BOOK REVIEW

Introduction to Hilbert Spaces with Applications, by Lokenath Debnath and Piotr Mikusiński, Academic Press, 1990, 508 pp., $49.95, ISBN 0-12-208435-7

Courses in functional analysis and in linear operator theory, taught in mathematics departments by mathematics faculty but designed for advanced graduated students in science and engineering, are commonplace. Such a course serves the mathematics graduate students well, too, for it introduces them to a perspective of functional analysis that is likely to be closer to the physical motivations of the subject. Especially if the primary interest is in linear operators on Hilbert spaces, it introduces some of the ideas from which less structured functional spaces generalize. The mathematics graduate students will see the motivation for a leaner structure. Students in science and engineering who take the course will see an elementary view of a mathematical structure in which to embed physical problems. Once science and engineering students understand that this course may be more likely to emphasize concepts rather than techniques, they often relish the overview with such gusto as to surprise the mathematics students and, even, the faculty.

An introductory course such as this should not require the Lebesque integral or even require that the students should have had undergraduate real analysis. But linear algebra is desired.

In order to be successful, the course should give many applications of the common ideas in Hilbert spaces to some of the following topics: ordinary and partial differential equations, integral equations, control problems, approximation theory, and optimization. An academic quarter is sufficient time to develop the uses of the Hilbert space theory in only a few of these areas of application, but some portion of these should be the recurring focus of many of the topics introduces in the class.

While this introductory class cannot treat any one of these topics in the detail that is accessible in Hilbert spaces, the students should come out of the course believing that there is information available in this structure that will be valuable for understanding applications of mathematics. An appreciation of Hilbert spaces should happen no matter which application has the student's attention. The students should marvel at the generality of the ideas, recognize that the period of one quarter has lifted their vision, and wish for more information that further develops the

ideas in the spirit of the course just finished.

What is needed is a good book to serve as a text and as a reference for more studies as ideas mature!

A candidate for such a book is this one by Debnath and Mikusiński. It presents the concepts of the subject and proceeds to illustrate these with examples that the science and engineering students will have encountered in studies of Fourier series, classical partial differential equations, optimal control problems, and quantum mechanics.

The first two chapters of this book should likely be omitted in such a course. A student will refer to the first chapter, entitled Normed Vector Spaces, as he encounters required concepts that the course assumes. The student may acquire these concepts from his private studies of Chapter 1 or from previous classes in linear algebra or, perhaps, from undergraduate analysis. The second chapter is entitled The Lebesque Integral. For such a one-quarter course, some of the ideas found in this chapter will be used without warning. There will be students who let these powerful ideas go by without notice, and others will raise their eyebrows. Having this summary of the integral in Chapter 2--compact, self-contained, introductory--gives credibility to the ideas that are used in class but not developed in the lectures. Mathematics students will be required to see more about integrals in some later course; engineering or science students may be stimulated to ask for more.

Chapter 3, Hilbert Spaces and Orthonormal Systems, and Chapter 4, Linear Operators on Hilbert Spaces, are the heart of the book. If a collection of analysts were asked what should be in an introductory course on Hilbert Spaces they would all list some basic ideas: projection theorems, orthogonality, modes of convergence, adjoint operators, compact operators... There would certainly be representation theorems for classes of linear operators.

I have taken the paradigm of a linear operator throughout such a graduate course to be

$$AX = \sum_{i=1}^{\infty} \lambda_i < x, \Phi_i > \Theta_i,$$

where each of $\{\Phi_i\}$ and $\{\Theta_i\}$ is a maximal orthonormal family in the space and $\{\lambda_i\}$ is a sequence of numbers. The goal has been to understand the implications of A having such a representation, to understand what A's could be represented this way, and to explore the utility of this representation. Chapters 3 and 4 give students the language and structure to achieve this goal. The applications are found in ordinary and partial differential equations, in integral equations, and in control problems. As an example, the Fredholm Alternative Theorems and surrounding ideas are easily dissected for linear transformations that satisfy this paradigm.

In developing and using such a paradigm, it is not necessary to follow the development of the material in Chapters 3 and 4 in detail. For a graduate course, is it not preferable that the

lecture be more than just a digest of the text? The text, rather, should provide an anchor and affirmation for the directions of the lectures. From time to time, proofs in class may be abbreviated: "See text, page whatever." The exercises that come with each chapter may be used to illustrate the material being developed. One may choose to bring the exercises into the context of the lectures, but the text has all the exercises at the end of each chapter. I would prefer to have problems integrated with the sections of the chapters as the ideas arise.

One other addition to the text that I think the students in my class found valuable was my giving illustrations of the utility of the material by introducing into the subject development some of the applications which occur in those last three chapters. Otherwise, science and engineering students might give up, thinking that this was going to be one more "theory" course for which they did not see the applicability.

I think the organization of the text is appropriate: the classroom presentation of the material can reach forward to the applications in Chapters 5, 6, 7, and 8 to illustrate the material as appropriate. Likely, there will not be time to complete Chapters 3 through 8 in one quarter. My reaction to this lack of time is: Not to worry! The students have in their hands a well-laid-out development of the elementary tools available in Hilbert spaces and examples of how the tools may be adapted to problems that they know and respect. Graduate students from electrical engineering seem to leave the course from this book with their finger in Chapter 8 where there are sections titled Minimization of a Quadratic Functional, Variational Inequalities, and Optimal Control Problems for a Dynamical System. The physics graduate students go away reading Chapter 7: Mathematical Foundations of Quantum Mechanics. Their desire for more will lead to further studies after the quarter has finished.

Who knows? Such a nice book may make taking another course in functional analysis irresistible to the graduate students in science and engineering.

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