Nonlinear Second Order System of Neumann Boundary Value Problems at Resonance*

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Abstract

Let $f : [0,\pi] \times \mathbf{R}^N \to \mathbf{R}^N$, $(N \geq 1)$ satisfy Caratheodory conditions, $e(x) \in L^1([0,\pi];\mathbf{R}^N)$. This paper studies the system of nonlinear Neumann boundary value problems

$$x''(t) + f(t, x(t)) = e(t), 0 < t < \pi,$$

$$x'(0) = x'(\pi) = 0.$$

This problem is at resonance since the associated linear boundary value problem

$$x''(t) = \lambda x(t), \quad 0 < t < \pi,$$

$$x'(0) = x'(\pi) = 0,$$

has $\lambda = 0$ as an eigenvalue. Asymptotic conditions on the nonlinearity $f(t, x(t))$ are offered to give existence of solutions for the nonlinear systems. The methods apply to the corresponding system of Lienard-type periodic boundary value problems.

Key words and phrases: Second-order system of Neumann boundary value problems, resonance at infinitely many eigenvalues, absence of L^∞-resonance, asymptotic resonance conditions, Fredholm operator

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1 Introduction

Let $\mathbf{R}^N$ denote the N-dimensional Euclidean space. For $x = (x_1, x_2, ..., x_N) \in \mathbf{R}^N$, and$y = (y_1, y_2, ..., y_N) \in \mathbf{R}^N$, let $|x| = \sqrt{x_1^2 + x_2^2 + ... + x_N^2}$, and $< x, y >= x_1y_1 + x_2y_2 + ... + x_Ny_N$ denote the Euclidean norm of x and the inner product of x and y in $\mathbf{R}^N$, respectively. Let

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$f = (f_1, f_2, ..., f_N) : [0,\pi] \times \mathbf{R}^N \to \mathbf{R}^N$ be a function satisfying Caratheodory's conditions, and let $e : [0,\pi] \to \mathbf{R}^N$ be a Lebesgue integrable function.

This paper is devoted to the study of systems of Neumann boundary value problems

$$-x''(t) + f(t, x(t)) = e(t), \quad 0 < t < \pi, \tag{1.1}$$

$$x'(0) = x'(\pi) = 0, \tag{1.2}$$

and

$$x''(t) + f(t, x(t)) = e(t), \quad 0 < t < \pi, \tag{1.3}$$

$$x'(0) = x'(\pi) = 0. \tag{1.4}$$

We obtain the existence of a solution for (1.1)-(1.2) when $\int_0^\pi e(t)dt = 0$ and when, for each $i = 1, 2, ..., N$, there exists a real number $r_i \geq 0$ such that

$$(i) \quad f_i(t,x)x_i \geq 0, \tag{1.5}$$

for a.e. $t \in [0,\pi]$ and all $x \in \mathbf{R}^N$ with $|x_i| \geq r_i$, and

$$(ii) \quad |f_i(t,x)| \leq \alpha_i(t), \tag{1.6}$$

for a.e. $t \in [0,\pi]$ and all $x \in \mathbf{R}^N$ with $|x_i| \leq r_i$. We give asymptotic conditions on the behavior of $x_i^{-1} f_i(t,x), i = 1, 2, ..., N$, at the first two eigenvalues 0 and 1 of the linear problem

$$x''(t) + \lambda x(t) = 0, \ 0 < t < \pi, \tag{1.7}$$

$$x'(0) = x'(\pi) = 0, \tag{1.8}$$

for the problem (1.3)-(1.4).

Our methods can be adapted and similar results obtained for Lienard's system of equations

$$\pm x''(t) + [\frac{d}{dt} gradF(x(t))] + f(t, x(t)) = e(t), \quad 0 < t < \pi, \tag{1.9}$$

$$x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0, \tag{1.10}$$

where $F : \mathbf{R}^N \to \mathbf{R}$ is in $C^2(\mathbf{R}^N, \mathbf{R}), f : [0, 2\pi] \times \mathbf{R}^N \to \mathbf{R}^N$ satisfies Caratheodory's conditions, and $e : [0, 2\pi] \to \mathbf{R}^N$ is Lebesgue integrable. The problem (1.9)-(1.10) was studied by Ianacci and Nkashama in [3], where they give sufficient non-resonance conditions for the existence of a solution. We provide in this paper sufficient resonance conditions for the existence of a solution for the problems (1.1)-(1.2) and (1.3)-(1.4) and accordingly for (1.9)-(1.10) in line with our remark above.

Our results and methods are inspired by the results of Gupta and Mawhin [2] for the problem (1.9)-(1.10) when $N = 1$. We present in Section 2 notations and definitions that we need in this paper. In Section 3 we present some lemmas that are extensions to systems of corresponding lemmas in [3]. We present in Section 4 our theorems giving the existence of solutions for the problems (1.1)-(1.2) and (1.3)-(1.4). Our conditions for the existence of solutions for (1.3)-(1.4) allow resonance at infinitely many eigenvalues of the linear problem (1.7)-(1.8). Finally, in Section 5 we present a theorem for the problem (1.3)-(1.4) sharpening the condition for resonance at infinitely many eigenvalues of the linear problem (1.7)-(1.8) in the absence of L^∞-resonance at the second eigenvalue $\lambda = 1$ of (1.7)-(1.8).

2 Notations and Definitions

Let $\mathbf{R}^N$, $N \geq 1$, denote the N-dimensional Euclidean space. For $x = (x_1, x_2, ..., x_n)$, let

$$|x| = (x_1^2 + x_2^2 + ... + x_N)^{1/2} \tag{2.1}$$

denote the Euclidean norm of x in $\mathbf{R}^N$; and for $x = (x_1, x_2, ..., x_N)$ and $y = (y_1, y_2, ..., y_N)$ in $\mathbf{R}^N$, let

$$< x, y >= \sum_{i=1}^{N} x_i y_i \tag{2.2}$$

denote the inner product of x and y in $\mathbf{R}^N$.

We shall use the following spaces:

(i) the Lebesgue spaces $L^p([0,\pi], \mathbf{R}^N), 1 \leq p \leq \infty$, with the norms defined by

$$||f||_{L^p_N} = [\sum_{i=1}^{N} (\frac{1}{\pi} \int_0^\pi |f_i|^p dt)^{2/p}]^{1/2}, \text{ for } 1 \leq p < \infty,$$

and

$$||f||_{L^\infty_N} = (\sum_{i=1}^{N} |f_i|^2_{L^\infty})^{1/2}, \text{ for } p = \infty;$$

(ii) the space of $C([0,\pi], \mathbf{R}^N)$ of continuous functions with its usual norm, the norm induced by the Lebesgue space $L^\infty([0,\pi], \mathbf{R}^N)$;

(iii) the Sobolev space $H^1([0,\pi], \mathbf{R}^N)$ defined by

$$H^1([0,\pi], \mathbf{R}^N) = \{x : [0,\pi] \to \mathbf{R}^N | x \text{ is absolutely continuous and } x' \in L^2([0,\pi], \mathbf{R}^N)\},$$

with the inner product defined by

$$(x, y)_{H^1_N} =< \frac{1}{\pi} \int_0^\pi x(t) dt, \frac{1}{\pi} \int_0^\pi y(t) dt > + \frac{1}{\pi} \int_0^\pi < x'(t), y'(t) > dt,$$

and the corresponding norm $|| \cdot ||_{H^1_N}$ defined by

$$||x||_{H^1_N} = (\frac{1}{\pi} \int_0^\pi |x'(t)|^2 dt + |\frac{1}{\pi} \int_0^\pi x(t) dt|^2)^{1/2};$$

(iv) the Sobolev space $\tilde{H}^1([0,\pi], \mathbf{R}^N)$ defined by

$$\tilde{H}^1([0,\pi], \mathbf{R}^N) = \{x \in H^1([0,\pi], \mathbf{R}^N) | \int_0^\pi x(t) dt = 0\}$$

with the norm induced by $H^1([0,\pi], \mathbf{R}^N)$; and

(v) the Sobolev space $W^{2,1}([0,\pi], \mathbf{R}^N)$ defined by

$$W^{2,1}([0,\pi], \mathbf{R}^N) = \{x : [0,\pi] \to \mathbf{R}^N | x \text{ and } x' \text{ absolutely continuous}\}$$

with the norm defined by

$$||x||_{W_N^{2,1}} = \sum_{j=0}^{2} ||x^{(j)}||_{L_N^1},$$

where $x^{(0)} \equiv x, x^{(1)} = x', x^{(2)} = x''$.

For the sake of simplicity in the notation of the space, we shall omit $\mathbf{R}^N$ when $N = 1$.

We note that for $x \in H^1([0,\pi];\mathbf{R}^N)$, $x = (x_1, x_2, ..., x_N)$ if and only if $x_i \in H^1[0,\pi]$, for $i = 1, 2, ..., N$. Also, every $x_i \in H^1[0,\pi]$ can be written in the form

$$x_i(t) = \bar{x}_i + \tilde{x}_i(t)$$

with $\tilde{x}_i \in \tilde{H}^1[0,\pi]$ and $\bar{x}_i = \frac{1}{\pi}\int_0^\pi x_i(t)dt$. Moreover,

$$||x_i||_{H^1} = (\bar{x}_i^2 + \frac{1}{\pi}\int_0^\pi (\tilde{x}_i'(t))^2 dt)^{1/2},$$

so that we have

$$||x||_{H_N^1} = (\sum_{i=1}^{N} ||x_i||_{H^1}^2)^{1/2}.$$

For $x = (x_1, x_2, ..., x_N) \in L^1[(0,\pi), \mathbf{R}^N]$, we write $\bar{x} = (\bar{x}_1, ..., \bar{x}_N)$, where $\bar{x}_i = \frac{1}{\pi}\int_0^\pi x_i(t)dt$, $i = 1, 2, ..., N$ and $\tilde{x} = x - \bar{x}$.

3 Technical Lemmas

Lemma 1 *Let* $\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_N) \in L^1([0,\pi], \mathbf{R}^N)$ *be such that for a.e.* $t \in [0,\pi]$,

$$\Gamma_i(t) \leq 1, \tag{3.1}$$

for $i = 1, 2, ..., N$ *with strict inequality holding on a subset of* $[0,\pi]$ *of positive measure. Then there exists a* $\delta = \delta(\Gamma) > 0$ *such that for all* $\tilde{x} \in \tilde{H}^1([0,\pi], \mathbf{R}^N)$ *with* $\tilde{x}'(0) = \tilde{x}'(\pi) = 0$,

$$\begin{aligned} B_\Gamma(\tilde{x}) &= \frac{1}{\pi}\int_0^\pi [|\tilde{x}'(t)|^2 - \sum_{i=1}^{N} \Gamma_i(t)\tilde{x}_i^2(t)]dt \\ &\geq \delta ||\tilde{x}||_{H_N^1}^2. \end{aligned} \tag{3.2}$$

Proof. Using (3.1), the method of expanding a scalar function $\tilde{x}_i \in \tilde{H}^1[0,\pi]$, with $\tilde{x}_i'(0) = 0$ and $\tilde{x}_i'(\pi) = 0$, into a cosine Fourier series, and Parseval's identities for $\tilde{x}_i$ and $\tilde{x}_i'$, we see that

$$\begin{aligned} B_\Gamma(\tilde{x}) &= \frac{1}{\pi}\int_0^\pi [|\tilde{x}'(t)|^2 - \sum_{i=1}^{N} \Gamma_i(t)\tilde{x}_i^2(t)]dt \\ &= \sum_{i=1}^{N} \frac{1}{\pi}\int_0^\pi [(\tilde{x}_i'(t))^2 - \Gamma_i(t)\tilde{x}_i^2(t)]dt \\ &\geq 0, \end{aligned} \tag{3.3}$$

for all $\tilde{x} \in \tilde{H}^1([0,\pi],\mathbf{R}^N)$ with $\tilde{x}'(0) = \tilde{x}'(\pi) = 0$. Moreover,

$$B_\Gamma(\tilde{x}) = 0, \tag{3.4}$$

if and only if

$$\tilde{x}(t) = A\cos t, \tag{3.5}$$

for some $A = (A_1, A_2, ..., A_N) \in \mathbf{R}^N$. But we then get from (3.4) and (3.5) that

$$0 = B_\Gamma(\tilde{x}) = \sum_{i=1}^{N} \frac{A_i^2}{\pi} \int_0^\pi (1 - \Gamma_i(t)) \cos^2 t dt;$$

so that by our assumption (3.1) on Γ_i we have $A_i = 0$ for every $i = 1, 2, ..., N$, and hence $\tilde{x} = 0$.

Let us next assume that the conclusion of the lemma is false. Then there exists a sequence $\{\tilde{x}_n\}$, $\tilde{x}_n \in \tilde{H}^1([0,\pi],\mathbf{R}^N)$, such that

$$\begin{aligned} B_\Gamma(\tilde{x}_n) &\to 0 \text{ as } n \to \infty, \\ \|\tilde{x}_n\|_{H_N^1} &= 1, \text{ for every } n = 1, 2, ... \end{aligned} \tag{3.6}$$

We may also assume, by going to a subsequence if necessary, that there exists an $\tilde{x} \in \tilde{H}^1([0,\pi],\mathbf{R}^N)$ such that

$$\begin{aligned} \tilde{x}_n &\to \tilde{x} \text{ weakly in } H^1([0,\pi],\mathbf{R}^N), \\ \tilde{x}_n &\to \tilde{x} \text{ in } C([0,\pi],\mathbf{R}^N). \end{aligned} \tag{3.7}$$

Using Theorem 5.2 in [4], we have that $B_\Gamma(\tilde{x}) \geq 0$, even though $\tilde{x}'(0)$ and $\tilde{x}'(\pi)$ may not be zero. Also, from (3.7) and the weak lower semicontinuity of the norm in $H^1([0,\pi],\mathbf{R}^N)$, we have that

$$\|\tilde{x}\|_{H_N^1} \leq \lim \inf_{n\to\infty} \|\tilde{x}_n\|_{H_N^1} = 1,$$

and hence

$$0 \leq B_\Gamma(\tilde{x}) \leq \lim \inf_{n\to\infty} B_\Gamma(\tilde{x}_n) = 0.$$

Thus $\tilde{x} = 0$, from the first part of the proof. We next see that

$$\begin{aligned} \frac{1}{\pi}\int_0^\pi |\tilde{x}_n'|^2 dt &= B_\Gamma(\tilde{x}_n) + \frac{1}{\pi}\sum_{i=1}^N \int_0^\pi \Gamma_i(t)|\tilde{x}_{ni}(t)|^2 dt \\ &\to \frac{1}{\pi}\sum_{i=1}^N \int_0^\pi \Gamma_i(t)|\tilde{x}_i(t)|^2 dt = \frac{1}{\pi}\int_0^\pi |\tilde{x}'(t)|^2 dt. \end{aligned}$$

Thus, $\tilde{x}_n \to \tilde{x}$ in $H^1([0,\pi],\mathbf{R}^N)$ and $\|\tilde{x}\|_{H_N^1} = 1$, which forms a contradiction. Hence the lemma. □

Lemma 2 *Let* $\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_N) \in L^1([0,\pi],\mathbf{R}^N)$ *and* $\Gamma_\alpha = (\Gamma_{\alpha_1}, ..., \Gamma_{\alpha_N}) \in L^1([0,\pi),\mathbf{R}^N)$. *Let* $\Gamma_\beta = (\Gamma_{\beta_1}, ..., \Gamma_{\beta_N}) \in L^1([0,\pi),\mathbf{R}^N)$ *and* $\Gamma_\infty = (\Gamma_{\infty_1}, ..., \Gamma_{\infty_N}) \in L^\infty([0,\pi],\mathbf{R}^N)$ *be such that*

$$(i) \qquad \Gamma = \Gamma_\alpha + \Gamma_\beta + \Gamma_\infty,$$

$$(ii) \quad \textit{for a.e. } t \in [0,\pi] \textit{ and every } i = 1, 2, ..., N, \Gamma_{\alpha_i}(t) \leq 1, \tag{3.8}$$

with strict inequality holding on a subset of $[0,\pi]$ *of positive measure,*

$$(iii)\quad \frac{\pi^2}{3}||\Gamma_\beta||_{L^1_N}+||\Gamma_\infty||_{L^\infty_N}<\delta(\Gamma_\alpha), \tag{3.9}$$

where $\delta(\Gamma_\alpha)>0$ *is given by Lemma 1.*

Then for every $\tilde{x}\in\tilde{H}^1([0,\pi],\mathbf{R}^N)$ *with* $\tilde{x}'(0)=\tilde{x}'(\pi)=0$,

$$B_\Gamma(\tilde{x})\geq[\delta(\Gamma_\alpha)-\frac{\pi^2}{3}||\Gamma_\beta||_{L^1_N}-||\Gamma_\infty||_{L^\infty_N}]||\tilde{x}||^2_{H^1_N}. \tag{3.10}$$

Proof. Using the fact that $H^1([0,\pi],\mathbf{R}^N)\subset C([0,\pi],\mathbf{R}^N)$ and the inequalities (see [8])

$$||\tilde{x}||_{L^2_N}\leq||\tilde{x}'||_{L^2_N}\leq||\tilde{x}||_{H^1_N},||\tilde{x}||_{L^\infty_N}\leq\frac{\pi}{\sqrt{3}}||\tilde{x}'||_{L^2_N}\leq\frac{\pi}{\sqrt{3}}||\tilde{x}||_{H^1_N}$$

for all $\tilde{x}\in\tilde{H}^1([0,\pi];\mathbf{R}^N)$, as well as Lemma 1, we see that

$$\begin{aligned}
B_\Gamma(\tilde{x}) &= \frac{1}{\pi}\int_0^\pi[|\tilde{x}'(t)|^2-\sum_{i=1}^N\Gamma_i(t)\tilde{x}_i^2(t)]dt\\
&= \frac{1}{\pi}\int_0^\pi[|\tilde{x}'(t)|^2-\sum_{i=1}^N\Gamma_{\alpha_i}(t)\tilde{x}^2(t)]dt\\
&\quad -\frac{1}{\pi}\sum_{i=1}^N\int_0^\pi[\Gamma_{\beta_i}(t)+\Gamma_{\infty_i}(t)]\tilde{x}_i^2(t)dt\\
&\geq \delta(\Gamma_\alpha)||\tilde{x}||^2_{H^1_N}-||\Gamma_\beta||_{L^1_N}||\tilde{x}||^2_{L^\infty_N}-||\Gamma_\infty||_{L^\infty_N}||\tilde{x}||^2_{L^2_N}\\
&\geq (\delta(\Gamma_\alpha)-\frac{\pi^2}{3}||\Gamma_\beta||_{L^1_N}-||\Gamma_\infty||_{L^\infty_N})||\tilde{x}||^2_{H^1_N}.
\end{aligned}$$

□

Definition 1 *For* $x=(x_1,x_2,...,x_N)$ *and* $y=(y_1,y_2,...,y_N)$ *in* $\mathbf{R}^N$, *we say* $x\leq y$ *if* $x_i\leq y_i$ *for every* $i=1,2,...,N$.

Lemma 3 *Let* $\gamma=(\gamma_1,\gamma_2,...,\gamma_N)\in L^1([0,\pi],\mathbf{R}^N)$ *and* $\Gamma=\Gamma_\alpha+\Gamma_\beta+\Gamma_\infty\in L^1([0,\pi],\mathbf{R}^N)$ *be as in Lemma 2, and let* $\delta(\Gamma_\alpha)$ *be given by Lemma 1. Then for all measurable functions* $p:[0,\pi]\to\mathbf{R}^N$ *such that* $\bar{\gamma}\leq\bar{p}$, $p(t)\leq\Gamma(t)$ *for a.e.* $t\in[0,\pi]$ *and all* $x\in W^{2,1}[(0,\pi),\mathbf{R}^N]$ *with* $x'(0)=x'(\pi)=0$,

$$\frac{1}{\pi}\int_0^\pi<\bar{x}-\tilde{x}(t),\tilde{x}''(t)+p^T(t)I\tilde{x}(t)>dt$$

$$\geq\eta|\bar{x}|^2+[\delta(\Gamma_\alpha)-\frac{\pi^2}{3}||\Gamma_\beta||_{L^1_N}-||\Gamma_\infty||_{L^\infty_N}]||\tilde{x}||^2_{H^1_N}. \tag{3.11}$$

Here $\eta=\min\{\bar{\gamma}_i|1\leq i\leq N\}$, *I denotes the* $N\times N$ *identity matrix,* $p^T(t)$ *denotes the transpose of the column vector col.*$\{p_1(t),p_2(t),...,p_N(t)\}$, *and all vectors are understood as column vectors for the purpose of matrix arithmetic.*

Proof. For $x = (x_1, x_2, ..., x_N) \in W^{2,1}([0,\pi], \mathbf{R}^N)$ with $x'(0) = x'(\pi) = 0$, we have (on integrating by parts and from Lemma 2) that

$$\begin{aligned}
\frac{1}{\pi}\int_0^\pi & < \bar{x} - \tilde{x}(t), \tilde{x}''(t) + p^T(t)I\tilde{x}(t) > dt \\
&= \frac{1}{\pi}\int_0^\pi |\tilde{x}'(t)|^2 dt + \sum_{i=1}^N \frac{1}{\pi}\int_0^\pi p_i(t)(\bar{x}_i^2 - \tilde{x}_i^2(t))dt \\
&= \frac{1}{\pi}\int_0^\pi [|\tilde{x}'(t)|^2 - \sum_{i=1}^N p_i(t)\tilde{x}_i^2(t)]dt + \sum_{i=1}^N \bar{p}_i\bar{x}_i^2 \\
&\geq \sum_{i=1}^N \bar{\gamma}_i\bar{x}_i^2 + [\delta(\Gamma_\alpha) - \frac{\pi^2}{3}||\Gamma_\beta||_{L^1_N} - ||\Gamma_\infty||_{L^\infty_N}]||\tilde{x}||^2_{H^1_N} \\
&\geq \eta|\bar{x}|^2 + [\delta(\Gamma_\alpha) - \frac{\pi^2}{3}||\Gamma_\beta||_{L^1_N} - ||\Gamma_\infty||_{L^\infty_N}]||\tilde{x}||^2_{H^1_N}.
\end{aligned}$$

Hence the lemma follows. □

4 Asymptotic Resonance Conditions for the Existence of Solutions

Let $f : [0,\pi] \times \mathbf{R}^N \to \mathbf{R}^N$ be a function satisfying Caratheodory's conditions, namely,

(i) for each $x \in \mathbf{R}^N$, the function $t \in [0,\pi] \to f(t,x) \in \mathbf{R}^N$ is measurable on $[0,\pi]$;

(ii) for a.e. $t \in [0,\pi]$, the function $x \in \mathbf{R}^N \to f(t,x) \in \mathbf{R}^N$ is continuous; and

(iii) for each $r > 0$, there exists a function $\alpha_r(t) \in L^1[0,\pi]$ such that $|f(t,x)| \leq \alpha_r(t)$ for a.e. $t \in [0,\pi]$ and all $x \in \mathbf{R}^N$ with $|x| \leq r$.

Let $x = C([0,\pi], \mathbf{R}^N)$ and $Y = L^1([0,\pi], \mathbf{R}^N)$. Also, let $Y_1 \subset Y$ denote the subspace of Y defined by

$$Y_1 = \{x \in L^1([0,\pi], \mathbf{R}^N) | x_i(t) \text{ is constant for a.e. } t \in [0,\pi], i = 1, 2, ..., N\}; \tag{4.1}$$

and let Y_2 be the closed subspace of Y such that $Y = Y_1 \oplus Y_2$. We define the canonical projections $P : Y \to Y_1$ and $Q : Y \to Y_2$ by setting, for $x \in Y$,

$$Px(t) = x(t) - \frac{1}{\pi}\int_0^\pi x(t)dt = \tilde{x}(t), \tag{4.2}$$

$$Qx(t) = \frac{1}{\pi}\int_0^\pi x(t)dt = \bar{x}, \tag{4.3}$$

for $t \in [0,\pi]$.

We next define a linear operator $L : D(L) \subset X \to Y$ by setting

$$D(L) = \{x \in W^{2,1}([0,\pi], \mathbf{R}^N) | x'(0) = x'(\pi) = 0\}; \tag{4.4}$$

and for $x \in D(L)$,

$$Lx = -x''. \tag{4.5}$$

Now, for $x \in D(L)$, we see, on integrating by parts, that

$$\begin{aligned}(Lx, x) &= \frac{1}{\pi}\int_0^{\pi} < -x''(t), x(t) > dt \\ &= \frac{1}{\pi}\int_0^{\pi} |x'(t)|^2 dt = ||\tilde{x}||^2_{H^1_N} \geq 0. \end{aligned} \tag{4.6}$$

Lemma 4 *For every given $y \in L^1([0,\pi], \mathbf{R}^N)$ with $\bar{y} = 0$, there exists a unique $x \in C([0,\pi], \mathbf{R}^N)$ with $\bar{x} = 0$ such that*

$$-x''(t) = y(t), \quad 0 < t < \pi, \tag{4.7}$$

$$x'(0) = x'(\pi) = 0. \tag{4.8}$$

Proof. It is easy to see that

$$x(t) = -\int_0^t (t-\tau)y(\tau)d\tau + \frac{1}{2\pi}\int_0^{\pi} (\pi-\tau)^2 y(\tau)d\tau, \tag{4.9}$$

for $t \in [0,\pi]$, is the unique solution for (4.7)-(4.8) with $\bar{x} = 0$. Hence the lemma follows. □

It follows from Lemma 4 that there is a bounded linear operator $K : Y_1 \to X$ such that for $y \in Y$,

$$KPy \in D(L), LKPy = Py, \quad (KPy, Py) \geq 0. \tag{4.10}$$

Now let $N : X \to Y$ be a nonlinear operator defined by

$$(Nx)(t) = f(t, x(t)), t \in [0,\pi], \tag{4.11}$$

where $f : [0,\pi] \times \mathbf{R}^N \to \mathbf{R}^N$ is a given function satisfying Caratheodory's conditions. It follows easily from the Arzela-Ascoli theorem that the operator $KPN : X \to X$ is a compact operator (i.e., it maps bounded subsets in X into relatively compact subsets of X) and $QN : X \to X$ is a bounded operator (i.e., QN maps bounded subsets in X into bounded subsets in X).

Theorem 1 *Let $f = (f_1, f_2, ..., f_N) : [0,\pi] \times \mathbf{R}^N \to \mathbf{R}$ be a function satisfying Caratheodory's conditions. Suppose that for each $i = 1, 2, ..., N$ there exist real numbers r_i, R_i, a_i and A_i with $r_i < 0 < R_i$ and $a_i \leq A_i$ such that*

(i) for a.e. $t \in [0,\pi]$ and all $x \in \mathbf{R}^N$ with $x_i \geq R_i$,

$$f_i(t, x) \geq A_i, \tag{4.12}$$

(ii) for a.e. $t \in [0,\pi]$ and all $x \in \mathbf{R}^N$ with $x_i \leq r_i$,

$$f_i(t, x) \leq a_i. \tag{4.13}$$

Suppose further that for every real number $r \geq 0$ and each $i = 1, 2, ..., N$ there exist functions $\alpha_{ri}(t) \in L^1[0,\pi]$ such that

$$|f_i(t, x)| \leq \alpha_{ri}(t), \tag{4.14}$$

for a.e. $t \in [0,\pi]$ *and all* $x \in \mathbf{R}^N$ *with* $|x_i| \le r$. *Then for every* $e \in L^1([0,\pi],\mathbf{R}^N)$ *with* $a_i \le \bar{e}_i \le A_i$ *for each* $i = 1,2,...,N$ *the boundary value problem*

$$-x''(t) + f(t,x(t)) = e(t), \quad 0 < t < \pi, \tag{4.15}$$

$$x'(0) = x'(\pi) = 0, \tag{4.16}$$

has at least one solution.

Proof. Define $F = (F_1, F_2, ..., F_N) : [0,\pi] \times \mathbf{R}^N \to \mathbf{R}^N$ by

$$F_i(t) = f_i(t) - \frac{A_i + a_i}{2}, \tag{4.17}$$

for $(t,x) \in [0,\pi] \times \mathbf{R}^N$ and $i = 1,2,...,N$. Also define $E : [0,\pi] \to \mathbf{R}^N$, $E = (E_1, E_2, ..., E_N)$, by

$$E_i(t) = e_i(t) - \frac{A_i + a_i}{2}, \tag{4.18}$$

for $t \in [0,\pi]$ and $i = 1,2,...,N$. Clearly F satisfies Caratheodory's conditions, and for $i = 1,2,...,N$ and for a.e. $t \in [0,\pi]$,

$$F_i(t,x) \ge \frac{A_i - a_i}{2} \ge 0, \tag{4.19}$$

for all $x \in \mathbf{R}^N$ with $x_i \ge R_i$, while

$$F_i(t,x) \le \frac{A_i - A_i}{2} \le 0, \tag{4.20}$$

for all $x \in \mathbf{R}^N$ with $x_i \le r_i$. Further, for every real number $r \ge 0$ and each $i = 1,2,...,N$ there exist functions $\beta_{ri} \in L^1[0,\pi]$ such that

$$|F_i(t,x)| \le \beta_{ri}(t), \tag{4.21}$$

for a.e. $t \in [0,\pi]$ and all $x \in \mathbf{R}^N$ with $|x_i| \le r$. Indeed, in view of (4.14), we have $\beta_{ri}(t) = \alpha_{ri}(t) + \frac{1}{2}|A_i + a_i|$ for $t \in [0,\pi]$. We also have for $i = 1,2,...,N$,

$$\frac{1}{2}(a_i - A_i) \le \bar{E}_i \le \frac{1}{2}(A_i - a_i). \tag{4.22}$$

Clearly, (4.15) is equivalent to

$$-x''(t) + F(t,x(t)) = E(t), 0 < t < \pi. \tag{4.23}$$

Let us next define the nonlinear operator $N : X \to Y$ by

$$(Nx)(t) = F(t,x(t)), \quad t \in [0,\pi],$$

while $x(t) \in X$. It is easy to see, in view of (4.19), (4.20), and (4.21), that for every $k \ge 0$ there exists a constant $C(k) \ge 0$ such that

$$(Nx, x) \ge k||Nx||_Y - C(k), \tag{4.24}$$

for $x \in X$. Now, if $L : D(L) \subset X \to Y$ is the linear operator defined by (4.4) and (4.5) and $K : Y_1 \to X$ is the linear operator as in (4.10), then the boundary value problem of (4.23) with (4.16) is equivalent to the operator equation

$$Lx + Nx = E, \quad x \in X, \tag{4.25}$$

which in turn is equivalent to the system of equations

$$\begin{aligned} Px + KPNx &= KPE \\ QNx &= QE, \end{aligned} \tag{4.26}$$

where P and Q are as defined by (4.2) and (4.3). Now, (4.26) is clearly equivalent to the single equation

$$Px + QNx + KPNx = KPE + QE, \tag{4.27}$$

which has the form of a compact perturbation of the Fredholm operator P of index zero. We can, therefore, apply the version given in [6: Theorem 1, Corollary 1] or [5: Theorem IV.4] or [7] of the Leray-Schauder Continuation theorem which ensures the existence of a solution for (4.27) if the set of all possible solutions of the family of equations

$$Px + (1-\lambda)Qx + \lambda QNx + \lambda KPN = \lambda KPE + \lambda QE, \tag{4.28}$$

$\lambda \in (0,1)$, is a priori bounded in X, independently of λ. Notice that (4.28) is equivalent to the system of equations

$$\begin{aligned} Px + \lambda KPNx &= \lambda KPE \\ (1-\lambda)Qx + \lambda QNx &= \lambda QE. \end{aligned} \tag{4.29}$$

If $x_\lambda \in X$ is a solution for (4.29) for some $\lambda \in (0,1)$, then $x_\lambda \in D(L)$ and

$$\begin{aligned} Lx_\lambda + \lambda PNx_\lambda &= \lambda PE, \\ (1-\lambda)Qx_\lambda + \lambda QNx_\lambda &= \lambda QE. \end{aligned} \tag{4.30}$$

Now we get from (4.30) that

$$(Lx_\lambda, Px_\lambda) + \lambda(PNx_\lambda, Px_\lambda) = \lambda(PE, Px_\lambda),$$

$$(1-\lambda)(Qx_\lambda, Qx_\lambda) + \lambda(QNx_\lambda, Qx_\lambda) = \lambda(QE, Qx_\lambda).$$

Since $(Lx_\lambda, Px_\lambda) = (Lx_\lambda, x_\lambda)$, and given (4.6) and (4.24), we obtain that

$$||Px_\lambda||^2_{H^1_N} + k||Nx_\lambda||_Y - C(k) \quad \le \quad ||PE||_Y \cdot ||Px_\lambda||_X + |QE| \cdot |Qx_\lambda|. \tag{4.31}$$

Now, the second equation in (4.30) gives for each $i = 1, 2, ..., N$ that

$$(1-\lambda)\frac{1}{\pi}\int_0^\pi x_{\lambda i}(t)dt + \lambda\frac{1}{\pi}\int_0^\pi F_i(t, x_\lambda(t))dt = \lambda\frac{1}{\pi}\int_0^\pi E_i(t)dt. \tag{4.32}$$

If $x_{\lambda i}(t) \ge R_i$ for every $t \in [0,\pi]$, we get from (4.32), in view of (4.19) and (4.22), that

$$(1-\lambda)R_i + \frac{\lambda}{2}(A_i - a_i) \le \frac{\lambda}{2}(A_i - a_i).$$

Thus, $(1-\lambda)R_i \leq 0$, and we have a contradiction. Similarly, $x_{\lambda i}(t) \leq r_i$ for every $t \in [0,\pi]$ leads to a contradiction. So for every $i = 1, 2, ..., N$ there exists a $\tau_i \in [0,\pi]$ such that $r_i \leq x_{\lambda i}(\tau_i) \leq R_i$. It follows that there exist constants $C_1 \geq 0$ and $C_2 \geq 0$, independent of $\lambda \in (0,1)$ such that

$$||x_\lambda||_X \leq C_1 + C_2||Px_\lambda||_{H_N^1}. \tag{4.33}$$

Finally, using the facts that $||Px_\lambda||_X \leq 2||x_\lambda||_X$ and $|Qx_\lambda| \leq ||x_\lambda||_X$, we have that

$$||Px_\lambda||_{H_N^1}^2 + k||Nx_\lambda||_Y - C(k) \leq C_3(C_1 + C_2||Px_\lambda||_{H_N^1}),$$

where $C_3 = 2||PE||_Y + |QE|$. Hence, there exists a constant $C > 0$, independent of $\lambda \in (0,1)$, such that

$$||Px_\lambda||_{H_N^1} \leq C,$$

which implies, from (4.33), that

$$||x_\lambda||_X \leq C_1 + C_2 C.$$

We have thus shown that the set of solutions of (4.28) is bounded in X independently of $\lambda \in (0,1)$. Hence the theorem follows. □

Theorem 2 *Let $\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_N) \in L^1([0,\pi], \mathbf{R}^N)$ be as in Lemma 2. Let $f = (f_1, f_2, ..., f_N) : [0,\pi] \times \mathbf{R}^N \to \mathbf{R}^N$ be as in Theorem 1. Assume, further, for each $i = 1, 2, ..., N$*

$$\limsup_{|x_i| \to \infty} \frac{f_i(t,x)}{x_i} \leq \Gamma_i(t), \tag{4.34}$$

uniformly a.e. in $t \in [0,\pi]$. Then, for every $e \in L^1([0,\pi], \mathbf{R}^N)$ with $a_i \leq \bar{e}_i \leq A_i$ for each $i = 1, 2, ..., N$ the boundary value problem

$$x''(t) + f(t, x(t)) = e(t), \quad 0 < t < \pi, \tag{4.35}$$

$$x'(0) = x'(\pi) = 0, \tag{4.36}$$

has at least one solution.

Proof. Define $F = (F_1, F_2, ..., F_n) : [0,\pi] \times \mathbf{R}^N \to \mathbf{R}^N$ and $E : [0,\pi] \to \mathbf{R}^N$, $E = (E_1, E_2, ..., E_N)$, as in the proof of Theorem 1, so that (4.19), (4.20), (4.21), and (4.22) hold. We have from (4.34) for each $i = 1, 2, ..., N$ that

$$\limsup_{|x_i| \to \infty} \frac{F_i(t,x)}{x_i} \leq \Gamma_i(t), \tag{4.37}$$

uniformly a.e. in $t \in [0,\pi]$. We also see for each $i = 1, 2, ...N$ for all $x \in \mathbf{R}^N$ with $|x_i| \geq \max(R_i, -r_i)$ that $F_i(t,x)x_i \geq 0$, so that $\Gamma_i(t) \geq 0$ a.e. in $[0,\pi]$. Moreover, the equation (4.35) is equivalent to

$$x''(t) + F(t, x(t)) = E(t), \quad 0 < t < \pi. \tag{4.38}$$

Now let $\eta = \frac{1}{2N}[\delta(\Gamma_\alpha) - \frac{\pi^2}{3}||\Gamma_\beta||_{L_N^1} - ||\Gamma_\infty||_{L_N^\infty}] > 0$. Then for each i there exists a $\rho_i > 0$ such that for a.e. $t \in [0,\pi]$ and all $x \in \mathbf{R}^N$ with $|x_i| \geq \rho_i$,

$$0 \leq \frac{F_i(t,x)}{x_i} \leq \Gamma_i(t) + \eta. \tag{4.39}$$

Next, set $\rho = \max\{\rho_i | 1 \leq i \leq N\}$ and define $\tilde{\gamma} : [0,\pi] \times \mathbf{R}^N \to \mathbf{R}^N$ by setting, for $(t,x) \in [0,\pi] \times \mathbf{R}^N$ and each $i = 1, 2, ...N$,

$$\begin{aligned}
\tilde{\gamma}_i(t,x) &= \frac{F_i(t,x)}{x_i} \text{ if } |x_i| \geq \rho, \\
\tilde{\gamma}_i(t,x) &= \frac{F_i(t,x_1,...,x_{i-1},\rho,x_{i+1},...,x_N)}{\rho}(\frac{x_i}{\rho}) \\
&\quad +(1-\frac{x_i}{\rho})\Gamma_i(t) \text{ for } 0 \leq x_i < \rho, \\
\tilde{\gamma}_i(t,x) &= \frac{F_i(t,x_1,...,x_{i-1},-\rho,x_{i+1},...,x_N)}{\rho}(\frac{x_i}{\rho}) \\
&\quad + (1+\frac{x_i}{\rho})\Gamma_i(t) \text{ for } -\rho \leq x_i < 0.
\end{aligned}$$

Then $\tilde{\gamma}$ satisfies Caratheodory's conditions and

$$0 \leq \tilde{\gamma}_i(t,x) \leq \Gamma_i(t) + \eta, \tag{4.40}$$

for a.e. $t \in [0,\pi]$, all $x \in \mathbf{R}^N$, and $i = 1, 2, ..., N$. If we next set $h = (h_1, h_2, ..., h_N)$ with

$$h_i(t,x) = F_i(t,x) - \tilde{\gamma}_i(t,x)x_i,$$

for $t \in [0,\pi]$, $x \in \mathbf{R}^N$, and $i = 1, 2, ..., N$, then we see from (4.21) and the definition of $\tilde{\gamma}_i$ that there exist functions $m_i(t) \in L^1[0,\pi]$ such that

$$|h_i(t,x)| \leq m_i(t), \tag{4.41}$$

for a.e. $t \in [0,\pi]$ and all $x \in \mathbf{R}^N$. Defining $g : [0,\pi] \times \mathbf{R}^N \to \mathbf{R}^N$, $g = (g_1, g_2, ..., g_N)$ by setting

$$g_i(t,x) = \tilde{\gamma}_i(t,x)x_i$$

for $(t,x) \in [0,\pi] \times \mathbf{R}^N$, $i = 1, 2, ..., N$, we see that the equation (4.38) is equivalent to

$$x''(t) + g(t,x(t)) + h(t,x(t)) = E(t). \tag{4.42}$$

We can next apply Theorem IV.4 of [5] to the boundary value problem posed by(4.42) and (4.36). It suffices to show that the set of solutions of the family of equations

$$x''(t) + (1-\lambda)\tilde{\Gamma}^T(t)Ix(t) + \lambda g(t,x(t)) + \lambda h(t,x(t)) = \lambda E(t), \tag{4.43}$$

$$x'(0) = x'(\pi) = 0,$$

$\lambda \in (0,1)$, is a priori bounded in $X = C([0,\pi],\mathbf{R}^N)$ independently of λ, where $\tilde{\Gamma}(t) = (\tilde{\Gamma}_1(t), ..., \tilde{\Gamma}_N(t))$ with $\tilde{\Gamma}_i(t) = \Gamma_i(t) + \eta$, $i = 1, 2, ..., N$.

If, now, $x(t)$ is a possible solution of (4.43) for some $\lambda \in (0,1)$, we see on integrating the equation obtained by taking the inner product of the equation in (4.43) with $\frac{1}{\pi}(\bar{x} - \tilde{x}(t))$ and using Lemma 3 with $\Gamma_{\infty i}$ replaced by $\Gamma_{\infty i} + \eta$, for $i = 1, 2, ..., N$ and $\gamma = (\gamma_1, ..., \gamma_N) \equiv 0$ that

$$0 = \frac{1}{\pi}\int_0^\pi < x''(t), \bar{x} - \tilde{x}(t) > dt$$

$$+\frac{1}{\pi}\int_0^\pi \{< \bar{x}-\tilde{x}(t),(1-\lambda)\tilde{\Gamma}^T(t)Ix(t)+\lambda g(t,x(t))+\lambda h(t,x(t))-\lambda E(t)>\}dt$$
$$\geq [\delta(\Gamma_\alpha)-\frac{\pi^2}{3}||\Gamma_\beta||_{L^1_N}-||\Gamma_\infty||_{L^\infty_N}-N\eta]||\tilde{x}||^2_{H^1_N}$$
$$-[\sum_{i=1}^{N}||m_i||_{L^1}+||E||_{L^1_N}](|\bar{x}|+||\tilde{x}||_{L^\infty_N})$$
$$\geq \eta N||\tilde{x}||^2_{H^1_N}-C(|\bar{x}|+||\tilde{x}||_{H^1_N}),$$

where $C>0$ is a constant independent of $\lambda\in(0,1)$. Hence,

$$||\tilde{x}||^2_{H^1_N}\leq(\frac{C}{\eta N})(|\bar{x}|+||\tilde{x}||_{H^1_N}). \tag{4.44}$$

Next, integrating each of the component equations in (4.43) over $[0,\pi]$, we see that

$$(1-\lambda)\frac{1}{\pi}\int_0^\pi(\Gamma_i(t)+\eta)x_i(t)dt+\lambda\frac{1}{\pi}\int_0^\pi F_i(t,x(t))dt$$
$$=\lambda\frac{1}{\pi}\int_0^\pi E_i(t)dt,$$

$i=1,2,...N$. As in the proof of Theorem 1, we see that there exist constants $C_1\geq 0$ and $C_2\geq 0$, such that

$$|\bar{x}|\leq||x||_X\leq C_1+C_2||\tilde{x}||_{H^1_N}. \tag{4.45}$$

It follows from (4.44) and (4.45) that there exists a constant C_3 independent of $\lambda\in(0,1)$ such that

$$||\tilde{x}||_{H^1_N}\leq C_3,$$

and hence

$$||x||_X\leq C_1+C_2C_3.$$

Thus we have shown that the set of solutions of (4.43) is bounded in X independently of λ. Hence the theorem holds. □

Remark 1. We say that the boundary value problem (4.35)-(4.36) has "no L^∞-resonance" at the second eigenvalue $\lambda=1$ of the linear eigenvalue problem (1.7)-(1.8) if $\Gamma_\alpha=\Gamma_\infty=0$ in Theorem 2. In the case of no L^∞-resonance, Theorem 2 implies the existence of a solution for the boundary value problem (4.35)-(4.36) if $||\Gamma_\beta||_{L^1_N}<\frac{3}{\pi^2}$. We give a sharpening of this result in Section 5.

5 Resonance Condition When No L^∞-Resonance Exists

We need the following lemma for a sharper resonance condition that gives the existence of a solution for the boundary value problem (4.35)-(4.36) when there is no L^∞-resonance.

Lemma 5 *Let $e\in L^1([0,\pi],\mathbf{R}^N)$ and $\Gamma=(\Gamma_1,\Gamma_2,...,\Gamma_N)\in L^1([0,\pi],\mathbf{R}^N)$ with $\bar{\Gamma}_i=\frac{1}{\pi}\int_0^\pi\Gamma_i(t)dt\geq 0$ for every $i=1,2,...,N$. Then every possible solution $x(t)$ of the linear boundary value problem*

$$x''(t)+p(t)^TIx(t)=e(t),0<t<\pi$$

$$x'(0) = x'(\pi) = 0, \tag{5.1}$$

with $p = (p_1, p_2, ..., p_N) \in L^1([0,\pi], \mathbf{R}^N)$ *such that*

$$\bar{p}_i \leq \bar{\Gamma}_i, \quad 0 \leq p_i(t) \tag{5.2}$$

for a.e. $t \in [0,\pi]$, $i = 1, 2, ..., N$, *satisfies the inequality*

$$(1 - \frac{\pi^2}{4}|\bar{\Gamma}|)||x''||^2_{L^1_N} \leq 2||e||_{L^1_N}||x''||_{L^1_N} + |\bar{\Gamma}|||e||_{L^1_N}||x||_{L^\infty_N}. \tag{5.3}$$

(Here $\bar{\Gamma} = (\bar{\Gamma}_1, \bar{\Gamma}_2, ..., \bar{\Gamma}_N)$.*)*

Proof. It follows from Lemma 4 of [1] that each solution $x_i(t)$ of the i-th component boundary value problem of (5.1), namely,

$$x_i''(t) + p_i(t)x_i(t) = e_i(t), \quad 0 < t < \pi$$

$$x_i'(0) = x_i'(\pi) = 0,$$

satisfies the inequality

$$(1 - \frac{\pi^2}{4}\bar{\Gamma}_i)||x_i''||^2_{L^1_N} \leq 2||e_i||_{L^1}||x_i''||_{L^1} + \bar{\Gamma}_i||e_i||_{L^1}||x_i||_{L^\infty}$$

for each $i = 1, 2, ..., N$. Noting that $\max_{1 \leq i \leq n} \bar{\Gamma}_i \leq |\bar{\Gamma}|$, we get

$$(1 - \frac{\pi^2}{4}|\bar{\Gamma}|)||x_i''||^2_{L^1} \leq 2||e_i||_{L^1}||x_i''||_{L^1} + |\bar{\Gamma}|||e_i||_{L^1}||x_i||_{L^\infty}$$

for each $i = 1, 2, ..., N$. On adding all these inequalities and using the Cauchy-Schwarz inequality in $\mathbf{R}^N$ we get that

$$(1 - \frac{\pi^2}{4}|\bar{\Gamma}|)||x''||^2_{L^1_N} \leq 2||e||_{L^1_N}||x''||_{L^1_N} + |\bar{\Gamma}|||e||_{L^1_N}||x||_{L^\infty_N}.$$

Hence the lemma follows. □

Theorem 3 *Let* $\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_N) \in L^1([0,\pi], \mathbf{R}^N)$ *be such that* $|\bar{\Gamma}| < \frac{4}{\pi^2}$. *Let* $f = (f_1, f_2, ..., f_N) : [0,\pi] \times \mathbf{R}^N \to \mathbf{R}^N$ *be as in Theorem 2. Then for every* $e \in L^1([0,\pi], \mathbf{R}^N)$ *with* $a_i \leq \bar{e}_i \leq A_i$ *for each* $i = 1, 2, ..., N$, *the boundary value problem*

$$x''(t) + f(t, x(t)) = e(t), 0 < t < \pi, \tag{5.4}$$

$$x'(0) = x'(\pi) = 0,$$

has at least one solution.

Proof. Define $F = (F_1, F_2, ..., F_N) : [0,\pi] \times \mathbf{R}^N \to \mathbf{R}^N$, $E : [0,\pi] \to \mathbf{R}^N$ and $E = (E_1, E_2, ..., E_N)$ as in the proof of Theorem 2. Then the boundary value problem (5.4) is equivalent to the boundary value problem

$$x''(t) + F(t, x(t)) = E(t), \quad 0 < t < \pi, \tag{5.5}$$

$$x'(0) = x'(\pi) = 0.$$

Also

$$\lim \sup_{|x_i|\to\infty} \frac{F_i(t,x)}{x_i} \le \Gamma_i(t),$$

uniformly a.e. in $t \in [0,\pi]$ and

$$F_i(t,x)x_i \ge 0,$$

for a.e. $t \in [0,\pi]$ and all $x \in \mathbf{R}^N$ with $|x_i| \ge \max(R_i, -r_i)$ so that $\Gamma_i(t) \ge 0$ for a.e. $t \in [0,\pi]$. Let $\eta = \frac{1}{2N}(\frac{4}{\pi^2} - |\bar{\Gamma}|)$ so that $|\bar{\Gamma}| + N\eta < \frac{4}{\pi^2}$. Proceeding as in the proof of Theorem 2, we can write the boundary value problem (5.5) in the equivalent form

$$x''(t) + g(t,x(t)) + h(t,x(t)) = E(t), \tag{5.6}$$

$$x'(0) = x'(\pi) = 0.$$

The same degree arguments will imply the existence of a solution for (5.6) if the set of all possible solutions of the family of equations

$$x''(t) + (1-\lambda)\Gamma^{*T}(t)Ix(t) + \lambda g(t,x(t)) + \lambda h(t,x(t)) = \lambda E(t) \tag{5.7}$$

$$x'(0) = x'(\pi) = 0,$$

$\lambda \in (0,1)$, is a priori bounded in $X = C([0,\pi],\mathbf{R}^N)$ independently of λ. Here $\Gamma^*(t) = (\Gamma_1^*(t), ..., \Gamma_N^*(t))$ with $\Gamma_i^*(t) = \Gamma_i(t) + \eta$, $i = 1,2,...,N$, $t \in [0,\pi]$.

We note that $g = (g_1, g_2, ..., g_N)$ in (5.6) is such that $g_i(t,x) = \tilde{\gamma}_i(t,x)x_i$. If we write $\tilde{\gamma}(t,x) = (\tilde{\gamma}_1(t,x), ..., \tilde{\gamma}_N(t,x))$, then

$$g(t,x) = \tilde{\gamma}(t,x)^T Ix.$$

We see that

$$0 \le (1-\lambda)\Gamma_i^*(t) + \lambda\tilde{\gamma}_i(t,x(t)) \le \Gamma_i^*(t)$$

for $i = 1,2,...,N$ in view of (4.40) with

$$\begin{aligned} |\bar{\Gamma}^*| &= \{\sum_{i=1}^N (\bar{\Gamma}_i + \eta)^2\}^{1/2} \le (\sum_{i=1}^N \bar{\Gamma}_i^2)^{1/2} + \sqrt{N}\eta \\ &\le |\bar{\Gamma}| + \eta N < \frac{4}{\pi^2}. \end{aligned}$$

Also, since

$$||E(t) - h(t,x(t))||_{L^1_N} \le ||E||_{L^1_N} + \sum_{i=1}^N ||m_i||_{L^1},$$

it follows from Lemma 5 that

$$\begin{aligned} (1 - \frac{\pi^2}{4}|\bar{\Gamma}^*|)||x''||^2_{L^1_N} &\le 2(||E||_{L^1_N} + \sum_{i=1}^N ||m_i||_{L^1})||x''||_{L^1_N} \\ &+ |\bar{\Gamma}^*|(||E||_{L^1_N} + \sum_{i=1}^N ||m_i||_{L^1})||x||_{L^\infty_N}. \end{aligned} \tag{5.8}$$

As in the proof of Theorem 2, we have that there exist constants $C_1 \geq 0$ and $C_2 \geq 0$, independent of $\lambda \in (0,1)$ such that

$$\begin{aligned} |\bar{x}| \leq ||x||_{L_N^\infty} &\leq C_1 + C_2||\tilde{x}||_{H_N^1} \\ &= C_1 + C_2||x'||_{L_N^2} \\ &\leq C_1 + C_2\frac{\pi}{2}||x''||_{L_N^1}. \end{aligned} \tag{5.9}$$

It then follows from (5.8) and (5.9) that there exists a constant $C_3 \geq 0$, independent of $\lambda \in (0,1)$, such that

$$||x''||_{L_N^1} \leq C_3$$

and, hence, from (5.9) again,

$$||x||_{L_N^\infty} \leq C_1 + C_2\frac{\pi}{2}C_3.$$

Hence the theorem holds. □

Remark 2. If there is no L^∞-resonance (i.e., $\Gamma_\alpha = \Gamma_\infty = 0$), Theorem 3 improves the condition on $\bar{\Gamma}$ to $|\bar{\Gamma}| \leq \frac{4}{\pi^2}$ compared to Theorem 2, where $|\bar{\Gamma}| < \frac{3}{\pi^2}$.

Remark 3. If $p(t) = (p_1(t), ..., p_N(t)) \in L^1([0,\pi], \mathrm{R}^N)$ in Lemma 5 satisfies, additionally, for a given $\eta > 0$, $p_i(t) \geq \eta > 0$ for a.e. $t \in [0,\pi]$, $i = 1, 2, ..., N$ and $|\bar{\Gamma}| < \frac{4}{\pi^2}$, it follows easily from the inequality (5.3) that the boundary value problem (5.1) has at most one solution.

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