AN EXISTENCE THEOREM FOR NONLINEAR DELAY DIFFERENTIAL EQUATIONS*

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ABSTRACT

In this paper we prove a theorem on the existence of solutions of nonlinear delay differential equations with implicit derivatives. The result is established using the measure of noncompactness of a set and Darbo's fixed point theorem.

Key words: Existence of solution, Delay differential equations, Darbo's theorem.

AMS subject classification: 34KO5.

1. INTRODUCTION

The theory of linear and nonlinear delay differential equations has been studied by several researchers [1,4]. Recently Dacka [5] proved an existence theorem for nonlinear delay differential equations with implicit derivatives using the measure of noncompactness of a set and Darbo's fixed point theorem. By the same method Balachandran and Somasundaram [2] established an existence theorem for nonlinear delay differential equations having implicit derivatives with delay depending on state variable. In [3] Banas proved a theorem about existence of solutions of some nonlinear Volterra integral equations with deviating argument without assuming the Lipschitz condition and using a technique similar to Dacka [5]. In this paper we shall prove an existence theorem for nonlinear delay differential equations with delay depending on implicit derivatives.

2. MATHEMATICAL PRELIMINARIES

Let $(X, \|\,.\,\|)$ be a Banach space and E be bounded set of X. In this paper the following definition of the measure of noncompactness of a set E is used [7].

$$\mu(E) = \inf \{r > 0: \text{ E can be covered by a finite number of balls whose radii are smaller than } r\}$$

The following version of Darbo's fixed point theorem being a generalization of Schauder's fixed point theorem shows the usefulness of the measure of noncompactness [6]. "If S is a nonempty bounded closed convex subset of X and $P : S \to S$ is a continuous mapping, such that for any set $E \subset S$ we have

$$\mu(PE) \leq k\, \mu(E)$$

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where k is a constant with $0 \le k < 1$, then P has a fixed point."

For the space of continuous functions $C_n[t_0,t_1]$ with norm

$$\|x\| = \max\{|x_i(t)| : i = 1,2,\ldots,n,\ t \in [t_0, t_1]\},$$

the measure of noncompactness of a set E is given by

$$\mu(E) = \frac{1}{2}\omega_0(E) = \frac{1}{2}\lim_{h\to 0^+} \omega(E,h)$$

where $\omega(E,h)$ is the common modulus of continuity of the functions which belong to the set E, that is,

$$\omega(E,h) = \sup_{x\in E} \quad [\sup |x(t) - x(s)| : |t - s| \le h]$$

where, as in the space of continuously differentiable functions $C[t_0,t_1]$ with norm

$$\|x\|_{C_n^1} = \|x\|_{C_n} + \|\dot{x}\|_{C_n},$$

we have

$$\mu(E) = (1/2)\,\omega_0(DE)$$

where

$$DE = \{\dot{x} : x \in E\}.$$

3. BASIC ASSUMPTIONS

Consider the following nonlinear delay differential equation with implicit derivative of the form

(1) $$\dot{x}(t) = f(x(t),x(t-r(x(t),\dot{x}(t),t)),t), \quad t \ge t_0$$

$$x(t) = \phi(t), \quad t \le t_0$$

where $x \in R^n$ and f is an n-vector function. Let $r(x(t),\dot{x}(t),t) \ge 0$. Set

$$\alpha(t) = t-r(x(t),\dot{x}(t),t), \text{ and}$$

$$a = \inf_{(x,\dot{x},t)} \alpha(t) \quad \text{and} \quad -\infty < a < t_0.$$

Then $x(t) = \phi(t)$ on $[a,t_0]$.

Assume that the functions $f(x,y,t)$ and $r(x,y,t)$ are continuous and satisfy the following conditions. For $y,\bar{y},x \in R^n$ and $t \in [t_0,t_1]$,

(2) $$| f(x,y,t) | \leq M$$

(3) $$| f(x(t),x(\alpha(t)),t)-f(x(t),x(\alpha(s)),t) | \leq N | \alpha(t)-\alpha(s) |$$

(4) $$| r(x,y,t) - r(x,\bar{y},t) \leq b | y-\bar{y} |$$

where M,N and b are positive constants such that $0 \leq Nb < 1$. Let $\phi(t)$ be a continuous R^n-valued function defined on $[a,t_0]$.

Definition: *The solution of* (1) *is the function* $x(t)$ *such that*:

i) $x(t)$ *is defined and continuous on the interval* $[a,t_1]$ *and is of class* C^1 *on* $[t_0,t_1]$ *such that at the point* t_0 *the right side derivative only is taken into account;*

ii) *The function* $x(t)$ *satisfies* (1) *on the interval* $[t_0,t_1]$, *whereas on the interval* $[a,t_0]$ *t the function* $x(t) = \phi(t)$.

Next we shall prove that the solution of (1) exists in the sense of the above definition.

4. EXISTENCE THEOREM

Theorem: *If the function* $f(x,y,t)$ *satisfies the conditions* (2) *and* (3), *and if* $r(x(t), \dot{x}(t),t) \geq 0$ *and satisfies the condition* (4), *then* (1) *has at least one solution for any initial function* $\phi \in C_n[a,t_0]$.

Proof: Consider the Banach space $C_n^1[t_0,t_1]$ and the set

$$H = \{x : x \in C_n^1[t_0,t_1], x(t_0) = \phi(t_0)\} .$$

For any function $x \in H$, $x(\alpha(t))$ will be the function defined in such a way that if $\alpha(t) < t_0$ for $t \in [t_0,t_1]$ then

(6) $$x(\alpha(t)) = \phi(\alpha(t)) .$$

Define the mapping T by

(7) $$T(x)(t) = \phi(t_0) + \int_{t_0}^{t} f(x(s), x(s - r(x(s), \dot{x}(s), s)),s)ds.$$

Moreover, consider the bounded closed set B in H as

(8) $$B = \{x \in H : \|x\| \leq L, \quad \|\dot{x}\| \leq M\}$$

where L and M are positive constants such that

$$L = |\phi(t_0)|+(t_1-t_0)M . \tag{9}$$

Since f is continuous, T is continuous and maps B into itself. Next let us estimate the modulus of continuity of the function $DT(x)(t)$ for $t,s \in [t_0,t_1]$. Since the only functions considered belong to some bounded subset of the space $C[t_0,t_1]$, and since these all have uniformly bounded derivatives, it follows that they are equicontinuous.

Thus

$$\begin{aligned} |DT(x)(t)-DT(x)(s)| &= |f(x(t),x(\alpha(t)),t)-f(x(s),x(\alpha(s)),s)| \\ &\le |f(x(t),x(\alpha(t)),t)-f(x(s),x(\alpha(t)),s)| \\ &\quad +|f(x(s),x(\alpha(t)),s)-f(x(s),x(\alpha(s)),s)| \\ &\le |f(x(t),x(\alpha(t)),t)--f(x(s),x(\alpha(t)),s)| \\ &\quad + N|t-s|+N|r(x(t),\dot{x}(t),t)-r(x(s),\dot{x}(t),s)| \\ &\quad + N|r(x(s),\dot{x}(t),s) - r(x(s),\dot{x}(s),s)| \end{aligned} \tag{10}$$

For the first term on the right of inequality (10), one can take the upper estimate as $\beta_0(|t-s|)$, where β_0 is some non-negative continuous function such that $\lim_{h\to 0^+} \beta_0(h) = 0$. This follows from the fact that it is formed by the composition of a finite number of functions having a uniformly bounded modulus of continuity. The second and third terms on the right of inequality (10) have the upper bound $Nh+N\beta_1(h)$; and the last term has upper bound $Nb\,|\dot{x}(t) - \dot{x}(s)|$. Letting $\beta = \beta_0+Nh+N\beta_1$ and $k = Nb$, one gets

$$\omega(DT(x),h) \le k\omega(Dx,h)+\beta(h) ;$$

hence it follows that

$$\mu(TE) \le k\mu(E)$$

for any bounded set $E \subset B \subset H$. Consequently, by Darbo's fixed point theorem, the mapping T has a fixed point $x \in C[t_0,t_1]$ such that

$$x(t) = T(x)(t) .$$

Clearly the extension of this function to the interval $[a,t_0]$ by means of the function ϕ is a solution of equation (1) having the following form:

$$x(t) = \phi(t_0) + \int_{t_0}^{t} f(x(s), x(s-r(x(s),\dot{x}(s),s)),s)ds, \qquad t \geq t_0 \tag{12}$$

$$x(t) = \phi(t), \ a < t < t_0$$

Remark: It should be observed that if one assumes that the function f and r satisfy also the Lipschitz condition, then the uniqueness of the solution of (1) can be established by standard techniques used in proving the uniqueness theorem.

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