**THE SQUARE-WAVE SPECTRAL DENSITY
OF A STATIONARY RENEWAL PROCESS ***

Marcel F. NEUTS and H. SITARAMAN

Department of Systems and
Industrial Engineering
University of Arizona
Tucson Arizona 85721

ABSTRACT

The power spectral density of a random square wave is a promising tool in the qualitative study of stationary point processes. This is illustrated for renewal processes and their superpositions.

Key Words: square wave, power spectral density, renewal process, phase-type distributions, random telegraph wave.

This research was supported in part by Grants Nrs. ECS-8803061 from the National Science Foundation and AFOSR-88-0076 from the Air Force Office of Scientific Research.

AMS Subject Classification: 60G55, 60K05

1. Introduction

We shall discuss, for the case of stationary renewal processes, a construction which holds much promise for the qualitative study of stationary point processes. Specifically, we examine the power spectral density of a square wave, which flipflops at each event in the renewal process. That spectral density has useful properties related to various classical operations on point processes. Explicit and easily computable formulas are now available for any finite state stationary Markov renewal process and will be discussed elsewhere. The square-wave spectral density is an analytically tractable and readily interpretable physical descriptor, which encodes *global* information about the fluctuations inherent in a point process. While random square waves have been used for a variety of purposes in signal processing, the square-wave spectral density is not a common tool in the study of point processes. An example of its use in studying photographic granularity may be found in Castro, Kemperman and Trabka [1].

This paper, which deals with renewal processes and their superpositions, is the first of a continuing investigation of the practical utility of square-wave spectra. Matrix-analytic expressions for more involved cases have now been derived, but we are awaiting the completion of extensive numerical studies to validate qualitative inferences based on square-wave spectra.

Let P be a stationary renewal process on the entire real line R , with underlying probability distribution $F(\cdot)$ of (necessarily) finite mean μ_1' . We shall denote by $\phi(\cdot)$, the Laplace-Stieltjes transform of $F(\cdot)$ and, in order to avoid discussion of special issues, we assume that $F(0+) = 0$, and that F is not of lattice type. The time origin 0 is an *arbitrary* point with reference to the process. This means, in particular, that the positive renewal times

$$S_n = \sum_{k=1}^n X_k, \quad \text{for } n \geq 1,$$

are the sums of independent random variables X_k , which for $k > 1$, have the common distribution $F(\cdot)$, whereas

$$P\{X_1 \leq x\} = F^*(x) = \int_0^x (\mu_1')^{-1} [1 - F(u)] du, \quad \text{for } x \geq 0.$$

We also consider the *counting process* which is the random counting measure N induced by the stationary renewal process on the Borel sets of R . In what follows, it will be sufficient to consider the random variables

$$N\{[0, t]\} = N(t) = \inf \{n : S_n \leq t\}, \quad \text{for } t \geq 0.$$

The *random square wave* corresponding to the renewal process P is the stochastic process defined by

$$Y_1(0) = 1, \text{ or } 0, \text{ each with probability } \frac{1}{2},$$

and

$$Y_1(t) = \max \{ 0, (-1)^{Y_1(0) + N([0,t]) + 1} \}, \text{ for } t \text{ real.}$$

The process $\{Y_1(\cdot)\}$ is clearly stationary and $E[Y_1(t)] = \frac{1}{2}$ for all t . The *centered square wave* is simply the process $Y(t) = Y_1(t) - \frac{1}{2}$.

The *square wave power spectral density* (SQSD) of the stationary renewal process is defined by

$$S(f) = \int_{-\infty}^{\infty} e^{-i\omega\tau} R(\tau) d\tau, \quad (1)$$

where $\omega = 2\pi f$ and $R(\tau)$ is the autocovariance function

$$R(\tau) = E[Y(0)Y(\tau)],$$

of the stationary process $Y(\cdot)$. The function $R(\cdot)$ is clearly an *even* function and for $\tau \geq 0$, we have

$$R(\tau) = E[Y_1(0)Y_1(\tau)] - \frac{1}{4} = \frac{1}{2}P\{N(\tau) \text{ is even}\} - \frac{1}{4}. \quad (2)$$

Formula (2) holds for every stationary point process without multiple arrivals. In terms of the probability generating function

$$P^*(z, \tau) = E\{z^{N(\tau)}\}, \text{ for } \tau \geq 0,$$

we may further rewrite (2) as

$$R(\tau) = \frac{1}{4}P^*(-1, \tau). \quad (3)$$

The preceding considerations are well-known, at least for the Poisson process [2]. For a Poisson process of rate λ , one easily obtains that

$$R(\tau) = \frac{1}{4} \exp\{-2\lambda|\tau|\}, \quad \text{for all real } \tau, \quad (4)$$

so that the *SQSD* is given by the Cauchy “density”

$$S(f) = \frac{\lambda}{4} [\lambda^2 + \pi^2 f^2]^{-1}. \quad (5)$$

For a general stationary renewal process, the analogous result is given by the following theorem:

Theorem 1: For a stationary non-lattice renewal process, the *SQSD* is given by

$$S(f) = \frac{1}{4\mu_1' \pi^2 f^2} \frac{1 - |\phi(i2\pi f)|^2}{1 + |\phi(i2\pi f)|^2 + 2\operatorname{Re} \phi(i2\pi f)}, \quad (6)$$

for all real f .

Proof: We have that

$$\begin{aligned} P\{N(\tau) = n\} &= 1 - F^*(\tau), \quad \text{for } n = 0, \\ &= F^* * F^{(n-1)}(\tau) - F^* * F^{(n)}(\tau), \quad \text{for } n \geq 1, \end{aligned}$$

so that by routine calculations, we obtain

$$P^*(z, \tau) = 1 - (1 - z) \sum_{n=0}^{\infty} F^* * F^{(n)}(\tau) z^n.$$

The autocovariance function $R(\cdot)$ is therefore given by

$$R(\tau) = \frac{1}{4} - \frac{1}{2} \sum_{n=0}^{\infty} F^* * F^{(n)}(\tau) (-1)^n.$$

The Laplace transform $\rho(\cdot)$ of $R(\cdot)$ for nonnegative values of the argument is routinely calculated and is given by

$$\rho(s) = \int_0^{\infty} e^{-s\tau} R(\tau) d\tau = \frac{1}{4s} - \frac{1}{2\mu_1' s^2} \frac{1 - \phi(s)}{1 + \phi(s)}, \quad (7)$$

for all s with $s \neq 0$ where $1 + \phi(s)$ is defined and non-vanishing. In particular, formula (7) holds for all purely imaginary, non-zero s . Writing

$$\int_{-\infty}^{\infty} e^{-i\omega\tau} R(\tau) d\tau = \rho(i\omega) + \rho(-i\omega),$$

and substituting, we obtain formula (6). •

For the important class of *phase type distributions* [3], the *SQSD* may be evaluated without the use of complex arithmetic. Let $F(\cdot)$ be a *PH*-distribution with representation (α, T) , then upon substitution in (6) and elimination of the complex terms, we obtain that

$$S(f) = \frac{1}{4\mu_1'} \frac{2\psi_1(f)[1 - 2\pi^2 f^2 \psi_1(f)] - \psi_2^2(f)}{[1 - 2\pi^2 f^2 \psi_1(f)]^2 + \pi^2 f^2 \psi_2^2(f)}, \quad (8)$$

where

$$\psi_1(f) = \alpha[4\pi^2 f^2 I + T^2]^{-1} \mathbf{e}, \quad \text{and} \quad \psi_2(f) = \alpha[4\pi^2 f^2 I + T^2]^{-1} T^{\circ}.$$

π is the stationary probability vector of the irreducible infinitesimal generator $T + T^{\circ}\alpha$. The vector π is explicitly given by $\pi = (\mu_1')^{-1} \alpha(-T)^{-1}$.

An alternative expression for $S(f)$ is obtained by recalling that for the *PH*-renewal process [3],

$$P^*(z, \tau) = \pi \exp[(T + z T^{\circ} \alpha) \tau] \mathbf{e},$$

Substituting in (3) and evaluating the inverse Fourier transform, we obtain that

$$S(f) = \pi T^{\circ} \alpha [4\pi^2 f^2 I + (T - T^{\circ} \alpha)^2]^{-1} \mathbf{e} \quad (9)$$

$$= (\mu_1')^{-1} \alpha [4\pi^2 f^2 I + (T - T^{\circ} \alpha)^2]^{-1} \mathbf{e}.$$

The existence of the inverse in (9) is shown if we establish that the matrix $T - T^o\alpha$ cannot have a purely imaginary eigenvalue. If we suppose, to the contrary, that for some non-zero vector u , we have

$$u(T - T^o\alpha - i\omega I) = 0,$$

then routine calculations lead to

$$u = (uT^o) \alpha(T - i\omega I)^{-1},$$

and this in turn implies that

$$(uT^o) [1 + \phi^*(-\omega)] = 0,$$

where $\phi^*(\cdot)$ is the characteristic function of the probability distribution of phase type. Since $F(\cdot)$ is not a lattice distribution, the second factor does not vanish for real values of ω . On the other hand, if uT^o vanishes, so does the vector u , which leads to a contradiction.

Theorem 2: The SQSD $S(\cdot)$ of a stationary renewal process determines the underlying probability distribution $F(\cdot)$ uniquely.

Proof: By the uniqueness theorem of the Fourier transform, $S(\cdot)$ uniquely determines the autocovariance function $R(\cdot)$. Since that function is even, it is determined by the Laplace transform $\rho(s)$. Letting s tend to $0+$ in (7), we obtain that $\mu_1' = 8\rho(0+)$, and clearly, (7) now leads to a unique expression for the Laplace-Stieltjes transform $\phi(s)$ in terms of $\rho(s)$ and $\rho(0+)$. Finally, $\phi(s)$ determines $F(\cdot)$ uniquely. •

Remark: Formula (7) may be rewritten as

$$\rho(s) = \frac{1}{4s} - \frac{1}{2} \frac{M_1^2(s)}{M_2(s)}, \quad (10)$$

where

$$M_1(s) = (\mu_1' s^2)^{-1} = \int_0^\infty e^{-st} E[N(t)] dt,$$

and

$$M_2(s) = \int_0^{\infty} e^{-st} E[N^2(t)] dt = \frac{1}{\mu_1' s^2} \frac{1 + \phi(s)}{1 - \phi(s)},$$

This last formula may be found, for example, in Takács [4]. Formula (10), which does not have a readily discernible probabilistic interpretation, may be substituted in the remaining derivations in the proof of Theorem 1 to relate the *SQSD* to the first two moments of the counting variable $N(t)$ of the renewal process.

Remark: Formula (10) shows that the *SQSD*, the renewal function with a renewal at time 0 (the Palm measure) and the variance of $N(t)$ express physical information about the stationary renewal process in mathematically equivalent forms. This remains the case for general stationary processes for the second and third descriptors, but not for the *SQSD*. In general, the *SQSD* contains information about the point process that is not implicit in the Palm measure and this is the principal motivation for our enquiry into the properties of that descriptor of point processes.

2. Superpositions

The *SQSD* is an important tool in studying the approach of superpositions of independent stationary (renewal) processes to the limiting Poisson process, as assured by the Palm-Khinchin theorem. The following theorem holds for stationary point processes with single arrivals:

Theorem 3: The square wave spectral density of the superposition of a finite number N of independent stationary renewal processes is 4^{N-1} times the *convolution* of the corresponding densities of the component processes.

Proof: By independence, the probability generating function $P^*(z, \tau)$ for the superposition is clearly given by the product $P_1^*(z, \tau) \cdot \dots \cdot P_N^*(z, \tau)$ of the generating functions for the component processes. By the same considerations as led to formula (3), we get that the autocovariance function

$$R(\tau; N) = \frac{1}{4} P^*(-1, \tau) = 4^{N-1} R_1(\tau) \cdot \dots \cdot R_N(\tau),$$

and the statement follows upon Fourier inversion. •

In generating numerical examples to illustrate the use of Theorem 3, we evaluated the autocovariance function $R_1(\cdot)$ for renewal processes with various underlying distributions, so as to examine the superposition under varying degrees of “burstiness” of (identical) component processes. Thereupon, we computed the autocovariance function

$$R(\tau; N) = 4^{N-1} [R_1(\tau)]^N,$$

for a large number of τ -values and performed a Fast Fourier inversion to obtain the *SQSD* of the superposition of N independent, identically distributed renewal processes. In Figures 1-7, we present graphs of the power spectral densities for several examples. In each graph the *SQSD* of the Poisson process of the same rate as the superposition was plotted for comparison.

In Figures 1-3, which correspond respectively to superpositions of 10, 20 and 30 renewal processes with an underlying Erlang distribution of order 8 and mean one, we see that convergence of the power spectral densities is rapid. This suggests that “morphological” differences between a superposition of such renewal processes and a Poisson process disappear rapidly.

An entirely different behavior is suggested by Figures 4-7, where the underlying distribution is hyper-exponential, so that the component processes are bursty. It is well-known, that such renewal processes are equivalent to “interrupted Poisson processes”. The significant differences in the spectral densities for low values of f reflect the greater prevalence of longer gaps in the superposition than in a Poisson process of the same rate. For many applications, the large differences in the high frequencies are, however, of greater consequence. We believe that they reflect the effect of increased overlapping of bursts in the component processes. The superposition of even a large number of bursty renewal processes exhibits significant local random fluctuations which are not captured by a Poisson approximation of the same rate. For many applications, the physical implications of this observation should not be overlooked.

It should be stressed that we have not matched means or rates between the first and second sets of examples, so that no inferences are made about differences between Erlang and hyper-exponential renewal processes of the same rate. Such comparisons are possible (and have been done) but a detailed discussion with full computational justification of these inferences is too lengthy to be included here.

Acknowledgements

The authors thank Dr. Bernard Lamond of the University of Arizona for many fruitful discussions and for assistance with software development. Reference [1] was brought to their attention by Dr. Teunis Ott.

References

- [1] Castro, P. E., Kemperman J. H. B. and Trabka, E. A. Alternating Renewal Model of Photographic Granularity. *J. Opt. Soc. America*, 63, 820-825, 1973.
- [2] Davenport Jr., W. B. and Root, W. L. *An Introduction to the Theory of Random Signals and Noise*. New York: McGraw Hill Book Company, 1958.
- [3] Neuts, M. F. *Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach*. Baltimore: The Johns Hopkins University Press, 1981.
- [4] Takács, L. *Introduction to the Theory of Queues*. New York: Oxford University Press, 1962.

Captions for the Figures:

Fig 1: Square wave spectra of the superposition of 10 stationary Erlang processes and a Poisson process of the same rate.

Fig 2: Square wave spectra of the superposition of 20 stationary Erlang processes and a Poisson process of the same rate.

Fig 3: Square wave spectra of the superposition of 30 stationary Erlang processes and a Poisson process of the same rate.

Fig 4: Square wave spectra of the superposition of 25 stationary hyper-exponential processes and a Poisson process of the same rate.

Fig 5: Square wave spectra of the superposition of 50 stationary hyper-exponential processes and a Poisson process of the same rate.

Fig 6: Square wave spectra of the superposition of 75 stationary hyper-exponential processes and a Poisson process of the same rate.

Fig 7: Square wave spectra of the superposition of 100 stationary hyper-exponential processes and a Poisson process of the same rate.