Existence and Uniqueness Theorems for a Third-Order Generalized Boundary Value Problem*

by

Chaitan P. Gupta†
Mathematics and Computer Science Division
Argonne National Laboratory
Argonne, IL 60439-4801

Abstract

Let $f:[0,1]\times R^3 \to R$ be a function satisfying Caratheodory's conditions, $e(x) \in L^1[0,1]$, $\eta \in [0,1]$, $h \geq 0$, $k \geq 0$, $h+k > 0$. This paper studies existence and uniqueness questions for the third-order three-point generalized boundary value problem

$$u''' + f(x,u,u',u'') = e(x), \quad 0 < x < 1,$$

$$u(\eta) = 0, \quad u''(0) - hu'(0) = u''(1) + ku'(1) = 0,$$

and the associated special cases corresponding to one or both of h and k equal to infinity. The conditions on the nonlinearity f turn out to be related to the spectrum of the linear boundary value problem $u''' = \lambda u'$, $u(\eta) = 0$, $u''(0) - hu'(0) = u''(1) + ku'(1) = 0$, in a natural way.

Keywords and Phrases: third order, three-point, generalized boundary value problems, degree theoretic methods, existence, uniqueness theorems, Leray Schauder continuation theorem.

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1. Introduction

The study of third-order multipoint boundary value problems emerges in a wide variety of applications. For example, a three-point third order boundary value problem arises in the analysis of a Sandwich beam ([13]). Such problems have been of considerable interest in recent times. (See [1] - [4], [8] - [13], [15] - [17]). The purpose of this paper is to obtain some existence and uniqueness results for the following three-point third-order generalized boundary value problem:

$$u''' + f(x, u, u', u'') = e(x), \, 0 < x < 1,$$

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†Permanent address: Department of Mathematical Sciences, Northern Illinois University, DeKalb, IL 60115.

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$$u(\eta) = 0, \; u''(0) - hu'(0) = u''(1) + ku'(1) = 0, \tag{1.1}$$

where $f: [0, 1] \times \boldsymbol{R} \times \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ is a function satisfying Caratheodory's conditions, $e(x) \in L^1[0, 1]$, $0 \le \eta \le 1$, $h \ge 0$, $k \ge 0$, $h + k > 0$.

Aftabizadeh and Wiener obtained existence theorems for the boundary value problems

$$u''' + f(x, u, u', u'') = 0, \quad 0 < x < 1,$$

$$u(0) = u_0, \; u'(0) = \bar{u}_0, \; u'(1) = u_1; \tag{1.2}$$

$$u''' + f(x, u, u') = 0, \quad 0 < x < 1,$$

$$u(0) = u'(0) = u'(1) = 0; \tag{1.3}$$

and

$$u''' + f(x, u, u') = 0, \quad 0 < x < 1,$$

$$u(0) = u_0, \; u''(0) - h\, u'(0) = u''(1) + ku'(1) = 0, \tag{1.4}$$

$h \ge 0$, $h \ge 0$, and $h + k > 0$, in [3] when f is a real valued continuous function of its argument under some additional conditions on f, (for example, f is bounded on $[0, 1] \times \boldsymbol{R}^k$, $k = 2$ *or* 3). We note that the inhomogeneous boundary value problem (1.2) or (1.4) can be transformed into a homogeneous boundary value problem with a zero right-hand side by a change of the dependent variable u to a new dependent variable v by setting

$$u = v + u_0 + \bar{u}_0 x + \frac{1}{2}(u_1 - \bar{u}_0)x^2$$

for the problem (1.2) and

$$u = v + u_0$$

for the problem (1.4). The methods used by Aftabizadeh and Wiener in [3] to get their existence theorems consist of reducing the boundary value problem in question to a second order integro-differential boundary value problem and then applying the Schauder fixed-point theorem following Cordenanu [7] and Bebernes-Gaines [5]. They also obtain an existence theorem for (1.2) when f satisfies a Lipschitz condition so as to enable the use of the Contraction mapping theorem. A generalization for their existence and uniqueness theorem for (1.3) was obtained by Aftabizadeh, Gupta, and Xu in [2].

We remark that the methods and assumptions of [3] depend heavily on the fact that each of the boundary value problems (1.2), (1.3), and (1.4) has a zero right-hand side and each is a two-point boundary value problem.

Our purpose in this paper is to obtain existence and uniqueness theorems for the boundary value problem (1.1) and the associated special cases corresponding to one or both of h and k equal to infinity. We use degree theoretic methods, and we impose conditions on the nonlinearity $f(x, u, u', u'')$ that are natural to the problem. More precisely, the conditions on $f(x, u, u', u'')$ are to be such that the nonlinearity represented by $f(x, u, u', u'')$ in (1.1) does not interfere with the spectrum of the linear boundary value problem

$$u'''(x) = \lambda u'(x), \quad 0 < x < 1,$$

$$u(\eta) = 0,\ u''(0) - hu'(0) = u''(1) + ku'(1) = 0. \tag{1.5}$$

We present our existence theorems in Section 2 and uniqueness theorems in Section 3.

2. Existence Theorems

Let $f : [0, 1] \times \boldsymbol{R}^3 \to \boldsymbol{R}$ be a function satisfying Caratheodory's conditions, that is, for a.e. x in $[0, 1]$, the function $(y, z, w) \in \boldsymbol{R}^3 \to f(x, y, z, w) \in \boldsymbol{R}$ is continuous; for every $(y, z, w) \in \boldsymbol{R}^3$, the function $x \in [0, 1] \to f(x, y, z, w) \in \boldsymbol{R}$ is measurable; and for every $r > 0$, there is a real valued function $g_r(x) \in L^1[0, 1]$ such that for a.e. x in $[0, 1]$,

$$|f(x, y, z, w)| \le g_r(x),$$

whenever, $|y| \le r,\ |z| \le r,\ |w| \le r$.

Let $e(x) \in L^1[0, 1]$, $h, k \in \boldsymbol{R}$, $h \ge 0$, $k \ge 0$, $h + k > 0$, and $0 \le \eta \le 1$ be given. We consider the following generalized third-order three-point boundary value problem:

$$u'''(x) + f(x, u(x), u'(x), u''(x)) = e(x), \quad 0 < x < 1,$$

$$u(\eta) = 0,\ u''(0) - hu'(0) = u''(1) + ku'(1) = 0. \tag{2.1}$$

LEMMA 1. *The linear boundary value problem,*

$$u'''(x) = e(x), \quad 0 < x < 1,$$

$$u(\eta) = 0,\ u''(0) - hu'(0) = u''(1) + ku'(1) = 0, \tag{2.2}$$

is uniquely solvable.

Proof. It is easy to see that

$$u(x) = \frac{1}{2}\int_0^x (x - \xi)^2 e(\xi)d\xi + A + Bx + Cx^2,$$

$x \in [0, 1]$, with

$$A = -\frac{1}{2}\int_0^\eta (\eta - \xi)^2 e(\xi)d\xi - B\eta - C\eta^2,$$

$$B = -\frac{1}{h+k+hk}\left[\int_0^1 e(\xi)d\xi + k\int_0^1 (1-\xi)e(\xi)d\xi\right],$$

$$C = -\frac{h}{2(h+k+hk)}\left[\int_0^1 e(\xi)d\xi + k\int_0^1 (1-\xi)e(\xi)d\xi\right],$$

is the unique solution of (2.2). Hence the lemma holds. □

Define a linear operator $L : D(L) \subset C^2[0, 1] \to L^1[0, 1]$ by setting

$$D(L) = \{u \in C^2[0, 1] \mid u,\ u',\ u'' \text{ are absolutely continuous on}$$

$$[0, 1],\ u(\eta) = 0,\ u''(0) - hu'(0) = 0,\ u''(1) + ku'(1) = 0\}, \tag{2.3}$$

and for $u \in D(L)$,

$$Lu = u'''. \tag{2.4}$$

Now, by Lemma 1, L is a 1-1, onto operator and the linear operator $K : L^1[0, 1] \to C^2[0, 1]$ defined, for $e \in L^1[0, 1]$, by

$$Ke = u, \tag{2.5}$$

where u is the unique solution of the linear boundary value problem (2.2), is such that, for $u \in D(L)$, $KLu = u$, and for $e \in L^1[0, 1]$, $Ke \in D(L)$ with $LKe = e$.

Define a nonlinear operator $N : C^2[0, 1] \to L^1[0, 1]$ by setting, for $u \in C^2[0, 1]$,

$$(Nu)(x) = f(x, u(x), u'(x), u''(x)),\ x \in [0, 1]. \tag{2.6}$$

Since f satisfies Caratheodory's conditions, one can easily see, by using the Arzela-Ascoli theorem, that the operator $KN : C^2[0, 1] \to C^2[0, 1]$ is a compact operator, that is, that KN maps bounded sets in $C^2[0, 1]$ into relatively compact subsets of $C^2[0, 1]$.

Definition 1. A function $F : [0, 1] \times R^2 \to R$ is said to satisfy L^2-Caratheodory's conditions if for a.e. x in $[0, 1]$, the function $(y, z) \in R^2 \to F(x, y, z) \in R$ is continuous; for every $(y, z) \in R^2$, the function $x \in [0, 1] \to F(x, y, z) \in R$ is measurable; and for every $r > 0$, there is a real valued function $g_r(x) \in L^2[0, 1]$ such that for a.e. x in $[0, 1]$,

$$|F(x, y, z)| \le g_r(x),$$

whenever $|y| \le r$ and $|z| \le r$.

THEOREM 1. *Let $f : [0, 1] \times R^3 \to R$ satisfy Caratheordory's conditions. Suppose that there exist functions $a(x)$, $b(x)$, and $c(x)$ in $L^1[0, 1]$, a function $d(x)$ in $L^2[0, 1]$, and an $m \in [0, 1)$ such that*

$$|f(x, y, z, w)| \leq |a(x)| + |b(x)|\ |y|^m + |c(x)|\ |z|^m + |d(x)|\ |w|^m, \tag{2.7}$$

for a.e. x in $[0, 1]$ *and all* $(y, z, w) \in R^3$. *Let* $h, k \in R$, $h \geq 0$, $k \geq 0$, $h + k > 0$, *and* $\eta \in [0, 1]$ *be given. Then for every* $e(x) \in L^1[0, 1]$ *(given) the boundary value problem (2.1) has at least one solution.*

Proof. It is clear that $u \in C^2[0, 1]$ is a solution to the boundary value problem (2.1) if and only if u is a solution of the operator equation

$$Lu + Nu = e.$$

The operator equation $Lu + Nu = e$ is equivalent to the equation

$$u + KNu = Ke.$$

We apply the Leray-Schauder continuation theorem (Theorem IV.5, [14]) to obtain the existence of a solution for $u + KNu = Ke$, or equivalently for the boundary value problem (2.1). It is then sufficient to verify that the set of all possible solutions of the family of equations

$$u'''(x) + \lambda f(x, u(x), u'(x), u''(x)) = \lambda e(x),\ 0 < x < 1,$$

$$u(\eta) = 0,\ u''(0) - hu'(0) = 0,\ u''(1) + ku'(1) = 0, \tag{2.8}$$

is, *a priori*, bounded in $C^2[0, 1]$ by a constant independent of $\lambda \in [0, 1]$. Now, we may assume, without loss of generality, that $h > 0$ in the following. We also note that if $u(x)$ is a solution of (2.8) for some $\lambda \in [0, 1]$, then

$$\|u\|_\infty \leq \|u'\|_\infty \leq |u'(0)| + \|u''\|_2. \tag{2.9}$$

Let $u(x)$ be a possible solution of (2.8) for some $\lambda \in [0, 1]$. On integrating by parts the equation obtained by multiplying the equation in (2.8) by $u'(x)$ and using (2.7) and (2.9), we get

$$0 = \int_0^1 u'''(x)u'(x)dx + \lambda\int_0^1 f(x, u(x), u'(x), u''(x))u'(x)dx - \lambda\int_0^1 e(x)u'(x)dx$$

$$= u''(1)u'(1) - u''(0)u'(0) - \int_0^1 [u''(x)]^2dx + \lambda\int_0^1 f(x, u(x), u'(x), u''(x))u'(x)dx$$

$$- \lambda\int_0^1 e(x)u'(x)dx$$

$$= -k[u'(1)]^2 - h[u'(0)]^2 - \int_0^1 [u''(x)]^2dx + \lambda\int_0^1 f(x, u(x), u'(x), u''(x))u'(x)dx$$

$$-\lambda\int_0^1 e(x)u'(x)dx$$

$$\leq -h[u'(0)]^2 - \int_0^1 [u''(x)]^2 dx + \lambda\int_0^1 |f(x, u(x), u'(x), u''(x))|\,|u'(x)|dx$$

$$+\lambda\int_0^1 |e(x)|\;|u'(x)|dx$$

$$\leq -h[u'(0)]^2 - \int_0^1 [u''(x)]^2 dx + \int_0^1 [\,|a(x)| + |b(x)|\,|u(x)|^m + |c(x)|\,|u'(x)|^m$$

$$+|d(x)|\,|u''(x)|^m]\,|u'(x)|dx + \int_0^1 |e(x)|\,|u'(x)|dx$$

$$\leq -h[u'(0)]^2 - \int_0^1 [u''(x)]^2 dx + [\,\|a\|_1 + \|b\|_1\|u\|_\infty^m + \|c\|_1\|u'\|_\infty^m$$

$$+\|d\|_2\|u''\|_2^m]\,\|u'\|_\infty + \|e\|_1\|u'\|_\infty$$

$$\leq -h[u'(0)]^2 - \int_0^1 [u''(x)]^2 dx + (\|b\|_1 + \|c\|_1)\|u'\|_\infty^{1+m}$$

$$+\|d\|_2\|u''\|_2^m(|u'(0)| + \|u''\|_2) + (\|a\|_1 + \|e\|_1)\|u'\|_\infty$$

$$\leq -h[u'(0)]^2 - \int_0^1 [u''(x)]^2 dx + (\|b\|_1 + \|c\|_1 + \|d\|_2)\,(|u'(0)| + \|u''\|_2)^{1+m}$$

$$+(\|a\|_1 + \|e\|_1)(|u'(0)| + \|u''\|_2)$$

$$\leq -h[u'(0)]^2 - \int_0^1 [u''(x)]^2 + 2^m(\|b\|_1 + \|c\|_1 + \|d\|_2)(|u'(0)|^{m+1} + \|u''\|_2^{m+1})$$

$$+(\|a\|_1 + \|e\|_1)(|u'(0)| + \|u''\|_2).$$

Hence,

$$\|u''\|_2^2 + h[u'(0)]^2 \le 2^m(\|b\|_1 + \|c\|_1 + \|d\|_2)(|u'(0)|^{m+1} + \|u''\|_2^{m+1})$$

$$+(\|a\|_1 + \|e\|_1)(|u'(0)| + \|u\|_2).$$

Now, $h > 0$ and $m \in [0, 1)$ so that $m + 1 < 2$. We easily see that there is a constant C, independent of $\lambda \in [0, 1]$, such that

$$\|u''\|_2 \le C, \ |u'(0)| \le C, \ \text{and} \ |u''(0)| \le hC. \tag{2.10}$$

Hence, we get from (2.9) and (2.10) that

$$\|u\|_\infty \le \|u'\|_\infty \le |u'(0)| + \|u''\|_2 \le 2C. \tag{2.11}$$

Next we have from the equation in (2.8), using (2.7) and (2.11), that

$$\|u'''\|_1 \le \|e\|_1 + \|a\|_1 + \|b\|_1\|u\|_\infty^m + \|c\|_1\|u'\|_\infty^m + \|d\|_2\|u''\|_2^m$$

$$\le \|e\|_1 + \|a\|_1 + (\|b\|_1 + \|c\|_1)(2C)^m + \|d\|_2 C^m. \tag{2.12}$$

Finally, since

$$u''(x) = u''(0) + \int_0^x u'''(t)dt,$$

we see from (2.11) and (2.12) that there is a constant C_1 independent of $\lambda \in [0, 1]$ such that

$$\|u''\|_\infty \le C_1;$$

and thus there exists a constant C_2, independent of $\lambda \in [0, 1]$, such that

$$\|u\|_{C^2[0,1]} \le C_2. \quad \square$$

Remark 1. We tacitly assumed that $0 < h < \infty$ in the proof of Theorem 1. Now, if $h = \infty$ and $k \ge 0$ is such that $0 < k < \infty$, then we can obtain

$$\|u''\|_2 \le C, \ |u'(1)| \le C, \ \text{and} \ |u''(1)| \le kC$$

instead of (2.10); and we can use $u''(x) = u''(1) - \int_x^1 u'''(t)dt$ to conclude $\|u''\|_\infty \le C_1$. If $h = \infty$ and $k = 0$ so that $u'(0) = 0$ and $u''(1) = 0$, then (2.10) gives $\|u''\|_2 \le C$. In this case we use $u''(x) = -\int_x^1 u'''(t)dt$ to conclude $\|u''\|_\infty \le C_1$. Finally, if $h = \infty$ and $k = \infty$, so that $u'(0) = u'(1) = 0$, we have an $\xi \in [0, 1]$ such that $u''(\xi) = 0$. In this case $u''(x) = \int_\xi^x u'''(t)dt$ helps us to get the desired conclusion. We have accordingly shown that Theorem 1 is valid in all cases.

Remark 2. We note that we can replace the assumption (2.7) in Theorem 1 by the following:

(i) there exist functions $a(x)$, $b(x)$, and $c(x)$ in $L^1[0, 1]$, a function $d(x)$ in $L^2[0, 1]$, and an $m \in [0, 1)$ such that

$$f(x, y, z, w)z \leq [|a(x)| + |b(x)| \; |y|^m + |c(x)| \; |z|^m + |d(x)| \; |w|^m]|z|$$

for a.e. x in $[0, 1]$ and all $(y, z, w) \in R^3$, and

(ii) there exist a function $F : [0, 1] \times R^2 \to R$ satisfying L^2-Caratheodory conditions and a function $\alpha(x) \in L^1[0, 1]$ such that

$$|f(x, y, z, w)| \leq |F(x, y, z)| \; |w| + \alpha(x)$$

for a.e. $x \in [0, 1]$ and all $(y, z, w) \in R^3$. This observation is useful for determining the existence of a solution in one of our uniqueness theorems in Section 3.

Remark 3. If $m = 1$ in (2.7), then it is easy to modify the proof of Theorem 1 to obtain at least one solution of the boundary value problem (2.1) provided that

$$2\|b\|_1 + 2\|c\|_1 + (\sqrt{2} - 1)\|d\|_2 < \min\{\max(h, k), 1\}.$$

The following theorem improves the condition in Remark 3 when $m = 1$ in (2.7) and $h = \infty$ (respectively, $k = \infty$) in the boundary value problem (2.1).

THEOREM 2. *Let $f : [0, 1] \times R^3 \to R$ satisfy Caratheodory's conditions. Suppose that there exist functions $a(x)$, $b(x)$, and $c(x)$ in $L^1[0, 1]$, and a function $d(x)$ in $L^2[0, 1]$ such that*

$$|f(x, y, z, w)| \leq |a(x)| + |b(x)| \; |y| + |c(x)| \; |z| + |d(x)| \; |w|, \tag{2.13}$$

for a.e. $x \in [0, 1]$ and all $(y, z, w) \in R^3$. Let $k \in R$, $k \geq 0$, and $\eta \in [0, 1]$ be given. Then for every $e(x) \in L^1[0, 1]$ (given), the boundary value problem

$$u'''(x) + f(x, u, (x), u'(x), u''(x)) = e(x), \; 0 < x < 1,$$

$$u(\eta) = 0, \; u'(0) = 0, \; u''(1) + ku'(1) = 0, \tag{2.14}$$

has at least one solution if

$$\|b\|_1 M_\eta + \|c\|_1 + \|d\|_2 < 1, \tag{2.15}$$

where $M_\eta = \max(\eta, 1-\eta)$.

Proof. It suffices to verify, as in the proof of Theorem 1, that the set of solutions of the family of equations

$$u'''(x) + \lambda f(x, u(x), u'(x), u''(x)) = \lambda e(x), \; 0 < x < 1,$$

$$u(\eta) = 0, \ u'(0) = 0, \ u''(1) + k\, u'(1) = 0, \tag{2.16}$$

is, *a priori,* bounded in $C^2[0, 1]$ by a constant independent of $\lambda \in [0, 1]$.

Let $u(x)$ be a possible solution of (2.16) for some $\lambda \in [0, 1]$. Then,

$$\|u\|_\infty \le M_\eta \|u'\|_\infty \text{ and } \|u'\|_\infty \le \|u''\|_2. \tag{2.17}$$

Next, as in the proof of Theorem 1, we integrate by parts the equation obtained by multiplying the equation in (2.16) by $u'(x)$. Using (2.13) and (2.17), we obtain that

$$0 = \int_0^1 u'''(x)u'(x)dx + \lambda\int_0^1 f(x,\, u(x),\, u'(x),\, u''(x))u'(x)dx - \lambda\int_0^1 e(x)u'(x)dx$$

$$= u''(1)u'(1) - \int_0^1 (u''(x))^2 dx + \lambda\int_0^1 f(x,\, u(x),\, u'(x),\, u''(x))u'(x)dx - \lambda\int_0^1 e(x)u'(x)dx$$

$$\le -k[u'(1)]^2 - \int_0^1 [u''(x)]^2 dx + \lambda\int_0^1 |f(x,\, u(x),\, u'(x),\, u''(x))|\ |u'(x)|dx$$

$$\le -k[u'(1)]^2 - \int_0^1 [u''(x)]^2 dx + \int_0^1 [\,|a(x)| + |b(x)|\ |u(x)| + |c(x)|\ |u'(x)|$$

$$+ |d(x)|\ |u''(x)|\,]u'(x)dx + \int_0^1 |e(x)|\ |u'(x)|dx$$

$$\le -k[u'(1)]^2 - \int_0^1 [u''(x)]^2 dx + [\,\|a\|_1 + \|b\|_1 \|u\|_\infty + \|c\|_1 \|u'\|_\infty + \|d\|_2 \|u''\|_2]\|u'\|_\infty$$

$$+ \|e\|_1 \|u'\|_\infty$$

$$\le -k[u'(1)]^2 - \|u''\|_2^2 + (\|b\|_1 M_\eta + \|c\|_1 + \|d\|_2)\|u''\|_2^2 + (\|a\|_1 + \|e\|_1)\|u''\|_2\,.$$

Hence,

$$k[u'(1)]^2 + [1 - (\|b\|_1 M_\eta + \|c\|_1 + \|d\|_2)]\|u''\|_2^2 \le (\|a\|_1 + \|e\|_1)\|u''\|_2,$$

and so

$$\|u''\|_2 \le \frac{\|a\|_1 + \|e\|_1}{1 - (\|b\|_1 M_\eta + \|c\|_1 + \|d\|_2)} \equiv C \tag{2.18}$$

because of our assumption (2.15).

It follows from (2.17), (2.18), and the equation in (2.16) that there exists a constant, independent of $\lambda \in [0, 1]$ and $k \geq 0$, such that

$$\|u\|_\infty \leq C_1, \quad \|u'\|_\infty \leq C_1, \quad \text{and } \|u'''\|_1 \leq C_1. \tag{2.19}$$

Three cases arise depending on $k = 0$, $0 < k < \infty$, and $k = \infty$. In the case $k = 0$, we have $u''(1) = 0$; using $u''(x) = -\int_x^1 u'''(t)dt$, we find that $\|u''\|_\infty \leq \|u'''\|_1 \leq C_1$. In the case $0 < k < \infty$, we have from (2.18) that $\sqrt{k}\,|u'(1)| \leq \sqrt{C_2}$ with $C_2 = (\|a\|_1 + \|e\|_1)C$. Thus, $|u''(1)| = |ku'(1)| \leq \sqrt{k}\sqrt{C_2}$, and we can now use $u''(x) = u''(1) - \int_x^1 u'''(t)dt$ to get $\|u''\|_\infty \leq \sqrt{k}\,\sqrt{C_2} + C_1$. Finally, in the case $k = \infty$, we have that $u'(1) = 0$; thus we get $\xi \in [0, 1]$ such that $u''(\xi) = 0$, since $u'(0) = 0$. Finally, we use $u''(x) = \int_\xi^x u'''(t)dt$ to get $\|u''\|_\infty \leq \|u'''\|_1 \leq C_1$.

We have thus shown that the set of all possible solutions of (2.16) is, *a priori*, bounded in $C^2[0, 1]$, by a constant independent of $\lambda \in [0, 1]$, in all cases. □

Remark 4. If the functions $b(x)$ and $c(x)$ in Theorem 2 are assumed to be in $L^2[0, 1]$, then the assumption (2.15) can be replaced by the (improved) estimate

$$4 M_\eta \|b\|_2 + 2\pi \|c\|_2 + \pi^2 \|d\|_2 < \pi^2, \tag{2.20}$$

because we can then use the sharper Wirtinger inequalities (see Lemmas 1.1, 1.2, and 1.3 in [2]),

$$\|u\|_2 \leq \frac{2}{\pi} M_\eta \|u'\|_2 \quad \text{and} \quad \|u'\|_2 \leq \frac{2}{\pi}\|u''\|_2, \tag{2.21}$$

for a solution $u(x)$ of (2.16).

Also, if the functions $b(x)$, $c(x)$, and $d(x)$ in Theorem 2 are assumed to be in $L^\infty[0, 1]$, then the assumption (2.15) can be replaced by the (improved) estimate

$$8M_\eta \|b\|_\infty + 4\pi \|c\|_\infty + 2\pi^2 \|d\|_\infty < \pi^3, \tag{2.22}$$

by our using the sharper Wirtinger inequalities (2.21).

Remark 5. If we assume that the function $f : [0, 1] \times R^3 \to R$ in Theorem 2 satisfies the condition

$$|f(x, y, z, w)| \leq |F(x, y, z)|\ |w| + \alpha(x),$$

for a.e. $x \in [0, 1]$ and all $(y, z, w) \in R^3$, where $\alpha(x) \in L^1[0, 1]$ and $F : [0, 1] \times R^2 \to R$ satisfies L^2-Caratheodory's conditions and the functions $b(x)$, $c(x)$, and $d(x)$ are measurable functions on $[0, 1]$ such that

$$b(x) \le b_0, \; c(x) \le c_0, \text{ and } d(x) \le d_0$$

for a.e. $x \in [0, 1]$ with

$$8M_\eta b_0 + 4\pi c_0 + 2\pi^2 d_0 < \pi^3,$$

then we can replace the assumption (2.13) by the weaker assumption

$$f(x, y, z, w)z \le a(x)|z| + b(x)|yz| + c(x)|z|^2 + d(x)|wz|,$$

to get the existence of a solution for the boundary value problem (2.14). In this way Theorem 2 generalizes Theorem 2.3 of [2], which corresponds to the case $k = 0$ in Theorem 2.

Remark 6. We remark that assumptions (2.15) and (2.20) allow the asymptotic values of $|z^{-1} f(x, y, z, w)|$ to cross infinitely many eigenvalues of the linear eigenvalue problem

$$u'''(x) = \lambda u'(x),$$

$$u(\eta) = 0, \; u'(0) = 0, \; u''(1) + ku'(1) = 0,$$

while the assumption (2.22) does not allow the asymptotic values of $|z^{-1} f(x, y, z, w)|$ to cross any of the eigenvalues of the same linear eigenvalue problem. This explains, in part, why (2.22) is sharper than both (2.15) and (2.20).

THEOREM 3. *Let $f : [0, 1] \times \mathbf{R}^3 \to \mathbf{R}$ satisfy Caratheodory's conditions. Assume that*

(i) there exist functions $a(x)$ and $b(x)$ in $L^1[0, 1]$, a function $d(x)$ in $L^2[0, 1]$, and an $m \in [0, 1)$ such that

$$|f(x, y, 0, w)| \le |a(x)| + |b(x)| \, |y|^m + |d(x)| \, |w|^m, \tag{2.23}$$

for a.e. $x \in [0, 1]$ and all $y, w \in \mathbf{R}$;

(ii) there exist a function $\alpha(x) \in L^1[0, 1]$ and a function $F : [0, 1] \times \mathbf{R}^2 \to \mathbf{R}$ satisfying L^2-Caratheodory's conditions such that

$$|f(x, y, z, w)| \le |F(x, y, z)| \, |w| + \alpha(x), \tag{2.24}$$

for a.e. $x \in [0, 1]$ and all $(y, z, w) \in \mathbf{R}^3$; and

(iii) for a.e. $x \in [0, 1]$ and all $(y, z, w) \in \mathbf{R}^3$, the partial derivative $\frac{\partial f}{\partial z}(x, y, z, w)$ exists and

$$\frac{\partial f}{\partial z}(x, y, z, w) \le 0. \tag{2.25}$$

Let $h, k \in \mathbf{R}$, $h \ge 0$, $k \ge 0$, $h + k > 0$, and $\eta \in [0, 1]$ be given. Then for every $e(x) \in L^1[0, 1]$ (given) the boundary value problem (2.1) has at least one solution.

Proof. It suffices to verify that the set of all possible solutions of the family of equations (2.8) is, *a priori*, bounded in $C^2[0, 1]$ by a constant independent of $\lambda \in [0, 1]$, as in the proof of Theorem 1. We may also assume that $h > 0$, as in the proof of Theorem 1. In fact, we shall assume that $0 < h < \infty$ in this proof; the other cases can be treated in a similar manner as indicated in Remark 1.

Let $u(x)$ be a possible solution of (2.8) for some $\lambda \in [0, 1]$, so that (2.9) holds. Using (2.23) and (2.25) we then have, as in the proof of Theorem 1, that

$$0 = \int_0^1 u'''(x)u'(x)dx + \lambda\int_0^1 f(x, u(x), u'(x), u''(x))u'(x)dx - \lambda\int_0^1 e(x)u'(x)dx$$

$$= -k[u'(1)]^2 - h[u'(0)]^2 - \int_0^1 [u''(x)]^2 dx$$

$$+ \lambda\int_0^1\int_0^1 \frac{\partial f}{\partial z}(x, u(x), su'(x), u''(x))(u'(x))^2 ds dx$$

$$+ \lambda\int_0^1 f(x, u(x), 0, u''(x))u'(x)dx - \lambda\int_0^1 e(x)u'(x)dx$$

$$\leq -h[u'(0)]^2 - \int_0^1 [u''(x)]^2 dx + \lambda\int_0^1 [|a(x)| + |b(x)|\ |u(x)|^m$$

$$+ |d(x)|\ |u''(x)|^m]|u'(x)|dx + \lambda\int_0^1 |e(x)|\,|u'(x)|dx$$

$$\leq -h[u'(0)]^2 - \int_0^1 [u''(x)]^2 dx + (\|a\|_1 + \|e\|_1)(|u'(0)| + \|u''\|_2)$$

$$+ 2^m(\|b\|_1 + \|d\|_2)(|u'(0)|^{1+m} + \|u''\|_2^{1+m}).$$

Hence,

$$\|u''\|_2^2 + h[u'(0)]^2 \leq 2^m(\|b\|_1 + \|d\|_2)(|u'(0)|^{m+1} + \|u''\|_2^{m+1})$$

$$+ (\|a\|_1 + \|e\|_1)(|u'(0)| + \|u''\|_2).$$

Thus there exists a constant C, independent of $\lambda \in [0, 1]$ such that

$$\|u''\|_2 \leq C, \quad |u'(0)| \leq C, \quad \text{and} \quad |u''(0)| \leq hC. \tag{2.26}$$

It follows from the equation in (2.8), if we use (2.9), (2.24), and (2.26), that there exists a constant C_1, independent of $\lambda \in [0, 1]$ such that

$$\|u'''\|_1 \le C_1. \tag{2.27}$$

Finally, using (2.26) and (2.27), we get from $u''(x) = u''(0) + \int_0^x u'''(t)dt$ that

$$\|u''\|_\infty \le hC + C_1.$$

Also, we have from (2.9) and (2.26) that

$$\|u\|_\infty \le \|u'\|_\infty \le |u'(0)| + \|u''\|_2 \le 2C.$$

Thus, we see that the set of all possible solutions of (2.8) is, *a priori*, bounded in $C^2[0, 1]$ by a constant independent of $\lambda \in [0, 1]$. □

Remark 7. Theorem 3 generalizes Theorem 3 of [3].

Remark 8. We remark that the conclusion of Theorem 3 holds if we replace (2.25) by the weaker assumption

$$\frac{\partial f}{\partial z}(x, y, z, w) \le \frac{1}{2} \min\{\max(h, k), 1\}.$$

THEOREM 4. *Let $f : [0, 1] \times \boldsymbol{R}^3 \to \boldsymbol{R}$ satisfy Caratheodory's conditions. Assume that*

(i) there exist functions $a(x)$ and $b(x)$ in $L^1[0, 1]$, and a function $d(x)$ in $L^2[0, 1]$ such that

$$|f(x, y, 0, w)| \le |a(x)| + |b(x)|\ |y| + |d(x)|\ |w|, \tag{2.28}$$

for a.e. $x \in [0, 1]$ and all y and w in $\boldsymbol{R}$;

(ii) there exist a function $\beta(x) \in L^2[0, 1]$ and a function $F : [0, 1] \times \boldsymbol{R}^2 \to \boldsymbol{R}$ satisfying L^2-Caratheodory's conditions such that

$$|f(x, y, z, w)| \le |F(x, y, z)|\ |w| + \beta(x), \tag{2.29}$$

for a.e. $x \in [0, 1]$ and all $(y, z, w) \in \boldsymbol{R}^3$, and

(iii) for a.e. $x \in [0, 1]$ and all $(y, z, w) \in \boldsymbol{R}^3$, the partial derivative $\frac{\partial f}{\partial z}(x, y, z, w)$ exists and there exists an $\alpha \ge 0$ such that

$$\frac{\partial f}{\partial z}(x, y, z, w) \le \alpha, \tag{2.30}$$

for a.e. $x \in [0, 1]$ and all $(y, z, w) \in \boldsymbol{R}^3$. Further, let $k \in \boldsymbol{R}$, $k \ge 0$ and $\eta \in [0, 1]$ be given. Then for every $e(x) \in L^2[0, 1]$ (given), the boundary value problem

$$u'''(x) + f(x, u(x), u'(x), u''(x)) = e(x), 0 < x < 1,$$

$$u(\eta) = 0,\ u'(0) = 0,\ u''(1) + ku'(1) = 0, \tag{2.31}$$

has at least one solution, provided

$$4\alpha + 2\pi\sqrt{M_\eta}\ \|b\|_1 + \pi^2\|d\|_2 < \pi^2, \tag{2.32}$$

where $M_\eta = \max\{\eta,\ 1-\eta\}$.

Proof. It suffices to verify, as in the proof of Theorem 2, that the set of solutions of the family of equations (2.16) is, *a priori*, bounded in $C^2[0, 1]$ by a constant independent of $\lambda \in [0, 1]$.

Let $u(x)$ be a possible solution of (2.16) for some $\lambda \in [0, 1]$. Then,

$$\|u\|_\infty \le \sqrt{M_\eta}\,\|u'\|_2,\quad \|u'\|_2^2 \le \frac{4}{\pi^2}\|u''\|_2^2,\ \text{ and }\ \|u'\|_\infty \le \|u''\|_2. \tag{2.33}$$

Next, as in the proof of Theorem 2, using (2.28), (2.30) and (2.33) we obtain that

$$0 = \int_0^1 u'''(x)u'(x)dx + \lambda\int_0^1 f(x,\ u(x),\ u'(x),\ u''(x))u'(x)dx - \lambda\int_0^1 e(x)u'(x)dx$$

$$= -k[u'(1)]^2 - \int_0^1 [u''(x)]^2dx + \lambda\int_0^1\int_0^1 \frac{\partial f}{\partial z}(x,\ u(z),\ su'(x),\ u''(x))[u'(x)]^2dsdx$$

$$+ \lambda\int_0^1 f(x,\ u(x),\ 0,\ u''(x))u'(x)dx - \lambda\int_0^1 e(x)u'(x)dx$$

$$\le -k[u'(1)]^2 - \int_0^1 [u''(x)]^2dx + \alpha\|u'\|_2^2 + [\|a\|_1 + \|b\|_1\|u\|_\infty + \|d\|_2\|u''\|_2]\|u'\|_\infty$$

$$+ \|e\|_1\|u'\|_\infty$$

$$\le -k[u'(1)]^2 - \int_0^1 [u''(x)]^2dx + \left[\frac{4}{\pi^2}\alpha + \frac{2}{\pi}\sqrt{M_\eta}\ \|b\|_1 + \|d\|_2\right]\|u''\|_2^2$$

$$+ (\|a\|_1 + \|e\|_1)\|u''\|_2,$$

and hence,

$$k[u'(1)]^2 + (1 - \frac{1}{\pi^2}[4\alpha + 2\pi\sqrt{M_\eta}\ \|b\|_1 + \pi^2\|d\|_2])\|u''\|_2^2$$

$$\le (\|a\|_1 + \|e\|_1)\|u''\|_2 \tag{2.34}$$

If we use (2.29), (2.32), and the equation in (2.16), it follows from (2.33) and (2.34) that there exists a constant C, independent of $\lambda \in [0, 1]$, such that

$$\|u'''\|_1 \le C, \ \|u\|_\infty \le C, \ \|u'\|_\infty \le C, \text{ and } \|u''\|_2 \le C. \tag{2.35}$$

Proceeding as in the last part of Theorem 2, we can find a constant C_1, independent of $\lambda \in [0, 1]$, such that

$$\|u''\|_\infty \le C_1.$$

We have thus shown that the set of solutions of (2.16) is, *a priori*, bounded in $C^2[0, 1]$, by a constant independent of $\lambda \in [0, 1]$. □

Remark 9. We can replace assumption (2.28) in Theorem 4 by the existence of partial derivatives $\frac{\partial f}{\partial w}(x, y, 0, w)$ and $\frac{\partial f}{\partial y}(x, y, 0, 0)$ for a.e. $x \in [0, 1]$ and all y and w in R with

$$\left|\frac{\partial f}{\partial y}(x, y, 0, 0)\right| \le |b(x)|,$$

and

$$\left|\frac{\partial f}{\partial w}(x, y, 0, w)\right| \le |d(x)|,$$

where $b(x)$ and $d(x)$ are as in Theorem 4 and $a(x) = f(x, 0, 0, 0)$.

Remark 10. We note that if $f(x, y, z, w)$ is independent of w then we do not need f to satisfy a condition of the form $|f(x, y, z, w)| \le |F(x, y, z)|\,|w| + \beta(x)$, with F satisfying L^2-Caratheodory's conditions as assumed in some of the theorems above.

3. Uniqueness Theorems

THEOREM 5. *Let $f : [0, 1] \times R^3 \to R$ satisfy Caratheodory's conditions. Assume that*

(i)

$$(f(x, y_1, z_1, w_1) - f(x, y_2, z_2, w_2))(z_1 - z_2) \le 0, \tag{3.1}$$

for a.e. $x \in [0, 1]$ and all $(y_i, z_i, w_i) \in R^3$, $i = 1, 2$; and

(ii) there exist a function $\alpha(x) \in L^2[0, 1]$ and a function $F : [0, 1] \times R^2 \to R$ satisfying L^2-Caratheodory's conditions such that

$$|f(x, y, z, w)| \le |F(x, y, z)|\,|w| + \alpha(x), \tag{3.2}$$

for a.e. $x \in [0, 1]$ and all $(y, z, w) \in R^3$. Let $h, k \in R$, $h \ge 0$, $k \ge 0$, $h + k > 0$, and $\eta \in [0, 1]$ be given. Then for every $e(x) \in L^1[0, 1]$ (given), the boundary value problem (2.1) has exactly one solution.

Proof. The existence of a solution for the boundary value problem (2.1) follows from Theorem 1 and Remark 1, since

$$f(x, y, z, w)z \le f(x, 0, 0, 0)z \le |f(x, 0, 0, 0)|\ |z|,$$

for a.e. $x \in [0, 1]$ and all $(y, z, w) \in \mathbf{R}^3$.

To verify uniqueness of a solution for (2.1), let $u_1(x)$ and $u_2(x)$ be two solutions for the boundary value problem (2.1) and set $v(x) = u_1(x) - u_2(x)$. Now, we have

$$v'''(x) + f(x, u_1(x), u_1'(x), u_1''(x)) - f(x, u_2(x), u_2'(x), u_2''(x)) = 0,$$

$$v(\eta) = 0,\ \ v''(0) - hv'(0) = 0,\ \ \text{and}\ \ v''(1) + kv'(1) = 0.$$

Hence,

$$0 = \int_0^1 v'''(x)v'(x)dx + \int_0^1 [f(x, u,(x), u_1'(x), u_1''(x)) - f(x, u_2(x), u_2'(x), u_2''(x))]v'(x)dx$$

$$\le -h[v'(0)]^2 - k[v'(1)]^2 - \int_0^1 [v''(x)]^2 dx.$$

Since $h + k > 0$, $h \ge 0$, and $k \ge 0$, we have that at least one of $v'(0)$ and $v'(1)$ is zero and $\int_0^1 [v''(x)]^2 dx = 0$. Accordingly, using the inequalities

$$\|v\|_\infty \le \|v'\|_\infty \le \|v''\|_2,$$

we get that $v(x) = 0$ for a.e. x in $[0, 1]$ and so for every x in $[0, 1]$, being continuous on $[0, 1]$. □

Remark 11. The assumption (3.2) is superfluous if $f(x, y, z, w)$ is independent of w in Theorem 5.

COROLLARY 1. *Let $f: [0, 1] \times \mathbf{R} \to \mathbf{R}$ satisfy Caratheodory's conditions. Suppose that for a.e. $x \in [0, 1]$ the function $u \in \mathbf{R} \to f(x, u) \in \mathbf{R}$ is monotonically decreasing. Let $h \ge 0$, $k \ge 0$, $h + k > 0$, and $\eta \in [0, 1]$ be given. Then for every $e(x) \in L^2[0, 1]$ (given), the boundary value problem*

$$u'''(x) + f(x, u'(x)) = e(x),\ 0 < x < 1,$$

$$u(\eta) = 0,\ u''(0) - hu'(0) = 0,\ u''(1) + ku'(1) = 0,$$

has exactly one solution.

The corollary is an immediate consequence of Theorem 5, since

$$(f(x, u_1) - f(x, u_2))(u_1 - u_2) \le 0$$

for a.e. $x \in [0, 1]$ and all $u_1, u_2 \in R$.

THEOREM 6. *Let $f : [0, 1] \times R^3 \to R$ satisfy Caratheodory's conditions. Suppose that there exist functions $b(x)$ and $c(x)$ in $L^1[0, 1]$, and a function $d(x)$ in $L^2[0, 1]$ such that*

$$|f(x, y_1, z_1, w_1) - f(x, y_2, z_2, w_2)| \tag{3.2}$$

$$\le |b(x)|\,|y_1 - y_2| + |c(x)|\,|z_1 - z_2| + |d(x)|\,|w_1 - w_2|,$$

for a.e. $x \in [0, 1]$ and all $(y_i, z_i, w_i) \in R^3$, $i = 1, 2$. Let $k \ge 0$, $h + k > 0$, and $\eta \in [0, 1]$ be given. Then for every $e(x) \in L^1[0, 1]$ (given), the boundary value problem

$$u'''(x) + f(x, u(x), u'(x), u''(x)) = e(x),\ 0 < x < 1,$$

$$u(\eta) = 0,\ \ u'(0) = 0,\ \ u''(1) + ku'(1) = 0, \tag{3.3}$$

has at exactly one solution, provided

$$\|b\|_1 M_\eta + \|c\|_1 + \|d\|_2 < 1. \tag{3.4}$$

Proof. The existence of a solution for (3.3) is immediate from Theorem 2, in view of (3.4) and

$$|f(x,y,z,w)| \le |f(x, 0,0,0)| + |b(x)|\,|y| + |c(x)|\,|z| + |d(x)|\,|w|,$$

for a.e. $x \in [0, 1]$ and all $(y,z,w) \in R^3$.

To verify uniqueness, let $u_1(x)$, and $u_2(x)$ be two solutions of (3.3). Set $v(x) = u_1(x) - u_2(x)$, and note that

$$\|v\|_\infty \le M_\eta \|v'\|_\infty \ \text{ and } \ \|v'\|_\infty \le \|v''\|_2. \tag{3.5}$$

It follows as in the proof of Theorem 5 that

$$\int_0^1 [v''(x)]^2 dx \le [M_\eta \|b\|_1 + \|c\|_1 + \|d\|_2] \|v''\|_2^2,$$

so that

$$[1 - (M_\eta \|b\|_1 + \|c\|_1 + \|d\|_2)] \|v''\|_2^2 \le 0,$$

and hence

$$\|v''\|_2 = 0,$$

in view of (3.4). Thus it follows from (3.5) that $\|v\|_\infty = 0$, so that $u_1(x) = u_2(x)$ for a.e. $x \in [0, 1]$ and hence for every $x \in [0, 1]$, because of continuity. □

Remark 12. If $b(x)$ and $c(x)$ in Theorem 6 are in $L^2[0, 1]$, then condition (3.4) is to be replaced by

$$4M_\eta \|b\|_2 + 2\pi \|c\|_2 + \pi^2 \|d\|_2 < \pi^2; \tag{3.6}$$

and if $b(x)$, $c(x)$, $d(x) \in L^\infty[0, 1]$ in Theorem 6, then the condition (3.4) is to be replaced by

$$8M_\eta \|b\|_\infty + 4\pi \|c\|_\infty + 2\pi^2 \|d\|_\infty < \pi^3, \tag{3.7}$$

for the conclusion of Theorem 6 to hold. These new conditions, as we noted in Remark 4, are needed for existence.

Remark 13. One can easily derive conditions similar to (3.4), (3.6), and (3.7) if f satisfies the following in place of (3.2):

$$(f(x,y_1,z_1,w_1) - f(x,y_2,z_2,w_2))(z_1-z_2)$$
$$\leq |b(x)|\,|y_1-y_2|\,|z_1-z_2| + |c(x)|\,|z_1-z_2|^2 + |d(x)|\,|w_1-w_2|\,|z_1-z_2|.$$

We conclude by remarking that our conditions could be sharpened somewhat if we were to use some of the sharper Wirtinger inequalities [6].

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