Solutions of Nonlinear Integrodifferential Equations of Mixed Type in Banach Spaces*

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ABSTRACT

In this paper, the author combines the topological degree theory and the monotone iterative technique to investigate the existence of solutions and also minimal and maximal solutions of the initial value problem for nonlinear integrodifferential equations of mixed type in Banach space. Two main theorems are obtained and two examples are given.

Key words: Integrodifferential equations, topological degree theory, monotone iterative technique, measure of noncompactness

AMS subject classifications: 45J, 47G.

1. INTRODUCTION

Consider the IVP of the nonlinear integrodifferential equation of mixed type in the Banach space

(1) $$u' = f(t,u,Tu,Su), \quad u(0) = u_0,$$

where $f \in C[I \times E \times E \times E, E]$, E is a real Banach space, $I = [0,a]$ with $a > 0$, $u_0 \in E$ and

(2) $$Tu(t) = \int_0^t k(t,s)\, u(s)ds, \quad Su(t) = \int_0^a h(t,s)\, u(s)ds.$$

In (2), $k \in C[D,R^1]$ and $h \in C[D_0,R^1]$, where $D = \{(t,s) \in R^2 \mid 0 \le s \le t \le a\}$ and $D_0 = \{(t,s) \in R^2 \mid 0 \le t,s \le a\}$. In the special case where f does not contain Tu and Su, the minimal and maximal solutions of (1) have been obtained by means of the monotone iterative technique in [1]. But, it is easy to see that the monotone iterative technique is not successful in the general case. Therefore, in this paper, we shall combine the topological

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degree theory and the monotone iterative technique to investigate the existence of solutions and also minimal and maximal solutions of the IVP (1). Two main theorems are obtained and two examples are given.

2. A FUNDAMENTAL LEMMA

Consider the nonlinear integral operator

$$Au(t) = p(t)\, u_0 + \int_0^t F(t,s,u(s),Tu(s),\, Su(s))ds, \tag{3}$$

where $p \in C[I,R^1]$ and $F \in C[D\times E\times E\times E,E]$. Let $k_0 = \max\{\, |k(t,s)| \mid (t,s) \in D\}$ and $h_0 = \max\{|h(t,s)| \mid (t,s) \in D_0\}$.

LEMMA 1. Let F be uniformly continuous on $D\times B_R\times B_R\times B_R$ for any $R > 0$, where $B_R = \{x\in E \mid \|x\| \le R\}$. Suppose that there exist $L_i > 0$ (i=1,2,3) with

$$a\left(L_1 + ak_0 L_2 + ah_0 L_3\right) < \frac{1}{2} \tag{4}$$

such that

$$\alpha(F(t,s,U,V,W)) \le L_1\, \alpha(U) + L_2\, \alpha(V) + L_3\, \alpha(W) \tag{5}$$

for any bounded $U,V,W \subset E$ and $(t,s) \in D$, where α denotes the Kuratowski measure of noncompactness. Then, A is a strict set contraction from $C[I,E]$ into $C[I,E]$, i.e. A is bounded and continuous from $C[I,E]$ into $C[I,E]$ and there exists a real number r strictly between 0 and 1 such that $\alpha(A(Q)) \le r\, \alpha(Q)$ for any bounded $Q \subset C[I,E]$.

PROOF. It is easy to see that the uniform continuity of F on $D\times B_R\times B_R\times B_R$ implies the boundedness of F on $D\times B_R\times B_R\times B_R$, and so A is a bounded and continuous operator from $C[I,E]$ into $C[I,E]$. The uniform continuity of F and (5) imply (see [2] Lemma 1.4.1)

$$\begin{aligned}\alpha(F(t,[0,t],U,V,W)) &= \sup\left\{\alpha(F(t,s,U,V,W)) \mid 0 \le s \le t\right\}\\ &\le L_1\, \alpha(U) + L_2\, \alpha(V) + L_3\, \alpha(W)\end{aligned} \tag{6}$$

for any bounded $U,V,W \subset E$. Let $Q \subset C[I,E]$ be bounded, i.e. there exists $R_0 > 0$ such that $\|u\|_C = \max\{\|u(t)\| \mid t \in I\} \le R_0$ for any $u \in Q$. By the uniform continuity and boundedness of F on $D\times B_R\times B_R\times B_R$ with $R = \max\{\, R_0, ak_0\, R_0, ah_0\, R_0\}$, we can easily show that the functions $\{Au \mid u \in Q\}$ are uniformly bounded and equicontinuous; hence (see [2] Lemma 1.4.1)

$$\alpha(A(Q)) = \sup\left\{\alpha(A(Q(t))) \mid t \in I\right\}, \tag{7}$$

where $A(Q(t)) = \{Au(t) \mid u \in Q, t \text{ is fixed}\}$. Using formula

$$\int_I x(t)dt \in |I| \ \overline{co}\{x(t) \mid t \in I\} \quad \text{for } x \in C[I,E]$$

and observing (6), we get

$$\begin{aligned} \alpha(A(Q(t))) &= \alpha\left(\left\{\int_0^t F(t,s,u(s),Tu(s),Su(s))\,ds \mid u \in Q\right\}\right) \\ &\le t\,\alpha\left(\overline{co}\left\{F(t,s,u(s),Tu(s),Su(s)) \mid s \in [0,t],\ u \in Q\right\}\right) \\ &= t\,\alpha\left(\left\{F(t,s,u(s),Tu(s),Su(s)) \mid s \in [0,t],\ u \in Q\right\}\right) \\ &\le t\,\alpha(F(t,[o,t],U,V,W)) \\ &\le t\left(L_1\,\alpha(U) + L_2\,\alpha(V) + L_3\,\alpha(W)\right), \end{aligned} \tag{8}$$

where $U = \{u(s) \mid s \in I, u \in Q\}$, $V = \{Tu(s) \mid s \in I, u \in Q\}$ and $W = \{Su(s) \mid s \in I, u \in Q\}$. Since $Tu(s) \in s\,\overline{co}\{k(s,y)\,u(y) \mid y \in [0,s]\}$, we have

$$\begin{aligned} V &\subset \overline{co}\{s\,k(s,y)\,u(y) \mid y \in [0,s], s \in I, u \in Q\} \\ &\subset \overline{co}\{ak_0 u(y), \theta, -ak_0\,u(y) \mid y \in I, u \in Q\}, \end{aligned}$$

where θ denotes the zero element of E, and consequently

$$\begin{aligned} \alpha(V) &\le \alpha\left(\left\{ak_0\,u(y), \theta, -ak_0\,u(y) \mid y \in I, u \in Q\right\}\right) \\ &= ak_0\,\alpha\left(\left\{u(y) \mid y \in I, u \in Q\right\}\right) \\ &= ak_0\,\alpha(U). \end{aligned} \tag{9}$$

Similarly, we find

$$\alpha(W) \le ah_0\,\alpha(U). \tag{10}$$

It follows from (7) - (10) that

$$\alpha(A(Q)) \le a\left(L_1 + ak_0\,L_2 + ah_0\,L_3\right)\alpha(U). \tag{11}$$

On the other hand, for given $\varepsilon > 0$, we can find a partition $Q = \bigcup_{i=1}^{n} Q_i$, satisfying

(12) $$\operatorname{diam}(Q_i) < \alpha(Q) + \varepsilon,\ i = 1,2,\ldots,n\ .$$

Choosing $u_i \in Q_i$ ($i = 1,2,\ldots,n$) and a partition $0 = t_0 < t_1 < \ldots < t_{j-1} < t_j < \ldots < t_m$ such that

(13) $$\| u_i(t) - u_i(s) \| < \varepsilon \text{ for } i = 1,2,\ldots,n\ , \text{ and } t,s \in I_j = [t_{j-1}, t_j],\quad j = 1,2,\ldots,m\ .$$

Clearly, $U = \bigcup_{j=1}^{m} \bigcup_{i=1}^{n} B_{ij}$, where $B_{ij} = \{u(s) \mid u \in Q_i\ , s \in I_j\}$. For any two $x,y \in B_{ij}$, we have $x = u(t)$ and $y = v(s)$ for some $u,v \in Q_i$ and $t,s \in I_j$. It follows from (12) and (13) that

$$\begin{aligned} \| x-y \| &\le \| u(t) - u_i(t) \| + \| u_i(t) - u_i(s) \| + \| u_i(s) - v(s) \| \\ &\le \| u - u_i \|_C + \varepsilon + \| u_i - v \|_C \\ &\le 2 \operatorname{diam}(Q_i) + \varepsilon < 2\,\alpha(Q) + 3\varepsilon. \end{aligned}$$

Consequently,

$$\operatorname{diam}(B_{ij}) \le 2\alpha(Q) + 3\varepsilon\ , i = 1,2,\ldots,n \text{ and } j = 1,2,\ldots,m\ ,$$

and therefore

$$\alpha(U) \le 2\alpha(Q) + 3\varepsilon\ ,$$

which implies

(14) $$\alpha(U) \le 2\alpha(Q),$$

since ε is arbitrary. Finally, it follows from (11), (14) and (4) that $\alpha(A(Q)) \le r\,\alpha(Q)$ with $r = 2a\,(L_1 + ak_0 L_2 + ah_0 L_3) < 1$. This shows that A is a strict set contraction and the Lemma is proved.

3. MAIN THEOREMS

THEOREM 1. Let f be uniformly continuous on $I \times B_R \times B_R \times B_R$ for any $R > 0$. Assume that there exist $L_i > 0$ ($i = 1,2,3$), which satisfy (4), such that

(15) $$\alpha(f(t,U,V,W)) \le L_1\,\alpha(U) + L_2\,\alpha(V) + L_3\,\alpha(W)$$

for any bounded $U,V,W \subset E$ and $t \in I$. Assume further

(16) $$\overline{\lim_{R\to+\infty}} \frac{M(R)}{R} < (aa_0)^{-1},$$

where $M(\mathrm{R}) = \sup \{ \| f(t,u,v,w) \| \mid (t,u,v,w) \in \mathrm{I} \times \mathrm{B}_{\mathrm{R}} \times \mathrm{B}_{\mathrm{R}} \times \mathrm{B}_{\mathrm{R}} \}$ and $a_0 = \max \{1, ak_0, ah_0\}$. Then IVP (1) has at least one solution in $\mathrm{C}^1[\mathrm{I,E}]$.

PROOF. It is well known that u is a solution of IVP (1) in $\mathrm{C}^1[\mathrm{I,E}]$ iff u is a solution in C[I,E] of the following integral equation:

$$u(t) = u_0 + \int_0^t f(s,u(s),Tu(s),Su(s))\, ds. \tag{17}$$

Let

$$Au(t) = u_0 + \int_0^t f(s,u(s),Tu(s),Su(s))\, ds. \tag{18}$$

By virtue of Lemma 1, A is a strict set contraction from C[I,E] into C[I,E]. On the other hand, (16) implies the existence of a real r strictly between 0 and $(aa_0)^{-1}$ and $\mathrm{R}_0 > 0$ such that

$$\frac{M(\mathrm{R})}{\mathrm{R}} < r \quad \text{for } \mathrm{R} \geq a_0 \mathrm{R}_0. \tag{19}$$

Let $\mathrm{R}^* = \max \{\mathrm{R}_0, \|u_0\|(1 - aa_0 r)^{-1}\}$. Then, for $u \in \mathrm{C}[\mathrm{I,E}]$ and $\|u\|_{\mathrm{C}} \leq \mathrm{R}^*$, we have

$$\|Tu\|_{\mathrm{C}} \leq ak_0 \|u\|_{\mathrm{C}} \leq ak_0 \mathrm{R}^*, \quad \|Su\|_{\mathrm{C}} \leq ah_0 \|u\|_{\mathrm{C}} \leq ah_0 \mathrm{R}^*,$$

and therefore, by (18) and (19),

$$\|Au\|_{\mathrm{C}} \leq \|u_0\| + a\, M(a_0 \mathrm{R}^*) < \|u_0\| + a\, a_0 \mathrm{R}^* r \leq \mathrm{R}^*.$$

Hence, by the Darbo fixed point theorem (see [3]), A has a fixed point in the ball $\{u \in \mathrm{C}[\mathrm{I,E}] \mid \|u\|_{\mathrm{C}} \leq \mathrm{R}^*\}$.

EXAMPLE 1. Consider the IVP of the infinite system of sublinear integrodifferential equations

$$\begin{aligned} u_n' &= \frac{3t^2}{n}\left(u_{n+1} + 2u_{2n} \int_0^t e^{ts} \sin(t-2s)\, u_n(s)\, ds - 1 \right)^{\frac{1}{3}} \\ &\quad - \frac{2}{n+1}\left(u_n^2 \int_0^1 \cos \pi (t-s)\, u_{n+1}(s)\, ds \right)^{\frac{1}{5}}, \quad 0 \leq t \leq 1; \\ u_n(0) &= \frac{1}{\sqrt{n}} \quad (n = 1,2,3...). \end{aligned} \tag{20}$$

Then IVP (20) has at least one continuously differentiable solution $\{u_1(t),u_2(t),\ldots,u_n(t),\ldots\}$ such that $u_n(t) \to 0$ as $n \to \infty$ for and $t \in [0,1]$. To show this, we let $a = 1$ and $E = c_0 = \{u = (u_1,u_2,\ldots,u_n,\ldots) \mid u_n \to 0\}$ with norm $\|u\| = \sup_n |u_n|$. Then, system (20) can be regarded as an equation of the form (1), where $u_0 = (1,\frac{1}{\sqrt{2}},\ldots,\frac{1}{\sqrt{n}},\ldots)$, $k(t,s) = e^{ts}\sin(t-2s)$, $h(t,s) = \cos\pi(t-s)$, $u = (u_1,u_2,\ldots,u_n,\ldots)$, $v = (v_1,v_2,\ldots,v_n,\ldots)$, $w = (w_1,w_2,\ldots,w_n,\ldots)$ and $f = (f_1,f_2,\ldots,f_n,\ldots)$, in which

$$f_n(t,u,v,w) = \frac{3t^2}{n}(u_{n+1} + 2u_{2u}v_n - 1)^{\frac{1}{3}} - \frac{2}{n+1}(u_n^2 w_{n+1})^{\frac{1}{5}}, \quad (n = 1,2,3,\ldots). \tag{21}$$

On account of (21), we have

$$|f_n(t,u,v,w)| \le \frac{3}{n}(\|u\| + 2\|u\|\cdot\|v\| + 1)^{\frac{1}{3}} + \frac{2}{n+1}\|u\|^{\frac{2}{5}}\cdot\|w\|^{\frac{1}{5}}, \quad (t \in I,\ n = 1,2,3,\ldots), \tag{22}$$

and so

$$\|f(t,u,v,w)\| \le 3(\|u\| + 2\|u\|\cdot\|v\| + 1)^{\frac{1}{3}} + \|u\|^{\frac{2}{5}}\cdot\|w\|^{\frac{1}{5}}. \tag{23}$$

Hence

$$M(R) \le 3(R + 2R^2 + 1)^{\frac{1}{3}} + R^{\frac{3}{5}},$$

and consequently,

$$\lim_{R\to+\infty}\frac{M(R)}{R} = 0. \tag{24}$$

This implies that (16) is satisfied since $1 < k_0 < e$, $h_0 = 1$ and $a_0 = k_0$. Obviously, f is uniformly continuous on $I \times B_R \times B_R \times B_R$ for any $R > 0$ and, by virtue of (22), it is easy to show that the set $f(t,U,V,W)$ is relatively compact in $E = c_0$ for any bounded $U,V,W \subset E$ and $t \in I = [0,1]$; and therefore (15) and (4) are satisfied for $L_1 = L_2 = L_3 = 0$. Hence, our conclusion follows from Theorem 1.

In the following, let P be a cone in E, and then P define a partial order in $E : x \le y$ iff $y - x \in P$ (see [4]). $u \in C^1[I,E]$ is called a lower (upper) solution of (1) if $u' \le f(t,u,Tu,Su)$ for $t \in I$ and $u(0) \le u_0$ ($u' \ge f(t,u,Tu,Su)$ for $t \in I$ and $u(0) \ge u_0$).

THEOREM 2. Let f be uniformly continuous on $I \times B_R \times B_R \times B_R$ for any $R > 0$, $k(t,s) \geq 0$ for $(t,s) \in D$ and $h(t,s) \geq 0$ for $(t,s) \in D_0$. Let cone P be normal and $y_0(t)$ and $z_0(t)$ be lower and upper solutions of (1) respectively with $y_0(t) \leq z_0(t)$ and $t \in I$. Suppose that there exist $M > 0$ and $L_i > 0$ $(i = 1,2,3,...)$ with

(25) $$a\,(M + L_1 + ak_0 L_2 + ah_0 L_3) < \frac{1}{2}$$

such that

(26) $$f(t,u,v,w) - f(t,\bar{u},\bar{v},\bar{w}) \geq -M(u-\bar{u})$$

$$\text{for } t \in I,\ y_0(t) \leq \bar{u} \leq u \leq z_0(t),\ Ty_0(t) \leq \bar{v} \leq v \leq Tz_0(t),$$

$$Sy_0(t) \leq \bar{w} \leq w \leq Sz_0(t), \text{ and}$$

(27) $$\alpha(f(t,U,V,W)) \leq L_1\, \alpha(U) + L_2\, \alpha(V) + L_3\, \alpha(W)$$

for any bounded $U,V,W \subset E$ and $t \in I$. Then IVP (1) has minimal solution $u_*(t)$ and maximal solution $u^*(t)$ in $[y_0,z_0]$; and $y_n(t) \to u_*(t)$ and $z_n(t) \to u^*(t)$ as $n \to \infty$ uniformly in $t \in I$, where

(28) $$y_n(t) = u_0\, e^{-Mt} + \int_0^t e^{-M(t-s)} \left[f(s,y_{n-1}(s),Ty_{n-1}(s),Sy_{n-1}(s)) + My_{n-1}(s)\right] ds$$

and

(29) $$z_n(t) = u_0\, e^{-Mt} + \int_0^t e^{-M(t-s)} \left[f(s,z_{n-1}(s),Tz_{n-1}(s),Sz_{n-1}(s)) + Mz_{n-1}(s)\right] ds,$$

$$n = 1,2,3, \ldots ,$$

which satisfy

(30) $$\begin{aligned} &y_0(t) \leq y_1(t) \leq \ldots \leq y_n(t) \leq \ldots \leq u_*(t) \leq u^*(t) \leq \ldots \\ &\leq z_n(t) \leq \ldots \leq z_1(t) \leq z_0(t),\ t \in I. \end{aligned}$$

PROOF. For any $x \in [y_0,z_0] \subset C[I,E]$, it is easy to see that the linear IVP

(31) $$u' = f(t,x,Tx,Sx) - M(u-x),\quad u(0) = u_0$$

has an unique solution in $C^1[I,E]$ given by

(32) $$u(t) = u_0\, e^{-Mt} + \int_0^t e^{-M(t-s)} (f(s,x(s),Tx(s),Sx(s)) + Mx(s))\, ds.$$

Define operator A by

$$Ax(t) = u_0\, e^{-Mt} + \int_0^t e^{-M(t-s)}(f(s,x(s),Tx(s),Sx(s)) + Mx(s))\, ds. \tag{33}$$

It is evident that u is a solution of (1) (in $C^1[I,E]$) iff u is a fixed point of A in $C[I,E]$. By virtue of (26), we see that $y_0 \le x_1 \le x_2 \le z_0$ implies that

$$f(t,x_1(t),Tx_1(t),Sx_1(t)) + Mx_1(t) \le f(t,x_2(t),Tx_2(t),Sx_2(t)) + Mx_2(t), \quad t \in I;$$

and therefore, observing (33), we have that A is a nondecreasing operator from $[y_0,z_0]$ into $C[I,E]$.

Now, let $y_1 = A\, y_0$ and $\bar{y} = y_1 - y_0$. Then

$$\begin{aligned}\bar{y}' = y_1' - y_0' &= f(t,y_0,Ty_0,Sy_0) - M(y_1 - y_0) - y_0' \\ &\ge f(t,y_0,Ty_0,Sy_0) - M\bar{y} - f(t,y_0,Ty_0,Sy_0) \\ &= -M\bar{y}, \quad t \in I;\end{aligned}$$

$$\bar{y}(0) = y_1(0) - y_0(0) \ge u_0 - u_0 = \theta; \text{ and}$$

$$(\bar{y}(t)\, e^{Mt})' = [\bar{y}'(t) + M\bar{y}(t)]\, e^{Mt} \ge \theta,\ t \in I.$$

And therefore

$$\bar{y}(t)\, e^{Mt} = \bar{y}(0) + \int_0^t \left(\bar{y}(s)\, e^{Ms}\right)' ds \ge \theta, \quad t \in I.$$

Hence, $\bar{y}(t) \ge \theta$, $t \in I$; i.e. $Ay_0 \ge y_0$. Similarly, we can show $Az_0 \le z_0$.

On the other hand, let

$$p(t) = e^{-Mt} \quad \text{and} \quad F(t,s,u,v,w) = e^{-M(t-s)} f_1(s,u,v,w),$$

where $f_1(s,u,v,w) = f(s,u,v,w) + Mu$. Then, F is uniformly continuous on $D \times B_R \times B_R \times B_R$ for any $R > 0$ and, observing (27) and the fact that $0 < e^{-M(t-s)} \le 1$ for $(t,s) \in D$, we have

$$\begin{aligned}\alpha(F(t,s,U,V,W)) &\le \alpha\left(\overline{co}\{f_1(s,U,V,W),\theta\}\right) \\ &= \alpha\left(f_1(s,U,V,W)\right) \\ &\le (L_1 + M)\,\alpha(U) + L_2\,\alpha(V) + L_3\,\alpha(W)\end{aligned}$$

for any bounded $U,V,W \subset E$ and $(t,s) \in D$. This inequality together with (25) implies, by Lemma 1, that A is a strict set contraction from $C[I,E]$ into $C[I,E]$. Finally, our conclusions follow from a fixed point theorem due to Amann (see [5], Theorem 3).

EXAMPLE 2. Consider the IVP of the infinite system of superlinear integrodifferential equations

$$u'_n = \frac{1}{8n}\left[(t-u_n)^3 + u^4_{2n+1}\right] + \frac{5}{\sqrt{n}}\left[\int_0^t \sin^2(t-3s)\, u_{2n}(s)\, ds\right]^2$$

(34)
$$+ \frac{1}{2n}\left[\int_0^1 e^{ts} u_{n+1}(s)\, ds\right]^3, \quad 0 \le t \le 1;$$

$$u_n(0) = 0 \quad (n = 1,2,3,\ldots).$$

Then IVP (34) has minimal and maximal continuously differentiable solutions satisfying $0 \le u_n(t) \le \frac{t}{n}$ $(t \in [0,1],\ n = 1,2,3,\ldots)$.

To show this let $a = 1$, $E = c_0 = \{u = (u_1, u_2, u_3, \ldots, u_n, \ldots) \mid u_n \to 0\}$ with norm $\|u\| = \sup_n |u_n|$ and $P = \{u = (u_1, u_2, \ldots, u_n, \ldots) \in c_0 \mid u_n \ge 0,\ n = 1,2,3,\ldots\}$. Then, P is a normal cone in c_0 and system (34) can be regarded as an equation of the form (1). In this situation $u_0 = (0,0,\ldots,0,\ldots)$, $k(t,s) = \sin^2(t-3s)$, $h(t,s) = e^{ts}$, $u = (u_1, u_2, \ldots, u_n, \ldots)$, $v = (v_1, v_2, \ldots, v_n, \ldots)$, $w = (w_1, w_2, \ldots, w_n, \ldots)$ and $f = (f_1, f_2, \ldots f_n, \ldots)$, in which

(35)
$$f_n(t,u,v,w) = \frac{1}{8n}\left[(t-u_n)^3 + u^4_{2n+1}\right] + \frac{5}{\sqrt{n}} v^2_{2n} + \frac{1}{2n} w^3_{n+1}$$
$$(n = 1,2,3,\ldots).$$

Equalities in (35) imply

(36)
$$|f_n(t,u,v,w)| \le \frac{1}{8n}\left[(t + \|u\|)^3 + \|u\|^4\right] + \frac{5}{\sqrt{n}}\|v\|^2 + \frac{1}{2n}\|w\|^3$$
$$(t \in I,\ I = [0,1],\ n = 1,2,3,\ldots).$$

It is clear that f is uniformly continuous on $I \times B_R \times B_R \times B_R$ for any $R > 0$; and by virtue of (35) and (36), we can easily prove that $f(t,U,V,W)$ is relatively compact in $E = c_0$ for any bounded $U,V,W \subset E$ and $t \in I$. Hence, (27) is satisfied for $L_1 = L_2 = L_3 = 0$. Let $y_0(t) = (0,0,\ldots,0,\ldots)$ and $z_0(t) = (t, \frac{t}{2}, \ldots, \frac{t}{n}, \ldots)$ for $t \in I$. Then $y_0(t) \le z_0(t)$, $t \in I$.

$$y_0(0) = z_0(0) = (0,0, \dots ,0, \dots) = u_0 \text{ and}$$
$$y_0'(t) = (0,0, \dots ,0, \dots),\ t \in \text{I}.$$
$$z_0'(t) = (1,\tfrac{1}{2}, \dots ,\tfrac{1}{n}, \dots),\ t \in \text{I}.$$

$$f_n\Big[t, y_0(t), Ty_0(t), Sy_0(t)\Big] = \frac{t^3}{8n} \geq 0 \quad (t \in \text{I},\ n = 1,2,3, \dots).$$

$$\begin{aligned} f_n\Big[t, z_0(t), Tz_0(t), Sz_0(t)\Big] &= \frac{1}{8n}\left((t - \frac{t}{n})^3 + (\frac{t}{2n+1})^4\right) \\ &\quad + \frac{5}{\sqrt{n}}\left(\int_0^t \frac{s}{2n} \sin^2(t - 3s)\, ds\right)^2 \\ &\quad + \frac{1}{2n}\left(\int_0^1 \frac{se^{ts}}{n+1} ds\right)^3 \\ &< \frac{1}{4n} + \frac{5}{16n^2\sqrt{n}} + \frac{e^3}{16n(n+1)^3} \\ &< \frac{1}{n} \qquad (t \in \text{I},\ n = 1,2,3, \dots). \end{aligned}$$

Consequently, y_0 and z_0 are lower and upper solutions of (34) respectively.

When $t \in \text{I},\ y_0(t) \leq \bar{u} \leq u \leq z_0(t),\ Ty_0(t) \leq \bar{v} \leq v \leq Tz_0(t)$, and $Sy_0(t) \leq \bar{w} \leq w \leq Sz_0(t)$, i.e.

$$0 \leq \bar{u}_n \leq u_n \leq \frac{t}{n},$$

$$0 \leq \bar{v}_n \leq v_n \leq \int_0^t \frac{s}{n} \sin^2(t - 3s)\, ds \leq \frac{t^2}{2n}, \text{ and}$$

$$0 \leq \bar{w}_n \leq w_n \leq \int_0^1 \frac{s}{n} e^{ts}\, ds \leq \frac{e}{2n} \qquad (t \in \text{I},\ n = 1,2,3, \dots),$$

we have

$$
\begin{aligned}
f_n(t,u,v,w) - f_n(t,\bar{u},\bar{v},\bar{w}) &= \frac{1}{8n}\left[(t-u_n)^3 + u_{2n+1}^4 - (t-\bar{u}_n)^3 - \bar{u}_{2n+1}^4\right] \\
&\quad + \frac{5}{\sqrt{n}}(v_{2n}^2 - \bar{v}_{2n}^2) + \frac{1}{2n}(w_{n+1}^3 - \bar{w}_{n+1}^3) \\
&\geq \frac{1}{8n}\left[(t-u_n)^3 - (t-\bar{u}_n)^3\right] \\
&\geq -\frac{3}{8n}(u_n - \bar{u}_n) \\
&\geq -\frac{3}{8}(u_n - \bar{u}_n) \\
&\quad (n = 1,2,3, \ldots)
\end{aligned}
$$

$$\left(\text{since } \frac{\partial}{\partial s}(t-s)^3 = -3(t-s)^2 \geq -3 \quad \text{for } 0 \leq s \leq t,\ 0 \leq t \leq 1\right).$$

Consequently, (26) is satisfied for $M = \frac{3}{8}$. Moreover

$$a(M + L_1 + ak_0 L_2 + ah_0 L_3) = a M = \frac{3}{8} < \frac{1}{2};$$

this shows that (25) is also satisfied. Hence, our conclusion follows from Theorem 2.

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