ON THE VARIANCE OF THE NUMBER OF REAL ROOTS OF A RANDOM TRIGONOMETRIC POLYNOMIAL*

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ABSTRACT

This paper provides an upper estimate for the variance of the number of real zeros of the random trigonometric polynomial $g_1 \cos\theta + g_2 \cos 2\theta + \dots + g_n \cos n\theta$. The coefficients g_i $(i = 1, 2, \dots, n)$ are assumed independent and normally distributed with mean zero and variance one.

Key words: random trigonometric polynomial, number of real roots, variance.

AMS subject classification: 60H, 42.

1. INTRODUCTION

Let

$$T(\theta) \equiv T_n(\theta, \omega) = \sum_{i=1}^{n} g_i(\omega) \cos i\theta ,$$

where $g_1(\omega), g_2(\omega), \dots, g_n(\omega)$ is a sequence of independent random variables defined on a probability space (Ω, A, P) each normally distributed with mathematical expectation zero and variance one. Denote by $N(\alpha,\beta)$ the number of real roots of the

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equation $T(\theta) = 0$ in the interval (α, β), where multiple roots are counted only once. Dunnage [3] showed that except for a set of functions of $T(\theta)$ of measure not larger than $(\log n)^{-1}$

$$N(0, 2\pi) = 2n/\sqrt{3} + O\{ n^{11/13} (\log n)^{3/13} \}.$$

Later Sambandham and Renganathan [9] and Farahmand [4] generalized this result to the case where the coefficients g_i have a non zero mean. They show that for n sufficiently large the mathematical expectation of the number of real roots, EN, satisfies

$$EN(0, 2\pi) \sim (2/\sqrt{3})\, n.$$

The results for the dependent coefficients with constant correlation coefficient or otherwise are due to Renganathan and Sambandham [6] and Sambandham [7] and [8]. A comprehensive treatment of the zeros of random polynomial constitutes the greater part of a book by Bharucha-Reid and Sambandham [1] which gives a rigorous and interesting survey of earlier works in this field.

Qualls [5] resolved the only known variance of the number of real roots of a random trigonometric polynomial. Indeed he considered a different type of random polynomial,

$$\sum_{i=0}^{n} (a_i \cos i\theta + b_i \sin i\theta)$$

which has the property of being stationary and for which a special theorem has been developed by Cramer and Leadbetter [2]. Here we shall prove the following theorem:

Theorem. Let $g_1(\omega), g_2(\omega), \ldots, g_n(\omega)$ be the independent random variables

corresponding to a Gaussian distribution with mean zero. Then the variance of the number of real roots of $T(\theta)$ satisfies

$$\mathrm{Var}\ N(0, 2\pi) = O[n^{24/13} (\log n)^{16/13}].$$

2. OVERVIEW OF PROOF OF THE THEOREM AND SOME LEMMAS

In general we make use of a delicate analysis suggested by the work of Dunnage in [3] with which we assume the reader is familiar. We divide the interval $(0, 2\pi)$ into intervals $I_1, I_2, \ldots, I_s$, each of equal length δ. Then with each I_j $(j = 1, 2, \ldots, s)$, we associate the following two functions:

$N_j(\omega)$ = number of zeros of $T(\theta)$ in I_j, counted according to their multiplicity

and

$$N_j^*(\omega) = \begin{cases} N_j(\omega) & \text{if } N_j(\omega) \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Now if $T(a)\,T(b) \leq 0$ we shall say, being prompted by a graphical idea, that $T(\theta)$ has a single crossover (s.c.o.) in (a, b), and let

$$\mu_j(\omega) = \begin{cases} 1 & \text{if } T(\theta) \text{ has a (s.c.o.) in } I_j \\ 0 & \text{otherwise} \end{cases}$$

clearly

$$(2.1) \quad 0 \le N_j(\omega) - \mu_j(\omega) \le N_j^*(\omega).$$

For the proof of the theorem we need the following lemmas.

Lemma 1. Provided that the interval of I, of length $\delta = o(1/n)$ does not overlap the ε - neighborhood of $0, \pi$ and 2π, where $\varepsilon \sim n^{-6/13} (\log n)^{-4/13}$, the probability that $T(\theta)$ has at least two zeros (counted according to their multiplicity) in I is $O(n^3, \delta^3)$.

Proof. This is lemma 11 of [3].

We denote by $N(\omega)$ the number of real zeros that $T(\theta)$ has in I and we define

$$N^*(\omega) = \begin{cases} N(\omega) & \text{if } N(\omega) \ge 2 \\ 0 & \text{otherwise}. \end{cases}$$

Lemma 2. For a constant A

$$E[N^*(\omega)]^2 < A n^3 \delta^3 \log n .$$

Proof. Suppose $T(\theta)$ has at least $k (\ge 2)$ zeros in I. Then if I is divided into 2^p equal parts where p is chosen as an integer satisfying $2^p < k < 2^{p+1}$ at least one part must contain two or more zeros, and by lemma 1, the probability of this occurring does not exceed

$$A\, 2^p\, n^3\, (\delta / 2^p)^3 = A\, n^3\, \delta^3\, 2^{-2p} < A\, n^3\, \delta^2 / k^2 .$$

Hence if q_k is the probability that $T(\theta)$ has at least k zeros in I, we have

$$q_k < A\, n^3\, \delta^3 / k^2 .$$

Now we find the mathematical expectation of N^{*2} as

$$E\,[N^{*2}] = \sum_{k=2}^{n} k^2 \operatorname{Prob}(n = k) \sum_{k=2}^{n} k^2 (q_k - q_{k+1})$$

$$= \sum_{k=2}^{n} k^2 q_k - \sum_{k=3}^{n+1} (k-1)^2 q_k$$

$$\leq 4 q_2 + \sum_{k=3}^{n+1} (2k - 1)\, q_k < A\, n^3\, \delta^3 \log n$$

which completes the proof of lemma 2.

Now we define

$$\alpha_j = E\,(N_j) \quad \text{and} \quad m_j = E\,(\mu_j) .$$

Lemma 3.

$$\sum m_j = (N / \sqrt{3}) + O\,\{N^{11/13}\,(\log n^{3/13}\} .$$

Proof. This is lemma 16 of [3].

3. PROOF OF THE THEOREM.

First we consider the interval $(\varepsilon, \pi - \varepsilon)$. We have

$$(3.1) \quad \mathrm{Var}\ N(\varepsilon, \pi - \varepsilon) \le 4E\left\{\sum_j (N_j - \mu_j)\right\}^2$$

$$+ 4E\left\{\sum_j (\mu_j - m_j)\right\}^2 + 4E\left\{\sum_j (m_j - \alpha_j)\right\}^2 .$$

From (2.1) and lemma 2 we have

$$(3.2) \quad E\left[\sum_j (N_j - \mu_j)^2\right] \le E\left[\sum_j N_j^*\right]^2 < s\,E\left[\sum_{j=1}^{s} (N_j^*)^2\right]$$

$$\le \frac{\pi}{\delta} \sum_{j=1}^{s} E(N_j^{*2}) < A\,\pi\, s\, n^3 \delta^2 \log n .$$

So far $\delta = o\,(1/n)$ has been an arbitrary constant; now since the total number of δ - intervals is $(\pi - 2\varepsilon) / \delta$, we choose δ such that

$$(\pi - 2\varepsilon) / \delta = n^{15/13} (\log n)^{-3/13} .$$

So from (3.2) we have

$$(3.3) \quad E \sum_j \{ (N_j - \mu_j) \}^2 < A\, n^{24/13} (\log n)^{16/13} .$$

Also from lemma 3 and the fact that

$$\sum_j \alpha_j = n/\sqrt{3} + O\{n^{11/13}(\log n)^{3/13}\}$$

we have

$$(3.4)\quad E\left\{\sum_j (m_j - \alpha_j)\right\}^2 = E[n/\sqrt{3} + O\{n^{11/13}(\log n)^{3/13}\} - n/\sqrt{3}$$

$$+ O\{n^{11/13}(\log n)^{3/13}\}]^2 = O\{n^{22/13}(\log n)^{6/13}\}.$$

Hence from (3.1), (3.2), (3.3) and since from [3, page 81]

$$E\left[\sum_j (\mu_j - m_j)\right]^2 = O\{n^{22/13}(\log n)^{6/13}\}$$

we have

$$(3.5)\quad \mathrm{Var}\, N(\varepsilon, \pi - \varepsilon) = O(n^{24/13}(\log n)^{16/13}).$$

To find the variance in the interval $(-\varepsilon, \varepsilon)$ let $\eta(r) = \eta(r, \omega)$ be the number of zeros of $T(\theta)$ in the circle $|z| \leq r$. From [3, page 83] we know that outside an exceptional set of measure at most $\exp(-n^2/2) + (2\pi)^{1/2}/n$

$$\eta(\varepsilon) \leq 1 + (2\log n + 2n\varepsilon)/\log 2.$$

Since the number of real roots in the segment of the real axis joining points $\pm\varepsilon$ does not exceed the number in the circle $|z| \leq \varepsilon$, we can obtain

$$(3.6)\quad N(-\varepsilon, \varepsilon) = O\{n^{7/13}(\log n)^{-4/13}\}$$

except for sample functions in an ω - set of measure not exceeding $\exp(-n^2/2)$ +

$(2\pi)^{1/2} / n$. Now let d be any integer of $O\{n^{7/13} (\log n)^{-4/13}\}$, then since the trigonometric polynomial has at most $2n$ zeros in $(0, 2\pi)$ from (3.6) we have

$$(3.7) \quad \operatorname{Var} N(-\varepsilon, \varepsilon) \leq \sum_{i=0}^{2n} i^2 \operatorname{Prob}(N = i)$$

$$= \sum_{i \leq d} i^2 \operatorname{Prob}(N = i) + \sum_{i > d}^{2n} i^2 \operatorname{Prob}(N = i)$$

$$< B\, n^{23/13} \operatorname{Prob}\{N < C\, n^{7/13} (\log n)^{-4/13}\}$$

$$+ 4 n^2 \operatorname{Prob}\{N > C' n^{7/13} (\log n)^{-4/13}\}$$

$$< D\, n^{23/13} + 4 n^2 \{\exp(-n^2/2) + (\sqrt{2/\pi})/n\}$$

$$= O(n^{23/13}),$$

where B, C, C' and D are constants. Finally from (3.5) and (3.7) we have proof of the theorem.

Remark. Although in this paper we assumed that the coefficients $g_i(\omega)$, $i = 1, 2, \ldots, n$ are independent with means zero and variance one, we can show that our theorem for the case of dependent coefficients with mean zero or non-zero (finite or infinite) and any finite variance would remain valid. However a subsequent study could be directed to reduce the upper bound obtained in our theorem, or further, to establish an asymptotic formula for the variance.

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