EXISTENCE OF POSITIVE SOLUTIONS FOR NEUTRAL DIFFERENTIAL EQUATIONS

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ABSTRACT

We establish sufficient conditions for the existence of positive solutions of the neutral delay differential equation

$$\frac{d}{dt}[y(t) + P(t)y(t-\tau)] + Q(t)y(t-\sigma) = 0.$$

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1. INTRODUCTION.

Consider the neutral delay differential equation

$$\frac{d}{dt}[y(t) + P(t)y(t-\tau)] + Q(t)y(t-\sigma) = 0 \tag{1}$$

where

(2) $P \in C^1([t_0,\infty),\mathbf{R})$, $Q \in C([t_0,\infty),\mathbf{R})$, $\tau \in (0,\infty)$ and $\sigma \in [0,\infty)$.

Our aim in this paper is to establish sufficient conditions for (1) to have a positive solution on $[t_1,\infty)$ for every $t_1 \geq t_0$.

Conditions for the existence of positive solutions are relatively scarce in the literature. See [1], [2] and [3] and the references cited therein. The main tool in our proofs is the Banach contraction principle.

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Let $m = \max\{\tau, \sigma\}$ and let $t_1 \geq t_0$. By a *solution* of (1) on $[t_1, \infty)$ we mean a function $y \in C([t_1 - m, \infty), \mathbf{R})$ such that $y(t) + P(t)y(t-\tau)$ is continuously differentiable for $t \geq t_1$ and such that (1) is satisfied for $t \geq t_1$. Let $t_1 \geq t_0$ and let $\phi \in C([t_1 - m, t_1], \mathbf{R})$ be a given initial function. Then one can show by the method of steps that (1) has a unique solution on $[t_1, \infty)$ satisfying the initial condition $y(t) = \phi(t)$, $t_1 - m \leq t \leq t_1$.

2. EXISTENCE OF POSITIVE SOLUTIONS.

In this section we will establish several sufficient conditions for (1) to have a positive solution on $[t_1, \infty)$ for every $t_1 \geq t_0$.

Our motivation for the first theorem stems from the observation that if we look for a positive solution of (1) of the form

$$y(t) = \exp(-\int_{t_1}^{t} \lambda(s)ds), \quad t \geq t_1$$

then $\lambda(t)$ satisfies the equation

$$\begin{aligned} \lambda(t) &= [-P(t)\lambda(t-\tau) + \dot{P}(t)]\exp(\int_{t-\tau}^{t} \lambda(s)ds) \\ &\quad + Q(t)\exp(\int_{t-\sigma}^{t} \lambda(s)ds). \end{aligned} \tag{3}$$

Theorem 1. Assume that (2) holds and that there exists a positive number μ such that

$$| P(t) | \mu e^{\mu\tau} + | \dot{P}(t) | e^{\mu\tau} + | Q(t) | e^{\mu\sigma} \leq \mu \quad \text{for} \quad t \geq t_0. \tag{4}$$

Then for every $t_1 \geq t_0$, (1) has a positive solution on $[t_1, \infty)$.

Proof. Let $m = \max\{\tau, \sigma\}$. Define the set of functions

$$\Lambda = \{\lambda \in C([t_1 - m, \infty), \mathbf{R}) : | \lambda(t) | \leq \mu \text{ for } t \geq t_1 - m\}.$$

For λ_1, $\lambda_2 \in \Lambda$ define

$$d(\lambda_1, \lambda_2) = \sup_{t \geq t_1 - m} | \lambda_1(t) - \lambda_2(t) | e^{-\eta t}$$

where η is chosen so large that

$$\frac{\mu}{\eta} + \mid P(t) \mid e^{(\mu-\eta)\tau} < \frac{1}{2} \quad \text{for} \quad t \geq t_1 - m. \tag{5}$$

Then (Λ, d) is a complete metric space. Define

$$(T\lambda)(t) = \begin{cases} [-P(t)\lambda(t-\tau) + \dot{P}(t)]\exp(\int_{t-\tau}^{t} \lambda(s)ds) & t \geq t_1 \\ \qquad +Q(t)\exp(\int_{t-\sigma}^{t} \lambda(s)ds) & \\ (T\lambda)(t_1) & t_1 - m \leq t < t_1. \end{cases}$$

Then $T\lambda$ is continuous on $[t_1 - m, \infty)$ and for $t \geq t_1 - m$,

$$\mid (T\lambda)(t) \mid \leq \mid P(t) \mid \mu e^{\mu\tau} + \mid \dot{P}(t) \mid e^{\mu\tau} + \mid Q(t) \mid e^{\mu\tau} \leq \mu$$

which shows that T maps Λ into Λ.

Next, we claim that T is a contraction mapping. Before we establish this property of T we need to observe that for λ_1, $\lambda_2 \in \Lambda$ and for $t \geq t_1$, by applying the mean value theorem we find,

$$\mid \exp(\int_{t-\tau}^{t} \lambda_1(s)ds) - \exp(\int_{t-\tau}^{t} \lambda_2(s)ds) \mid \leq e^{\mu\tau} \int_{t-\tau}^{t} \mid \lambda_1(s) - \lambda_2(s) \mid ds.$$

Also observe that

$$\begin{aligned} & \mid \lambda_1(t-\tau)\exp(\int_{t-\tau}^{t} \lambda_1(s)ds - \lambda_2(t-\tau)\int_{t-\tau}^{t} \lambda_2(s)ds) \mid \\ = \; & \mid \lambda_1(t-\tau)[\exp(\int_{t-\tau}^{t} \lambda_1(s)ds - \exp\int_{t-\tau}^{t} \lambda_2(s)ds)] \\ & + \exp(\int_{t-\tau}^{t} \lambda_2(s)ds)[\lambda_1(t-\tau) - \lambda_2(t-\tau) \mid \\ \leq \; & \mu e^{\mu\tau} \int_{t-\tau}^{t} \mid \lambda_1(s) - \lambda_2(s) \mid ds + e^{\mu\tau} \mid \lambda_1(t-\tau) - \lambda_2(t-\tau) \mid . \end{aligned}$$

Hence if λ_1, $\lambda_2 \in \Lambda$ and $t \geq t_1 - m$,

$$\begin{aligned} & \mid (T\lambda_1)(t) - (T\lambda_2)(t) \mid \\ \leq \; & \mid P(t) \mid\mid \lambda_1(t-\tau)\exp(\int_{t-\tau}^{t} \lambda_1(s)ds) - \lambda_2(t-\tau)\exp(\int_{t-\tau}^{t} \lambda_2(s)ds) \mid \\ & + \mid \dot{P}(t) \mid\mid \exp(\int_{t-\tau}^{t} \lambda_1(s)ds) - \exp(\int_{t-\tau}^{t} \lambda_2(s)ds) \mid \end{aligned}$$

$$
\begin{aligned}
& + \mid Q(t) \mid\mid \exp(\int_{t-\sigma}^{t} \lambda_1(s)ds) - \exp(\int_{t-\sigma}^{t} \lambda_2(s)ds) \mid \\
\leq\ & \mid P(t) \mid \mu e^{\mu\tau} \int_{t-\tau}^{t} \mid \lambda_1(s) - \lambda_2(s) \mid ds \\
& + \mid P(t) \mid e^{\mu\tau} \mid \lambda_1(t-\tau) - \lambda_2(t-\tau) \mid \\
& + \mid \dot{P}(t) \mid e^{\mu\tau} \int_{t-\tau}^{t} \mid \lambda_1(s) - \lambda_2(s) \mid ds \\
& + \mid Q(t) \mid e^{\mu\sigma} \int_{t-\sigma}^{t} \mid \lambda_1(s) - \lambda_2(s) \mid ds \\
=\ & \mid P(t) \mid \mu e^{\mu\tau} \int_{t-\tau}^{t} (\mid \lambda_1(s) - \lambda_2(s) \mid e^{-\eta s}) e^{\eta s} ds \\
& + \mid P(t) \mid e^{\mu\tau} (\mid \lambda_1(t-\tau) - \lambda_2(t-\tau) \mid e^{-\eta(t-\tau)}) e^{\eta(t-\tau)} \\
& + \mid \dot{P}(t) \mid e^{\mu\tau} \int_{t-\tau}^{t} (\mid \lambda_1(s) - \lambda_2(s) \mid e^{-\eta s}) e^{\eta s} ds \\
& + \mid Q(t) \mid e^{\mu\tau} \int_{t-\tau}^{t} (\mid \lambda_1(s) - \lambda_2(s) \mid e^{-\eta s}) e^{\eta s} ds \\
\leq\ & \mid P(t) \mid \mu e^{\mu\tau} \cdot d(\lambda_1, \lambda_2) \cdot \frac{1}{\eta} (e^{\eta t} - e^{\eta(t-\tau)}) \\
& + \mid P(t) \mid e^{\mu\tau} \cdot d(\lambda_1, \lambda_2) \cdot e^{\eta(t-\tau)} \\
& + \mid \dot{P}(t) \mid e^{\mu\tau} \cdot d(\lambda_1, \lambda_2) \cdot \frac{1}{\eta} (e^{\eta t} - e^{\eta(t-\tau)}) \\
& + \mid Q(t) \mid e^{\mu\sigma} \cdot d(\lambda_1, \lambda_2) \cdot \frac{1}{\eta} (e^{\eta t} - e^{\eta(t-\sigma)}) \\
\leq\ & \frac{1}{\eta} d(\lambda_1 \lambda_2) \cdot e^{\eta t} [\mid P(t) \mid \mu e^{\mu\tau} + \eta \mid P(t) \mid e^{(\mu-\eta)\tau} \\
& + \mid \dot{P}(t) \mid e^{\mu\tau} + \mid Q(t) \mid e^{\mu\sigma}.
\end{aligned}
$$

From this inequality and by using (4) and (5), we see that for $t \geq t_1 - m$,

$$
\begin{aligned}
\mid (T\lambda_1(t) - (T\lambda_2(t) \mid e^{-\eta t} &\leq \frac{1}{\eta} d(\lambda_1, \lambda_2)[\mu + \eta \mid P(t) \mid e^{\mu(\tau-\eta)}] \\
&= d(\lambda_1, \lambda_2)[\frac{\mu}{\eta} + \mid P(t) \mid e^{(\mu-\eta)\tau}] \\
&< \frac{1}{2} d(\lambda_1, \lambda_2).
\end{aligned}
$$

It follows that $d(T\lambda_1, T\lambda_2) \leq (1/2)d(\lambda_1, \lambda_2)$. Therefore, there exists a (unique) solution $\lambda \in \Lambda$ of $T\lambda = \lambda$. To complete the proof it suffices

to show that the positive function

$$y(t) = \exp(-\int_{t_1}^{t} \lambda(s)ds), \quad t \geq t_1 - m$$

is a solution of (1) on $[t_1, \infty)$. To this end observe that for $t \geq t_1$, $\dot{y}(t) = -\lambda(t)y(t)$ and that

$$\begin{aligned} y(t-\tau) &= \exp(-\int_{t_1}^{t-\tau} \lambda(s)ds) \\ &= \exp(-\int_{t_1}^{t} \lambda(s)ds) \exp(\int_{t-\tau}^{t} \lambda(s)ds) \\ &= y(t) \exp(\int_{t-\tau}^{t} \lambda(s)ds). \end{aligned}$$

Similarly, $y(t-\sigma) = y(t)\exp(\int_{t-\sigma}^{t} \lambda(s)ds)$. Hence, for $t \geq t_1$,

$$\begin{aligned} & \frac{d}{dt}[y(t) + P(t)y(t-\tau)] + Q(t)y(t-\sigma) \\ = \; & -\lambda(t)y(t) - P(t)\lambda(t-\tau)y(t-\tau) + \dot{P}(t)y(t)\exp(\int_{t-\tau}^{t} \lambda(s)ds) \\ & + Q(t)y(t)\exp(\int_{t-\sigma}^{t} \lambda(s)ds) \\ = \; & y(t)[-\lambda(t) + (T\lambda)(t)] = 0 \end{aligned}$$

and the proof is complete.

The next lemma will be used to obtain several sufficient conditions for the existence of a positive solution of (1). As in the case of Theorem 1, the motivation behind this result stems from the observation that if we look for a positive solution of (1) of the form

$$y(t) = \exp(\int_{t_1}^{t} \lambda(s)ds), \quad t \geq t_1$$

then $\lambda(t)$ satisfies the equation

$$\lambda(t) = [-P(t)\lambda(t-\tau) - \dot{P}(t)]\exp(-\int_{t-\tau}^{t} \lambda(s)ds) - Q(t)\exp(-\int_{t-\sigma}^{t} \lambda(s)ds).$$

Lemma 1. Assume that (2) holds and that the functions P, $\dot{P}$ and Q are all bounded on $[t_0, \infty)$. Suppose also that there exists a positive number

μ such that for every function λ in the set $\Lambda = \{\lambda \in C[[t_0 - m, \infty), \mathbf{R}^+] : \lambda(t) \leq \mu\}$ the following inequality holds:

$$(6) \qquad 0 \leq [-P(t)\lambda(t-\tau) - \dot{P}(t)]\exp(-\int_{t-\tau}^{t} \lambda(s)ds) \\ -Q(t)\exp(-\int_{t-\sigma}^{t} \lambda(s)ds) \leq \mu, \quad t \geq t_0.$$

Then for every $t_1 \geq t_0$, (1) has a (positive) increasing solution on $[t_1, \infty)$.

Proof. Let B and η be positive constants chosen in such a way that $\mid P(t) \mid \mu + \mid \dot{P}(t) \mid + \mid Q(t) \mid \leq B$ and

$$(7) \qquad \frac{B}{\eta} + \mid P(t) \mid e^{-\eta\tau} \leq \frac{1}{2} \quad \text{for} \quad t \geq t_1 - m$$

where $m = \max\{\tau, \sigma\}$.

For λ_1, $\lambda_2 \in \Lambda$ define $d(\lambda_1, \lambda_2) = \sup_{t \geq t_1 - m} \mid \lambda_1(t) - \lambda_2(t) \mid e^{-\eta t}$. Then (Λ, d) is a complete metric space. Define

$$(T\lambda)(t) = \begin{cases} [-P(t)\lambda(t-\tau) - \dot{P}(t)]\exp(-\int_{t-\tau}^{t} \lambda(s)ds) & t \geq t_1 \\ \qquad -Q(t)(-\int_{t-\sigma}^{t} \lambda(s)ds) & \\ T(\lambda)(t_1) & t_1 - m \leq t < t_1. \end{cases}$$

Then $T\lambda$ is continuous on $[t_1 - m, \infty)$, and by (6) $0 \leq (T\lambda)(t) \leq \mu$. Hence $T\lambda$ maps Λ into Λ. Moreover, T is a contraction mapping. For if λ_1, $\lambda_2 \in \Lambda$ and $t \geq t_1 - m$,

$$\begin{aligned} & \mid (T\lambda_1)(t) - (T\lambda_2)(t) \mid \\ \leq\ & \mid P(t) \mid\mid \lambda_1(t-\tau)\exp(-\int_{t-\tau}^{t} \lambda_1(s)ds) - \lambda_2(t-\tau)\exp(-\int_{t-\tau}^{t} \lambda_2(s)ds) \mid \\ & + \mid \dot{P}(t) \mid\mid \exp(-\int_{t-\tau}^{t} \lambda_1(s)ds) - \exp(-\int_{t-\tau}^{t} \lambda_2(s)ds) \mid \\ & + \mid Q(t) \mid\mid \exp(-\int_{t-\sigma}^{t} \lambda_1(s)ds) - \exp(-\int_{t-\sigma}^{t} \lambda_2(s)ds) \mid \\ =\ & \mid P(t) \mid\mid \lambda_1(t-\tau)[\exp(-\int_{t-\tau}^{t} \lambda_1(s)ds) - \exp(-\int_{t-\tau}^{t} \lambda_2(s)ds)] \\ & + \exp(-\int_{t-\tau}^{t} \lambda_2(s)ds)[\lambda_1(t-\tau) - \lambda_2(t-\tau)] \mid \end{aligned}$$

$$
\begin{aligned}
& + \mid \dot{P}(t) \mid\mid \exp(-\int_{t-\tau}^{t} \lambda_1(s)ds) - \exp(-\int_{t-\tau}^{t} \lambda_2(s)ds) \mid \\
& + \mid Q(t) \mid\mid \exp(-\int_{t-\sigma}^{t} \lambda_1(s)ds) - \exp(-\int_{t-\sigma}^{t} \lambda_2(s)ds) \mid \\
\leq\ & \mid P(t) \mid \mu \int_{t-\tau}^{t} \mid \lambda_1(s) - \lambda_2(s) \mid ds + \mid P(t) \mid\mid \lambda_1(t-\tau) - \lambda_2(t-\tau) \mid \\
& + \mid \dot{P}(t) \mid \int_{t-\tau}^{t} \mid \lambda_1(s) - \lambda_2(s) \mid ds + \mid Q(t) \mid \int_{t-\sigma}^{t} \mid \lambda_1(s) - \lambda_2(s) \mid ds \\
\leq\ & d(\lambda_1, \lambda_2)\{[\mid P(t) \mid \mu + \mid \dot{P}(t) \mid] \int_{t-\tau}^{t} e^{\eta s} ds + \mid Q(t) \mid \int_{t-\sigma}^{t} e^{\eta s} ds \\
& + \mid P(t) \mid e^{\eta(t-\tau)}\} \\
\leq\ & d(\lambda_1, \lambda_2) \cdot e^{\eta t} \frac{1}{\eta} [\mid P(t) \mid \mu + \mid \dot{P}(t) \mid + \mid Q(t) \mid + \eta \mid P(t) \mid e^{-\eta t}] \\
\leq\ & d(\lambda_1, \lambda_2) \cdot e^{\eta t} [\frac{B}{\eta} + \mid P(t) \mid e^{-\eta \tau}].
\end{aligned}
$$

From this inequality and by using (7) we see that for $t \geq t_1 - m$, $d(T\lambda_1, T\lambda_2) \leq (1/2)d(\lambda_1, \lambda_2)$. Hence there exists a (unique) solution $\lambda \in \Lambda$ of $T\lambda = \lambda$. To complete the proof it suffices to observe by direct substitution into (1) that

$$y(t) = \exp(\int_{t_1}^{t} \lambda(s)ds), \quad t \geq t_1 - m$$

is a positive (increasing) solution of (1) on $[t_1, \infty)$. The proof is complete.

Theorem 2. Assume that (2) holds and suppose that the functions $P(t)$, $\dot{P}(t)$ and $Q(t)$ are all bounded on $[t_0, \infty)$. Then in each of the following three cases and for every $t_1 \geq t_0$, (1) has a positive (increasing) solution on $[t_1, \infty)$.

(i) For $t \geq t_0$ and for some positive number α the following inequalities hold: $P(t) \leq 0$, $\dot{P}(t) + Q(t) \leq 0$, $(\sigma - \tau)Q(t) \geq 0$ and

$$[-1 + \mid P(t) \mid]\alpha + \mid \dot{P}(t) \mid + \mid Q(t) \mid \leq 0.$$

(ii) For $t \geq t_0$ and for some positive number β the following inequalities hold: $\mid P(t) \mid \beta + \dot{P}(t) + Q(t) \leq 0$, $(\sigma - \tau)Q(t) \geq 0$ and

$$[-1 + \mid P(t) \mid]\beta + \mid \dot{P}(t) \mid + \mid Q(t) \mid \leq 0.$$

(iii) For $t \geq t_0$ and for some positive number γ the following inequalities hold: $P(t) \leq 0$, $\dot{P}(t) \leq 0$, $Q(t) \leq 0$ and

$$[1 + P(t)]\gamma + \dot{P}(t) + Q(t) \geq 0.$$

Proof. In view of Lemma 1, it suffices to show that in each of the three cases (i), (ii) and (iii), condition (6) is satisfied.

First, assume that (i) holds. We will prove that (6) holds with μ in the definition of the set Λ of Lemma 1 taken to equal α. To simplify the notation, set

$$I = [-P(t)\lambda(t-\tau) - \dot{P}(t)]\exp(-\int_{t-\tau}^{t} \lambda(s)ds) - Q(t)\exp(-\int_{t-\sigma}^{t} \lambda(s)ds).$$

We must prove that $0 \leq I \leq \alpha$. Indeed,

$$I \leq \mid P(t) \mid \alpha + \mid \dot{P}(t) \mid + \mid Q(t) \mid \leq \alpha.$$

Furthermore,

$$\begin{aligned} I &\geq -\dot{P}(t)\exp(-\int_{t-\tau}^{t} \lambda(s)ds) - Q(t)\exp(-\int_{t-\sigma}^{t} \lambda(s)ds) \\ &\geq Q(t)[\exp(-\int_{t-\tau}^{t} \lambda(s)ds) - \exp(-\int_{t-\sigma}^{t} \lambda(s)ds)] \geq 0 \end{aligned}$$

and the proof in this case is complete.

Next, assume that (ii) holds. Here μ in the definition of Λ is taken equal to β. Clearly, $I \leq \beta$. Furthermore,

$$\begin{aligned} I &\geq [- \mid P(t) \mid \beta - \dot{P}(t)]\exp(-\int_{t-\tau}^{t} \lambda(s)ds) - Q(t)\exp(-\int_{t-\sigma}^{t} \lambda(s)ds) \\ &\geq Q(t)[\exp(-\int_{t-\tau}^{t} \lambda(s)ds) - \exp(-\int_{t-\sigma}^{t} \lambda(s)ds) \\ &\geq 0 \end{aligned}$$

and the proof in case (ii) is complete.

Finally, assume that (iii) holds. Here μ in the definition of Λ is taken equal to γ. Clearly in this case $I \geq 0$ and

$$I \leq -\gamma P(t) - \dot{P}(t) - Q(t) \leq \gamma.$$

The proof of the theorem is complete.

Example 1. The neutral differential equation

$$\frac{d}{dt}[y(t) + \frac{t}{2e(t+1)}y(t-1)] + \frac{t}{e^3(t+1)}y(t-2) = 0 \tag{8}$$

satisfies the hypotheses of Theorem 1 with $t_0 = 0$ and $\mu = 1$. Hence for every $t_1 \geq 0$, (8) has a positive solution on $[t_1, \infty)$.

Example 2. The neutral differential equation

$$\frac{d}{dt}[y(t) - \frac{1}{2}(1 - \frac{1}{t})y(t-1)] + \frac{1}{2t^2}y(t-2) = 0 \tag{9}$$

satisfies the hypotheses of Theorem 2(i) with $t_0 = 1$ and $\alpha = 1$. Therefore for every $t_1 \geq 1$, (9) has a positive solution on $[t_1, \infty)$.

Example 3. The neutral differential equation

$$\frac{d}{dt}[y(t) + \frac{1}{4}(\sin^3 t)y(t-3)] - 2(\sin^2 t)y(t-1) = 0 \tag{10}$$

satisfies the hypotheses of Theorem 2(ii) where t_0 equals any real number and with $\beta = 4$. Hence for every $t_1 \in R$, (10) has a positive solution on $[t_1, \infty)$.

Example 4. The neutral differential equation

$$\frac{d}{dt}[y(t) + \frac{1}{2}(\frac{1}{t} - 1)y(t-1)] - \frac{1}{2t^2}y(t-2) = 0 \tag{11}$$

satisfies the hypotheses of Theorem 2(iii) with $t_0 = 1$ and $\gamma = 1$. Therefore for every $t_1 \geq 1$, (11) has a positive solution on $[t_1, \infty)$.

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