{l-n}\int_{-1}^{+1} D^{(l-n)}P(x)D^{(0)}Q(x)\,dx \quad (7)$$

For $D^{(l-n)}P(x)$ it can be written:

$$D^{(l-n)}\left\{\sum_{k=1}^{l-n} A_k[(-x)^{k-1}x^{l-n+1-k} + \cdots + 1]\right\}$$
$$+ D^{(l-n)}\{A_0(1-x)^{-1}(1+x)^{l-n+1}\} \quad (7a)$$

† Leibniz's rule is applied, the symbol $D^{(n)}$ stands for d^n/dx^n and in the sum only k values leading to non-negative factorials are allowed: $m-n \leqq k \leqq l+m$ for $n<m$.

where

$$A_k = \binom{l-n}{k}(-1)^{l-n-k}\frac{(l+m)!\,(l-m)!}{(l+m-k)!\,(n-m+k)!}$$

according to Eq. (2).

Two cases can be distinguished:

(i) $n < m$: A_0 does not exist and:

$$D^{(l-n)}P(x) = (-1)^{l-n-1}\{(l-n)!\}^2 \sum_k \binom{l+m}{k}\binom{l-m}{l-n-k} \tag{8a}$$

(ii) $n \geqq m$: the k-sum occurring in Eq. (8a) *must* start at $k=0$ now. Consequently†

$$\begin{aligned} D^{(l-n)}P(x) &= D^{(l-n)}\{A_0(1-x)^{-1}(l+x)^{l-n+1}\} \\ &\quad + \text{the result of Eq. (8a)} \\ &\quad - (-1)^{(l-n-1)}\{(l-n)!\}^2\binom{l+m}{0}\binom{l-m}{l-n} \\ &= (-1)^{l-n}\frac{(l-m)!}{(n-m)!}D^{(l-n)}\{(1-x)^{-1}(1+x)^{l-n+1}\} \\ &\quad + (-1)^{l-n-1}(l-n)!\,(l-n)!\left\{\binom{2l}{l-n} - \binom{l-m}{l-n}\right\} \end{aligned} \tag{8b}$$

Using

$$\int_{-1}^{+1}(1-x)^p(1+x)^q\,dx = \frac{p!\,q!}{(p+q+1)!}2^{p+q+1} \quad (p \geqq 0; q \geqq 0) \tag{8c}$$

it follows for the $n < m$ case for Eq. (7)

$$\frac{2}{2l+1}\left\{\frac{(l+m-2)!\,(l-m+2)!}{(l+m)!\,(l-m)!}\right\}^{1/2} \tag{9}$$

For the $n \geqq m$ case evaluation of Eq. (7) requires calculation of the

† Use has been made of the well-known formula $\sum_{t=0}^{r}\binom{s}{t}\binom{u}{r-t} = \binom{s+u}{r}$ where r, s, t and u are integers. For $u-r<0$ the sum starts at $t=r-u$ but the result expression remains the same.

integral

$$\int_{-1}^{+1} Q(x) D^{(l-n)}\{(1-x)^{-1}(1+x)^{l-n+1}\}\,dx \qquad (10)$$

Repeatedly integrating by parts it can be shown that the general stock term

$$(-1)^{q-1} D^{(q-1)} Q(x) D^{(l-n-q)}\{(1-x)^{-1}(1+x)^{l-n+1}\}\big|_{-1}^{+1}$$

$(1 \leqq q \leqq l-n)$ does not exhibit negative exponents of $(1-x)$ or $(1+x)$. Consequently, it equals zero. As a result, Eq. (10) reads:

$$(-1)^{l-n}\int_{-1}^{+1} D^{(l-n)} Q(x) D^{(0)}\{(1-x)^{-1}(1+x)^{l-n+1}\}\,dx \qquad (10a)$$

Using Leibniz's rule and Eq. (8c) it is found for Eq. (10a):

$$(l-n)!\,\frac{2^{2l+1}}{2l+1}\sum_{k=0}^{l-n}(-1)^k \frac{\binom{l+m-2}{k}\binom{l-m+2}{n-m+2+k}}{\binom{2l}{n-m+1+k}} \qquad (11)$$

It can be shown ("trial and error" methods, no proof found yet) that the sum in Eq. (11) equals:

$$\frac{(l+m-2)!\,(l-n+1)!\,(n-m+1)!}{(2l)!}\left\{\binom{2l+1}{l-n+1} - \binom{l-m+2}{l-n+1}\right\} \qquad (11a)$$

For the $n \geqq m$ case it follows (via Eqs. (8b) and (11a)) for Eq. (7):

$$\frac{2}{2l+1}\left\{\frac{(l+m-2)!\,(l-m+2)!}{(l+m)!\,(l-m)!}\right\}^{1/2}\left\{1 - \frac{(2l+1)(n-m+1)}{(l-m+1)(l-m+2)}\right\} \qquad (12)$$

Equations (9) and (12) have been checked using the Fourier-series development of the P_l^{mn} for $l = 4(1)23$, $m = 2(2)l$, $n = 0(2)l$.

References

Bunge, H. J. (1982). *Texture analysis in materials science,* Butterworths, London.