

ON THE INVERSE PROBLEM FOR A HEAT-LIKE EQUATION*

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ABSTRACT

Using the integral representation of the solution of the boundary value problem for the equation with one time-dependent coefficient at the highest space-derivative three inverse problems are solved. Depending on the property of the coefficient we consider cases when the equation is of the parabolic type and two special cases of the degenerate/mixed type. In the parabolic case the corresponding inverse problem is reduced to the nonlinear Volterra integral equation for which the uniqueness of the solution is proved. For the special cases explicit formulae are derived. Both "minimal" and overspecified boundary data are considered.

Key words: heat equation, degenerate/mixed type, non-constant coefficient, boundary value problem, integral representation, inverse problem, nonlinear integral equation.

AMS subject classifications: 35R25, 35K70

INTRODUCTION

Problems of determining unknown coefficients in differential equations, whether partial or ordinary, arise from various applications when only some functionals of their solutions can be considered as given data. In a sense, such problems are inverse to the direct ones when the equation and boundary data are given. Usually these problems involve the restoration of one coefficient or physical parameter. Considerable literature exists on this subject. The important feature of inverse problems is their incorrectness in a classical or Hadamard sense. Since the existence of the solution of such problems is often guaranteed by the physical nature of the problem, it is sufficient to establish the uniqueness of the solution and its continuous dependence on the data inside some functional class (see [1] and [2].)

A physical "Gedanken" experiment leading to such a situation may appear from the need to assess and completely restore whenever possible a source function [3] or some physical parameters of heat propagation in the media where "material constants" are changing in time. This task can only be done indirectly by means of additional measurements of heat or heat flux at a specified point inside the conductor or at its boundary.

* Received 5/1/87, revised 7/11/87, accepted 7/20/87

We shall consider three inverse problems associated with the direct boundary value problem for a "heat-like" equation in a semi-infinite domain with an unknown coefficient $\alpha(t)$ of the following type:

$$(1) \quad L_{\alpha} u = \frac{\partial u}{\partial t} - \alpha(t) \frac{\partial^2 u}{\partial x^2} = 0, \quad x > 0, \quad t > 0;$$

$$(2) \quad u(x,0) = \varphi(x), \quad \varphi(0) = 0, \quad x \geq 0;$$

$$(3) \quad u(0,t) = 0, \quad t \geq 0.$$

Depending on the coefficient $\alpha(t)$, equation (1) will be of the mixed/degenerate type in parts 3.1 and 4 of this paper and of the parabolic type in parts 2 and 3.2.

The additional information necessary for the restoration of the coefficient $\alpha(t)$ is given in the form:

$$(4) \quad u(q,t) = h(t), \quad t \geq 0, \quad q > 0; \quad u(q,0) = h(0) = \varphi(q) \neq 0$$

(that is, for all t the "direct solution" is being measured additionally at the chosen point $x = q > 0$, where the initial condition is not 0).

To solve an inverse problem one has to find such a coefficient $\alpha(t)$ starting from the *a priori* knowledge that it belongs to some specified class (in this case it is a continuous function depending only on one variable t with the property (7) or (8)), such that a solution of the direct problem (1)-(3) (or a similar one) will satisfy condition (4).

Under different conditions on φ and α we obtain explicit formulas for $\alpha(t)$ in two special cases and a nonlinear Volterra integral equation for the unknown coefficient $\alpha(t)$ in the general case. All problems are linked by the common technique based on the integral representation of the direct solution discussed in part 1. Parabolic potentials (see [4]) play a central role in bypassing the usual non-well-posedness of inverse problems, reducing the general case (part 2) to the nonhomogeneous Volterra integral equation of the second kind which then can be solved by the method of successive approximations.

Throughout the paper we shall denote by capital letters functions oddly extended into a region $x < 0$ (ϑ , U , etc.), and we also set all functions $\equiv 0$ for $t < 0$. $M[0,T]$ will denote the space of vanishing for $t < 0$ functions bounded in any strip $(-\infty < x < +\infty) \times [0,T]$ with the norm:

$$\|u\|_T = \sup_{x,t} |u(x,t)| < \infty.$$

1. INTEGRAL REPRESENTATION

To make the presentation self-contained, a fundamental solution of (1) is written explicitly and the integral representation of the solution of a boundary value problem is given.

First, the fundamental solution of (1) is given in the form

$$(5) \quad E_{\alpha_1}(x - \xi, t) = \frac{H(t)}{\sqrt{\pi \alpha_1(t)}} \exp \left[-\frac{(x - \xi)^2}{4\alpha_1(t)} \right],$$

where $H(t)$ is a Heaviside function and

$$(6) \quad \alpha_1(t) = \int_0^t \alpha(z) dz.$$

This can be accomplished easily by the Fourier transform in the space variable (as in [5]) combined with the variation of parameters method.

For now, the only restriction on $\alpha(t)$ is as follows:

$$(7) \quad \alpha_1(t) > 0 \text{ for all } t > 0.$$

Thus, as long as (7) remains true, we can use the fundamental solution (5) and other formulas derived from it for the mixed/degenerate type of (1).

If we further constrain that

$$(8) \quad \alpha(t) > 0 \text{ for all } t \geq 0,$$

(thus preserving the parabolic type of the equation), then by introducing the new variable

$$(9) \quad \tau = \int_0^t \alpha(z) dz,$$

the direct problem of the (1)-(3) type

$$\begin{aligned}
 (10) \quad L_{\alpha} u &= \frac{\partial u}{\partial t} - \alpha(t) \frac{\partial^2 u}{\partial x^2} = f(x,t), \quad x > 0, t > 0; \\
 u(x,0) &= \varphi(x), \quad \varphi(0) = 0, \quad x \geq 0; \\
 u(0,t) &= r(t), \quad t \geq 0;
 \end{aligned}$$

is reduced to the "standard" heat equation and to a corresponding boundary value problem for it. Using the well known Green's formula for such a problem and, taking into consideration the fact that the substitution (9) under condition (8) is a one-to-one correspondence, we can return to the original problem and thus obtain the integral representation for its solution. For the "odd extension into $x < 0$ case," where $f(x, t)$ extends into $F(x, t)$, mentioned in the introduction, we have:

$$\begin{aligned}
 (11) \quad U(x,t) &= \int_0^t d\tau \int_{-\infty}^{+\infty} F(\xi, \tau) E_{\beta_1}(x - \xi, t - \tau) d\xi + \int_{-\infty}^{+\infty} \vartheta(\xi) E_{\alpha_1}(x - \xi, t) d\xi \\
 &+ \int_0^t \alpha(\tau) r(\tau) \frac{\partial}{\partial \xi} (E_{\beta_1}(x - \xi, t - \tau))_{\xi=0} d\tau,
 \end{aligned}$$

which acquires the form

$$(12) \quad U(x,t) = \int_{-\infty}^{+\infty} \vartheta(\xi) E_{\alpha_1}(x - \xi, t) d\xi,$$

in case of our problem (1)-(3). (See Appendix 1).

The integrals involved in (11) are called parabolic potentials. Their properties are inherited from those of standard heat potentials. In each case, only the first two integrals will be utilized.

Remark. The second integral in (11) preserves its form and may be used even in the case of mixed/degenerate type of (1) under condition (7). Since this result cannot be obtained directly from (11), that holds only for a parabolic type of equation with α satisfying (8), we will discuss it in Appendix 1.

2. GENERAL CASE

Let us consider the inverse problem (1)-(4) oddly extending all functions into $x < 0$ and letting them $\equiv 0$ for $t < 0$. We start with the *a priori* assumption that the coefficient $\alpha(t)$ is a continuous function of one variable t and positive for all $t \geq 0$ since we want to deal now with (1) of the parabolic type. We shall not impose any extra conditions on the initial data, except for the necessary smoothness. At the same time we shall keep (4) minimal in the sense that for the restoration of one function $\alpha(t)$ we use information (in addition to what is necessary for the direct problem) of the same dimension. In many cases this information is not sufficient (see [6],[7], and part 3 of this paper).

Suppose φ is at least twice continuously differentiable and φ is bounded together with all its derivatives. Let us introduce the new function

$$(13) \quad W(x,t) = U(x,t) - \vartheta(x) H(t),$$

being a solution of the problem

$$(14) \quad \begin{cases} \frac{\partial W}{\partial t} - \alpha(t) \frac{\partial^2 W}{\partial x^2} = \alpha(t) \vartheta''(x) H(t); \\ W(x,0) = 0; \\ W(0,t) = 0, \end{cases}$$

where u satisfies (1)-(3). Then using the integral representation for the solution of this nonhomogeneous problem (see part 1 and Appendix 1) we get

$$(15) \quad W(x,t) = \int_0^t \alpha(\tau) d\tau \int_{-\infty}^{+\infty} \vartheta''(\xi) E_{\beta_1}(x - \xi, t - \tau) d\xi = U(x,t) - \vartheta(x)$$

for all x and $t \geq 0$, and from (4) for $x = q$ and all $t \geq 0$:

$$(16) \quad \int_0^t \alpha(\tau) d\tau \int_{-\infty}^{+\infty} \vartheta''(\xi) E_{\beta_1}(q - \xi, t - \tau) d\xi = h(t) - \varphi(q).$$

This is a nonlinear Volterra integral equation of the first kind with unknown $\alpha(t)$. Our approach to recover the coefficient $\alpha(t)$ will now be based on finding a solution of this equation which leads to the following result:

Theorem 1. Let the function φ from the initial condition (2) be four times continuously differentiable, bounded with all its derivatives.

Let $x = q > 0$ be a point with $\varphi''(q) \neq 0$.

Let $h(t)$ in (4) $\in C^1(\mathbf{R}_+)$ and be a value at the specified point $x = q$ of the solution of the boundary value problem (1)-(3) for some coefficient $\alpha(t)$ which is *a priori* known to be a positive (to preserve the parabolic type of the equation) continuous function of only one variable t and bounded from above by a constant A_0 :

$$(17) \quad 0 < \alpha(t) \leq A_0 \text{ for all } t \geq 0.$$

Then there exists a unique solution of the inverse problem (1)-(4) (one coefficient $\alpha(t)$ in its class) which can be found from the equation (16) for all $t \geq 0$.

Proof. We can rewrite equation (16) in the form

$$(18) \quad \int_0^t \alpha(\tau) K_{\beta_1}(t, \tau) d\tau = F(t),$$

where $F(t)$ is the right side of (16) and

$$K_{\beta_1}(t, \tau) = \int_{-\infty}^{+\infty} \vartheta''(\xi) E_{\beta_1}(q - \xi, t - \tau) d\xi.$$

Then, since

$$K_{\beta_1}(t, t) = \int_{-\infty}^{+\infty} \vartheta''(\xi) E_{\beta_1}(q - \xi, 0) d\xi = \vartheta''(x) * \delta(q - x) = \vartheta''(q) \neq 0$$

the integral equation (18) of the first kind can be regularized (by mere differentiation in t and integration by parts using properties of the fundamental solution (5)) into equivalent equation of the second kind:

$$(19) \quad \alpha(t) \varphi''(q) + \alpha(t) \int_0^t \alpha(\tau) d\tau \int_{-\infty}^{+\infty} \vartheta^{(4)}(\xi) E_{\beta_1}(q - \xi, t - \tau) d\xi = h'(t).$$

Then, defining an operator B as

$$(20) \quad B \alpha(t) = \int_0^t \alpha(\tau) d\tau \int_{-\infty}^{+\infty} \vartheta^{(4)}(\xi) E_{\beta_1}(q - \xi, t - \tau) d\xi,$$

and inserting the variable y into (20) (see also Appendix 1)

$$y = \frac{1}{4 \beta_1 (t - \tau)}, \quad \frac{1}{4 y^2} dy = \alpha(\tau) d\tau,$$

we express it in the form:

$$B \alpha = \frac{1}{4 \sqrt{\pi}} \int_{-\infty}^{+\infty} \vartheta^{(4)}(\xi) d\xi \int_{\frac{1}{4 \alpha_1 (1)}}^{+\infty} y^{-3/2} e^{-(q - \xi)^2 y} dy.$$

This allows us to get:

$$(21) \quad \begin{aligned} |vB\mu - \mu Bv| &\leq |\mu| |B\mu - Bv| + |Bv| |\mu - v| \\ &\leq \|\varphi^{(4)}\|_T (\|\mu\|_T + \|v\|_T) \|\mu - v\|_T. \end{aligned}$$

Then we can rewrite the equation (19) in the form

$$(22) \quad A \alpha(t) = \alpha(t), \quad 0 \leq t \leq T,$$

where the operator A

$$A \alpha(t) = \frac{1}{\vartheta''(q)} h'(t) - \frac{1}{\vartheta''(q)} \alpha(t) B \alpha(t)$$

leaves the space $C[0, T]$ invariant. Now, the immediate consequence is:

$$\|A\mu - A\nu\|_T \leq p \|\mu - \nu\|_T, \text{ where } p = \frac{T \|\vartheta^{(4)}\|_T}{|\vartheta''(q)|} (\|\mu\|_T + \|\nu\|_T).$$

If $p < 1$, the operator A is a contraction on $C[0, T]$. This implies the existence and uniqueness of the solution of the equation (22), which can be found by the method of successive approximations. Under the condition (17) we shall assume that

$$(23) \quad T < \frac{|\vartheta''(q)|}{2 A_0 \|\vartheta^{(4)}\|_T}$$

in order to get $p < 1$. It is evident that the right side of (23) is a constant not depending on time since φ is a function of x only. This means that we have found the width T of the strip in which the unique solution of (19) is guaranteed, and this value of T is fixed (it is an important fact for the continuation of this process for t -values beyond the initial strip).

The continuation of the process of " α -restoration" into the next strip of the T -width can be performed in a way similar to that discussed in [8]. After finding $\alpha(t)$ from (19) we evaluate the function $u(x, t)$ using its integral representation (12) and then by means of $u(x, t)$ and its derivatives at $t = T - \epsilon$ (it may be necessary to make a small ϵ - step backward, inside the first strip) we define the new initial and boundary conditions similar to (2)-(4) and repeat the whole procedure again. Due to the fact that T is a time-independent constant (it has to be calculated only once from (23)) we can find the unique solution of the inverse problem for any given value t in a finite number of steps, that concludes the proof.

Remarks.

1. The function $h'(t)$ in equation (19) can not be $\equiv 0$ in any continuum Δ ($t \in \Delta$); otherwise, it contradicts (17). Moreover, starting from such first Δ on the time axis, the problem degenerates into the steady state case.

2. The $\varphi'' = 0$ case (see (16)-(18)) leads to the steady state problem (since $u(x, t) = \varphi(x) \equiv Cx$), which is not informative concerning α and is not considered here.

3. The case when $\varphi^{(4)} = 0$ is trivial and both the equation (19) and representation (12) under the condition (4) lead to the same formula for α .

4. Under the conditions of Theorem 1, the solution of the inverse problem (1)-(4) continuously depends on the given data in the sense of M-norm.

3. THE FIRST INVERSE PROBLEM

3.1. Now let us consider simultaneously two problems: (1)-(3) (its solution is denoted by $u_1(x,t)$) and a similar problem with the same unknown coefficient $\alpha(t)$:

$$(24) \quad \begin{aligned} \frac{\partial u_2}{\partial t} - \alpha(t) \frac{\partial^2 u_2}{\partial x^2} &= 0, \quad x > 0, \quad t > 0; \\ u_2(x,0) &= \varphi''(x), \quad x \geq 0; \quad \varphi''(x) \in C[0,\infty]; \\ u_2(0,t) &= 0, \quad t \geq 0. \end{aligned}$$

These problems differ only in their initial conditions. Coefficient α belongs to the class of functions satisfying (7) as in part 2.

Our goal is to recover the coefficient $\alpha(t)$ from the "experimental" data obtained as additional measurements of the solutions of (1)-(3) and (24) at a specified point $x = q$. The situation like this one will occur if we specially design the " α -restoration experiment." That is, we have to produce two physically identical experiments with initial data φ and φ'' , respectively, measuring the outcome in both cases at a point $x = q$ with $\varphi''(q) \neq 0$. We have the following statement:

Theorem 2. Let two additionally "collected" data (similar to (4)) be

$$(25) \quad \begin{aligned} u_1(q,t) &= h_1(t); \\ u_2(q,t) &= h_2(t), \quad t \geq 0. \end{aligned}$$

Also let φ be twice differentiable with all its derivatives bounded, $h_1 \in C^1(\mathbf{R}_+)$, $h_2 \in C(\mathbf{R}_+)$, $h_2(t) \neq 0$ for all t ($\varphi''(q) = h_2(0) \neq 0$) and

$$(26) \quad \int_0^t \frac{h_1'(\tau)}{h_2(\tau)} d\tau > 0 \text{ for all } t > 0 \text{ (}=0 \text{ for } t = 0).$$

Then there is a unique solution for the inverse problem (1)-(3), (24), (25) (with one continuous coefficient $\alpha(t)$ of the class (7)) which satisfies the expression:

$$(27) \quad \alpha(t) = \frac{h_1'(t)}{h_2(t)}.$$

Proof. (See Appendix 2.)

Remarks.

1. It is easily seen that (26) is a necessary and sufficient condition for $\alpha(t)$ to be of class (7). The necessity follows from the *a priori* knowledge of the "experimental" fact that h_1 and h_2 represent solutions of (1)-(3) and (24) respectively at $x = q$ for the same coefficient α that is a continuous function of t only and satisfies (7).

2. We were able to find the coefficient $\alpha(t)$ explicitly due to the overspecified data: two functions $h_{1,2}(t)$ versus one unknown $\alpha(t)$.

3. As in part 2, if $\varphi'' \equiv 0$ then $\varphi(x) = Cx$, $u_2 \equiv 0$, and by the direct calculation from (12) we get the only solution $u_1(x,t) = \varphi(t) \equiv Cx$ (a steady state case which gives no information about α).

4. If functions $h_{1,2}(t)$ satisfy the conditions of Theorem 2, the case $h_1'(t) \equiv 0$ (for $h_2(0) = \varphi''(q) \neq 0$, otherwise, see Remark 3 above) contradicts assumption (26). It can be also proved that, if $h_1'(t) \equiv 0$ then locally (close to $t=0$) $\alpha(t) \equiv 0$ and $u(x,t) \equiv \varphi(x)$. That is, we obtain a steady-state case locally, and for the problem (1)-(3) without sources, - globally for all t . Again, this case is not informative concerning α and it is not considered here.

5. If the functions $h_{1,2}$ are inserted in (27) with an absolute error then it can be proved directly from (27) that for any time-interval Δt , where the error for h_1 may be considered as a constant (time-independent) value, the relative error for the solution $\alpha(t)$ is of the same magnitude as the corresponding relative error Δ of h_2 .

3.2. Now let us consider the strictly parabolic type of the equation (1) imposing further restrictions on the coefficient $\alpha(t) \in C(\mathbf{R}_+)$, namely, $\alpha(t) > 0$ for all $t > 0$. Then, since $\alpha(t) > 0$, we change the restriction on the given data (26) into:

$$(28) \quad h_1'(t) / h_2(t) > 0 \text{ for all } t \geq 0.$$

Theorem 3. Suppose φ satisfies all the conditions of Theorem 2, functions $h_1, h_2 \in C^1(\mathbf{R}_+)$, $h_2(t) \neq 0$ for all t ($\varphi''(q) = h_2(0) \neq 0$) and represent the "outcome of the experiment" formally described by (1)-(3), (24), (25) with some positive coefficient $\alpha(t)$.

Then h_1, h_2 satisfy the following identity:

$$(29) \quad h_1(t) - \varphi(q) = \int_0^t \frac{h_1'(\tau)}{h_2(\tau)} h_2(t - \tau) d\tau, \quad t \geq 0,$$

and $\alpha(t)$ can be found as a unique solution of the integral equation:

$$(30) \quad h_1'(t) = \alpha(t) h_2(0) + \int_0^t \alpha(\tau) (h_2(t - \tau))'_t d\tau, \quad t \geq 0,$$

in the form (27).

The solution of the inverse problem depends continuously on the given data in M-norm.

Proof. (See Appendix 2).

Remark. It is interesting to compare the equations (30) and (19). We can see that due to the overspecified data in case of (30) the kernel does not contain the unknown function. As a result equation (30) is a linear one versus the nonlinear equation (19) arising from the problem with "minimal" data.

4. THE SECOND INVERSE PROBLEM

Let us consider the second special case when the function φ from (2) is a solution of the problem

$$(31) \quad \frac{d^2 \vartheta}{dx^2} = \lambda \vartheta; \quad \vartheta(0) = 0,$$

for any fixed $\lambda < 0$. Then, since there are no sources in the problem (1)-(3) and $u(0,t) = 0$, the integral representation for the solution of (1)-(3) can be written in the form (12).

In this section, as in section 3.1, we consider again continuous functions $\alpha(t)$ satisfying only the condition (7). A utilization of (12) here leads to the following result:

Theorem 4. Let φ be a nontrivial solution of (31) for some fixed $\lambda < 0$ ($\varphi(q) \neq 0$) and $h(t)$ in (4) $\neq 0$ for all $t \geq 0$ and belongs to $C^1(\mathbf{R}_+)$. If there are two coefficients $\alpha(t)$ and $a(t)$ (in the class of functions satisfying (7)) such that both solutions of the

boundary value problem (1)-(3) with operators L_α and L_a , respectively, satisfy the same condition (4), then

$$\alpha(t) \equiv a(t) \text{ for all } t \geq 0.$$

The unique coefficient $\alpha(t)$ satisfies the formula

$$(32) \quad \alpha(t) = \frac{h'_t(t, \lambda)}{\lambda h(t, \lambda)}, \quad t \geq 0.$$

It can also be shown that $\alpha(t)$ given by (32) preserves the property (7) if and only if

$$(33) \quad \left| \frac{h(t)}{h(0)} \right|^{\text{sgn} \lambda} > 1,$$

Proof. Suppose there are two solutions of the inverse problem (1)-(4), (31). Then (4) and (12) lead to

$$(34) \quad \int_{-\infty}^{+\infty} \vartheta(\xi) E_{\alpha_1}(q - \xi, t) d\xi = \int_{-\infty}^{+\infty} \vartheta(\xi) E_{a_1}(q - \xi, t) d\xi,$$

where E_{a_1} is obtained from E_{α_1} by substitution a for α .

Then t -differentiation of (34) and integration by parts give

$$\alpha(t) \int_{-\infty}^{+\infty} \vartheta''(\xi) E_{\alpha_1}(q - \xi, t) d\xi = a(t) \int_{-\infty}^{+\infty} \vartheta(\xi) E_{a_1}(q - \xi, t) d\xi,$$

and from (31) and (34):

$$\alpha \lambda u_\alpha(q, t) = a \lambda u_a(q, t), \quad \text{or } \alpha(t) \equiv a(t), \quad t \geq 0.$$

Then, substituting $h(t)$ in the right side of (34) and performing the same sequence of steps, we get

$$\alpha(t) \lambda \int_{-\infty}^{+\infty} \vartheta(\xi, \lambda) E_{\alpha_1}(q - \xi, t) d\xi = h'_t(t, \lambda)$$

and

$$\lambda \alpha(t) h(t, \lambda) = h'_t(t, \lambda), \text{ for all } t \geq 0$$

Since $h(t) \neq 0$ we obtain (32) and it follows that

$$\alpha_1(t) = \int_0^t \alpha(z) dz = \int_0^t \frac{h'_z(z, \lambda)}{\lambda h(z, \lambda)} dz = \frac{1}{\lambda} \ln \left| \frac{h(t, \lambda)}{h(0, \lambda)} \right|$$

is positive for $t > 0$ if and only if

$$\left| \frac{h(t)}{h(0)} \right|^{\text{sgn}\lambda} > 1,$$

this implies that for continuous h with $h(0) \neq 0$: $h(t) \neq 0$ for all $t \geq 0$.

This and the uniqueness of $\alpha(t)$ in its class complete the proof.

Remark. The inverse problem of this part can be reduced to the problem considered in part 3. Thus all the remarks made there are relevant here. On the other hand, due to the specific condition (31) we were able to obtain an explicit description of the permissible data (see (33)) instead of the implicit formula (26).

APPENDIX 1

As indicated in part 1, the general problem (10) in the case when the coefficient $\alpha(t)$ satisfies (8), can be reduced to

$$\frac{\partial v}{\partial \tau} - \frac{\partial^2 v}{\partial x^2} = \frac{f(x, t(\tau))}{\alpha(t(\tau))} \equiv F(x, \tau), \quad x > 0, \quad \tau > 0;$$

$$(35) \quad \begin{aligned} v(x, 0) &= \varphi(x), \quad \varphi(0), \quad x \geq 0; \\ v(0, \tau) &= r(t(\tau)) \equiv R(\tau), \quad \tau \geq 0; \end{aligned}$$

by the substitution of (9), where $v(x, \tau) \equiv u(x, t(\tau))$. The solution of (35) can be written in the form

$$(36) \quad \begin{aligned} v(x, \tau) &= \int_0^\tau dz \int_0^{+\infty} F(\xi, z) E(x - \xi, \tau - z) d\xi + \int_0^{+\infty} v(\xi, 0) E(x - \xi, \tau) d\xi \\ &+ \int_0^\tau v(0, z) \frac{\partial}{\partial \xi} (E(x - \xi, \tau - z)) \Big|_{\xi=0} dz - \int_0^\tau \frac{\partial v}{\partial \xi} (0, z) E(x, \tau - z) dz, \end{aligned}$$

where $E(x, \tau)$ is a standard fundamental solution for a heat equation. Then, since the substitution of (9) under condition (8) defines a monotonously increasing function $\tau(t)$, we are able to return to the original variables and the functions in (10) and (36) by introducing $z = \tau(y)$, where $0 \leq z \leq \tau(t)$ for $0 \leq y \leq t$:

$$(37) \quad \begin{aligned} u(x, t) &= \int_0^t d\tau \int_0^{+\infty} f(\xi, \tau) E_{\beta_1}(x - \xi, t - \tau) d\xi + \int_0^{+\infty} u(\xi, 0) E_{\alpha_1}(x - \xi, t) d\xi \\ &+ \int_0^t \alpha(\tau) u(0, \tau) \frac{\partial}{\partial \xi} E_{\beta_1}(x, t - \tau) d\tau - \int_0^t \alpha(\tau) \frac{\partial u}{\partial \xi}(0, \tau) E_{\beta_1}(x, t - \tau) d\tau, \end{aligned}$$

where

$$E_{\beta_1}(x - \xi, t - \tau) = \frac{H(t - \tau)}{2\sqrt{\pi\beta_1(t - \tau)}} \exp\left\{-\frac{(x - \xi)^2}{4\beta_1(t - \tau)}\right\}$$

and

$$\beta_1(t - \tau) = \alpha_1(t) - \alpha_1(\tau), \quad \alpha(t) = \int_0^t \alpha(z) dz, \quad \alpha_1(0) = 0.$$

In (37) all functions are considered to be $\equiv 0$ for $x < 0$. In the case when we prefer an odd extension into $x < 0$, formula (37) does not contain the last integral, and such extended solution of (10) is represented by formula (11), part 1.

Remark. As mentioned in part 1, in the case of α satisfying only (7), the integral representation (12) for the solution of the problem (1)-(3) can not be obtained as a part of the general formula (11). Instead we have to start from the heat equation (35) with $F = R \equiv 0$, derive its solution in the form

$$V(x, \tau) = \int_{-\infty}^{+\infty} v(\xi, 0) E(x - \xi, \tau) d\xi$$

and then measure it at the point $\tau = \alpha_1(t)$ (this may be done independently of whether (9) implies existence of the inverse function $t = t(\tau)$ or not).

Then we immediately get that the function $u(x, t) \equiv v(x, \tau(t))$ satisfies (1)-(3) and is represented by the formula (12).

APPENDIX 2

Proof of Theorem 2. If there are two such coefficients a and α that for both of them we get the same values (25) at the end of the "experiment", then

$$U_{1,\alpha}(q, t) = \int_{-\infty}^{+\infty} \vartheta(\xi) E_{\alpha_1}(q - \xi, t) d\xi = h_1(t) = U_{1,a}(q, t) = \int_{-\infty}^{+\infty} \vartheta(\xi) E_{a_1}(q - \xi, t) d\xi$$

$$U_{2,\alpha}(q, t) = \int_{-\infty}^{+\infty} \vartheta''(\xi) E_{\alpha_1}(q - \xi, t) d\xi = h_2(t) = U_{2,a}(q, t) = \int_{-\infty}^{+\infty} \vartheta''(\xi) E_{a_1}(q - \xi, t) d\xi,$$

and differentiation in t gives:

$$h'_1(t) = \int_{-\infty}^{+\infty} \vartheta(\xi) \frac{\partial}{\partial t} E_{\alpha_1}(q - \xi, t) d\xi = \alpha(t) \int_{-\infty}^{+\infty} \vartheta''(\xi) E_{\alpha_1}(q - \xi, t) d\xi = \alpha(t) h_2(t)$$

which is the same for $a(t)$. Both uniqueness and the formula (27) immediately follow from the above formulas for $t \geq 0$.

Proof of Theorem 3.

Since we have here all the conditions of the Theorem 2., formula (27) is still valid. Then, introducing a new function (13) which satisfies the boundary value problem (14) and using the integral representation for the solution of the nonhomogeneous problem we get the formula (15) and at $x = q$ (using (25) and the proof of the Theorem 2):

$$h_1(t) - \varphi(q) = \int_0^t \alpha(\tau) h_2(t - \tau) d\tau, \quad (t \geq 0)$$

From this relation and (27) we immediately get (29) and (30). The equation (30) is a linear Volterra integral equation of the second kind with a continuous kernel and, due to (28), it is a nonhomogeneous equation. Hence, it has a unique continuous solution $\alpha(t)$ ($t \in [0, T]$ for any finite T), and α given by the formula (27) definitely satisfies this equation (just differentiate (29) in t). Its continuous dependence on the given data naturally follows from the Volterra integral equation (30), since it can be easily shown that some power of the operator K

$$K\alpha = -\frac{1}{h_2(0)} \int_0^t \alpha(\tau) (h_2(t - \tau))'_t d\tau$$

is a contraction on $C(0, T)$.

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