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Short Communication

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A conjecture by Brakman (1987) concerning a sum involving binomial coefficients is proved by using formulae for hypergeometric functions.

KEY WORDS: Hypergeometric functions, binomial coefficients.

In a recent paper Brakman (1987) conjectured that

$$\sum_{k=0}^{l-n}(-1)^k\frac{\binom{l+m-2}{k}\binom{l-m+2}{n-m+k+2}}{\binom{2l}{n-m+k+1}} = \frac{(l+m-2)!(l-n+1)!(n-m+1)!}{(2l)!} \times\left[\binom{2l+1}{l-n+1}-\binom{l-m+2}{l-n+1}\right],$$

for $l, m, n \in \mathbb{N}$, $m \leq n \leq l$. We present a proof of this formula, using hypergeometric functions.

The hypergeometric functions ${}_2F_1$ and ${}_3F_2$ are defined by

$${}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}\,\middle|\, z\right)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}z^k$$

and

$${}_3F_2\left(\begin{matrix}a, b, c\\ d, e\end{matrix}\,\middle|\, z\right)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k(c)_k}{(d)_k(e)_k k!}z^k,$$

where $(a)_k = a(a+1)\dots(a+k-1)$.

We shall only use the case where $a, b, c, d, e \in \mathbb{Z}$ and $z = 1$. For a positive integer n we may write

$$(n)_k = \frac{(n+k-1)!}{(n-1)!}$$

and

$$(-n)_k = (-1)^k \frac{n!}{(n-k)!} \quad \text{for} \quad k \leq n, \qquad (-n)_k = 0 \quad \text{for} \quad k > n.$$

$$\sum_{k=0}^{l-n} (-1)^k \frac{\binom{l+m-2}{k}\binom{l-m+2}{n-m+k+2}}{\binom{2l}{n-m+k+1}}$$

$$= \sum_{k=0}^{l-n} (-1)^k \frac{(l+m-2)!(l-m+2)!(n-m+k+1)!\,(2l+m-n-k-1)!}{k!(l+m-k-2)!(n-m+k+2)!(l-n-k)!(2l)!}$$

$$= \frac{(l-m+2)!(n-m+1)!(2l+m-n+1)!}{(2l)!(n-m+2)!(l-n)!} \times \sum_{k=0}^{l-n} \frac{(n-l)_k(2-l-m)_k(n-m+2)_k}{(n-m-2l+1)_k(n-m+3)_k}$$

$$= \frac{(l-m+2)!(n-m+1)!(2l+m-n-1)!}{(2l)!(n-m+2)!(l-n)!} \times {}_3F_2\left(\begin{matrix} n-l, 2-l-m, n-m+2 \\ n-m-2l+1, n-m+3 \end{matrix} \,\middle|\, 1\right).$$

We now use the formula (Luke, 1969, p. 111, formula (38))

$${}_3F_2\left(\begin{matrix} a, b, c \\ d+1, c+1 \end{matrix} \,\middle|\, z\right) = \frac{c}{c-d}\,{}_2F_1\left(\begin{matrix} a, b \\ d+1 \end{matrix} \,\middle|\, z\right) - \frac{d}{c-d}\,{}_3F_2\left(\begin{matrix} a, b, c \\ d, c+1 \end{matrix} \,\middle|\, z\right)$$

to write

$$
{}_3F_2\left(\begin{matrix} n-l, 2-l-m, n-m+2 \\ n-m-2l+2, n-m+3 \end{matrix} \,\middle|\, 1\right)
= \frac{n-m+2}{2l+1}\,{}_2F_1\left(\begin{matrix} n-l, 2-l-m \\ n-m-2l+2 \end{matrix} \,\middle|\, 1\right) - \frac{n-m-2l+1}{2l+1}
\times {}_3F_2\left(\begin{matrix} n-l, 2-l-m, n-m+2 \\ n-m-2l+1, n-m+3 \end{matrix} \,\middle|\, 1\right)
$$

or

$$
{}_3F_2\left(\begin{matrix} n-l, 2-l-m, n-m+2 \\ n-m-2l+1, n-m+3 \end{matrix} \,\middle|\, 1\right)
= \frac{2l+1}{2l+m-n-1}\,{}_3F_2\left(\begin{matrix} n-l, 2-l-m, n-m+2 \\ n-m-2l+2, n-m+3 \end{matrix} \,\middle|\, 1\right)
- \frac{n-m+2}{2l+m-n-1}\,{}_2F_1\left(\begin{matrix} n-l, 2-l-m \\ n-m-2l+2 \end{matrix} \,\middle|\, 1\right).
$$

The hypergeometric functions on the right can be evaluated by using the formulae

$$
{}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix} ; 1\right) = \frac{(c-b)_n}{(c)_n} \quad \text{(Luke, 1969, p. 99 formula (3))}
$$

and

$$
{}_3F_2\left(\begin{matrix} -n, a, b \\ a+b-n-c+1, c \end{matrix} \,\middle|\, 1\right) = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n} \quad \text{(Luke, 1969, p. 103 formula (2)).}
$$

We obtain

$$
{}_3F_2\left(\begin{matrix} n-l, 2-l-m, n-m+2 \\ n-m-2l+1, n-m+3 \end{matrix} \,\middle|\, 1\right)
= \frac{(2l+1)(n+l+1)_{l-n}(1)_{l-n}}{(2l+m-n-1)(n-m+3)_{l-n}(l+m-1)_{l-n}}
- \frac{(n-m+2)(n-l)_{l-n}}{(2l+m-n-1)(n-m-2l+2)_{l-n}}
$$

$$
\begin{aligned}
&= \frac{(2l+1)(l-n)!(2l)!(l+m-2)!(n-m+2)!}{(2l+m-n-1)(l+n)!(l-m+2)!(2l+m-n-2)!} \\
&\quad - \frac{(n-m+2)(l-n)!(l+m-2)!}{(2l+m-n-1)(2l+m-n-2)!} \\
&= \frac{(l-n)!(l+m-2)!(n-m+2)!}{(2l+m-n-1)!} \\
&\quad \times \left[\frac{(2l+1)!}{(l+n)!(l-m+2)!} - \frac{1}{(n-m+1)!}\right].
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\sum_{k=0}^{l-n} (-1)^k \frac{\binom{l+m-2}{k}\binom{l-m+2}{n-m+k+2}}{\binom{2l}{n-m+k+1}} \\
&\quad = \frac{(l+m-2)!(n-m+1)!(l-n+1)!}{(2l)!} \\
&\quad \times \left[\frac{(2l+1)!}{(l+n)!(l-n+1)!} - \frac{(l-m+2)!}{(l-n+1)!(n-m+1)!}\right] \\
&\quad = \frac{(l+m-2)!(n-m+1)!(l-n+1)!}{(2l)!} \\
&\quad \times \left[\binom{2l+1}{l-n+1} - \binom{l-m+2}{l-n+1}\right],
\end{aligned}
$$

which is the desired result.

References

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