AN ABSTRACT INVERSE PROBLEM¹

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ABSTRACT

In this paper we consider an inverse problem that corresponds to an abstract integrodifferential equation. First, we prove a local existence and uniqueness theorem. We also show that every continuous solution can be locally extended in a unique way. Finally, we give sufficient conditions for the existence and a stability of the global solution.

Key words: inverse problem, abstract integrodifferential equation, existence, uniqueness, stability.

AMS (MOS) subject classification: 35R.

1. INTRODUCTION

Let X, Y be two Banach spaces, and let $A: D(A) \subset X \rightarrow X$ be a linear operator. Let $T > 0, F_1, F_2: [0, T] \times X \times Y \rightarrow X, L: X \rightarrow Y, v: [0, T] \rightarrow Y$, and $x \in X$ be given data.

We consider the following problem: find $(u, p): [0, T] \rightarrow X \times Y$ such that

$$\begin{aligned} (1) \quad & u'(t) = Au(t) + F_1(t, u(t), p(t)) + \int_0^t F_2(s, u(s), p(t-s))ds, 0 \leq t \leq T, \\ (2) \quad & u(0) = x, \\ (3) \quad & Lu(t) = v(t), 0 \leq t \leq T. \end{aligned}$$

Such a problem has been considered previously by Prilepko, Orlovskii in [6,7], Lorenzi, Sinestrari in [4], and the author in [1].

The local existence and uniqueness result is obtained by Prilepko, Orlovskii for the case $F_2 = 0$, and by Lorenzi, Sinestrari for the case Y is a subspace of $L(X), F_1(t, u, p) = pBx$, and $F_2(t, u, p) = pBu$, where B is some given linear operator in X . The stability problem has been studied by Lorenzi and Sinestrari in [5].

In [1] the author treats the case of $Y = C[0, T]^n (n \geq 1), F_1(t, u, (p_1, \dots, p_n)) = \sum_{i=1}^n p_i y_i, y_i$ in $X (1 \leq i \leq n)$ and $F_2 = 0$. Then a global existence and uniqueness theorem is obtained.

¹Received: January 1990; Revised: October 1990.

The present work is concerned with a generalization of those results.

Throughout this paper we assume:

- (H1) A is a closed linear operator with a dense domain generating a strongly continuous semi-group e^{At} . Without loss of generality, we suppose that e^{At} is equibounded:

$$\|e^{At}\| \leq M, t \geq 0 \text{ for some } M \geq 1.$$

(H2) $x \in D(A),$

(H3) $L \in L(X, Y),$

(H4) $v \in C^1([0, T]; Y),$ and $v(0) = Lx.$

(H5,1) F_1 and AF_1 are continuous in $[0, T] \times D(A) \times Y.$

For each $r > 0$, there exist positive continuous real valued functions $g_{1,i}(r, \cdot), i = 0, 1$ such that

(H5,2) $\|F_1(t, u_1, p_1)\|_{D(A)} \leq g_{1,0}(r, t),$

(H5,3) $\|F_1(t, u_1, p_1) - F_1(t, u_2, p_2)\|_{D(A)} \leq g_{1,1}(r, t)(\|u_1 - u_2\|_{D(A)} + \|p_1 - p_2\|_Y),$

for each $(u_i, p_i) \in \{(u, p) \in D(A) \times Y, \|u\|_{D(A)} + \|p\|_Y \leq r\}, i = 1, 2,$ and $t \in [0, T].$

(H6,1) $\int_0^t F_2$ and $A \int_0^t F_2$ are continuous in $[0, T] \times D(A) \times Y.$

For each $r > 0$, there exist positive continuous real valued functions $g_{2,i}(r, \cdot), i = 0, 1$, such that

(H6,2) $\left\| \int_0^t F_2(s, u_1(s), p_1(t-s))ds \right\|_{D(A)} \leq \int_0^t g_{2,0}(r, s)ds,$

(H6,3) $\left\| \int_0^t (F_2(s, u_1(s), p_1(t-s)) - F_2(s, u_2(s), p_2(t-s)))ds \right\|_{D(A)}$
 $\leq \int_0^t g_{2,1}(r, s)(\|u_1(s) - u_2(s)\|_{D(A)} + \|p_1(s) - p_2(s)\|_Y)ds,$

for each $(u_i, p_i) \in \{(u, p) \in C([0, T]; D(A) \times Y), \sup_{0 \leq s \leq t} (\|u(s)\|_{D(A)} + \|p(s)\|_Y) \leq r\}, i = 1, 2,$ and $t \in [0, T].$

There exist continuous function $H_1: [0, T] \times Y \times Y \rightarrow Y$ with the following properties. For each $r > 0$ there exist positive continuous real valued functions $C(r, \cdot)$ such that

(H7,1) $\|H_1(t, u_1, p_1) - H_1(t, u_2, p_2)\|_Y \leq C(r, t)(\|u_1 - u_2\|_{D(A)} + \|p_1 - p_2\|_Y),$ for each
 $(u_i, p_i) \in \{(u, p) \in Y \times Y, \|u\|_Y + \|p\|_Y \leq r\}, i = 1, 2,$ and $t \in [0, T].$

$K: p \rightarrow H_1(t, v(t), p)$ has an inverse $\Phi(t, \cdot)$ continuous map $t \rightarrow \Phi(t, w)$, and there exist positive continuous real valued function k , such that

$$(H7,2) \quad \|\Phi(t, w_1) - \Phi(t, w_2)\|_Y, t \in [0, T], w_i \in Y, i = 1, 2.$$

$$(H7,3) \quad LF_1(t, u, p) = H_1(t, Lu, p), (u, p) \in D(A) \times Y, \text{ and } t \in [0, T].$$

2. EXISTENCE OF THE LOCAL SOLUTION

In this section we prove that the local solution of our inverse problem is obtained by a fixed point theorem. Let

$$a(t) = M \|x\|_{D(A)} + \|\Phi(t, 0)\|_Y + k(t) \|v'(t) - Le^{At}Ax\|_Y, t \in [0, T], r_0 = 2 \sup_{0 \leq t \leq T} a(t),$$

$$g_i(r_0, t, s) = M(1 + k(t) \|L\|)(g_{1,i}(r_0, s) + (t-s)g_{2,i}(r_0, s)) + k(t) \|L\| g_{2,i}(r_0, s), \quad 0 \leq s \leq t \leq T, \\ i = 0, 1, \text{ and let } T_0 \in [0, T] \text{ be such that}$$

$$T_0 \sup_{0 \leq s \leq t \leq T} g_0(r_0, t, s) \leq \frac{r_0}{2}, \text{ and } T_0 \sup_{0 \leq s \leq t \leq T} g_1(r_0, t, s) = \gamma < 1.$$

Let $Z(T_0) = C([0, T_0]; D(A) \times Y)$ equipped with the norm

$$\|(u, p)\|_{Z(T_0)} = \sup_{0 \leq t \leq T_0} (\|u(t)\|_{D(A)} + \|p(t)\|_Y).$$

Then, we define the mapping

$$\Psi: Z(T_0) \rightarrow Z(T_0): (u, p) \rightarrow (U, P),$$

where

$$U(t) = e^{At}x + \int_0^t e^{A(t-s)} F_1(s, u(s), p(s)) ds \\ + \int_0^t e^{A(t-s)} \int_0^s F_2(\sigma, u(\sigma), p(s-\sigma)) d\sigma ds, \\ P(t) = \Psi(t, v'(t) - Le^{At}Ax - \int_0^t LF_2(s, u(s), p(t-s)) ds \\ - \int_0^t Le^{A(t-s)} AF_1(s, u(s), p(t-s)) ds \\ - \int_0^t Le^{A(t-s)} A \int_0^s F_2(\sigma, u(\sigma), p(s-\sigma)) d\sigma ds), 0 \leq t \leq T_0.$$

Proposition 1. There exists a unique (u_0, p_0) in $B(r_0, T_0)$ satisfying $(u_0, p_0) = \Psi(u_0, p_0)$, where $B(r_0, T_0)$ denotes the closed ball of $Z(T_0)$ with the center 0 and radius r_0 .

Proof. We claim that Ψ is a strict contraction from $B(r_0, T_0)$ into itself. Hence, according to the fixed point theorem, there is a unique (u_0, p_0) in $B(r_0, T_0)$ such that $(u_0, p_0) = \Psi(u_0, p_0)$. Let (u_i, p_i) in $B(r_0, T_0)$, $(U_i, P_i) = \Psi(u_i, p_i)$, $i = 1, 2$, and t in $[0, T_0]$.

We have then

$$\begin{aligned} \|U_1(t)\|_{D(A)} &\leq M \|x\|_{D(A)} + M \int_0^t \|F_1(s, u(s), p(s))\|_{D(A)} ds \\ &\quad + M \int_0^t \left\| \int_0^s F_2(\sigma, u(\sigma), p(s-\sigma)) d\sigma \right\|_{D(A)} ds. \end{aligned}$$

Using (H5) and (H6) we obtain

$$\begin{aligned} \|U_1(t)\|_{D(A)} &\leq M \|x\|_{D(A)} + M \int_0^t g_{1,0}(r_0, s) ds + M \int_0^t \int_0^s g_{2,0}(r_0, \sigma) d\sigma ds \\ &\leq M \|x\|_{D(A)} + M \int_0^t (g_{1,0}(r_0, s) + (t-s)g_{2,0}(r_0, s)) ds. \end{aligned}$$

From (H7, 2) we deduce

$$\begin{aligned} \|P_1(t)\|_Y &\leq \|\Phi(t, 0)\|_Y + k(t) \|v'(t) - Le^{At}Ax \\ &\quad - \int_0^t LF_2(s, u(s), p(t-s)) ds - \int_0^t Le^{A(t-s)}AF_1(s, u(s), p(t-s)) ds \\ &\quad - \int_0^t Le^{A(t-s)}A \int_0^s F_2(\sigma, u(\sigma), p(s-\sigma)) d\sigma ds\|_Y. \end{aligned}$$

Hence

$$\begin{aligned} \|P_1(t)\|_Y &\leq \|\Phi(t, 0)\|_Y + k(t) \|v'(t) - Le^{At}Ax\| + \|L\| k(t) \int_0^t g_{2,0}(r_0, s) ds \\ &\quad + M \|L\| k(t) \int_0^t (g_{1,0}(r_0, s) + (t-s)g_{2,0}(r_0, s)) ds. \end{aligned}$$

Thus

$$\begin{aligned} \|U_1(t)\|_{D(A)} + \|P_1(t)\|_Y &\leq \|x\|_{D(A)} + \|\Phi(t, 0)\|_Y + k(t) \|v'(t) - Le^{At}Ax\| \\ &\quad + \|L\| k(t) \int_0^t g_{2,0}(r_0, s) ds \\ &\quad + M(1 + \|L\| k(t)) \int_0^t (g_{1,0}(r_0, s) + (t-s)g_{2,0}(r_0, s)) ds \\ &\leq a(t) + \int_0^t g_1(r_0, t, s) ds. \end{aligned}$$

This implies that

$$\|(U_1, P_1)\|_{Z(T_0)} \leq r_0.$$

On the other hand, in the same way as above, it is easily seen that

$$\begin{aligned} & \|U_1(t) - U_2(t)\|_{D(A)} + \|P_1(t) - P_2(t)\|_Y \\ & \leq \int_0^t g_2(r_0, t, s) (\|u_1(s) - u_2(s)\|_{D(A)} + \|p_1(s) - p_2(s)\|_Y) ds \\ & \leq \gamma \sup_{0 \leq s \leq t} (\|u_1(s) - u_2(s)\|_{D(A)} + \|p_1(s) - p_2(s)\|_Y) ds. \end{aligned}$$

It follows that

$$\|(U_1, P_1) - (U_2, P_2)\|_{Z(T_0)} \leq \gamma \|(u_1, p_1) - (u_2, p_2)\|_{Z(T_0)}.$$

Our claim is proven.

Proposition 2. (u, p) is a solution of the inverse problem (1)–(3) in $[0, T]$ iff $(u, p) = \Psi(u, p)$.

Proof. It is well known that the solution of Cauchy problem (1) and (2) is given by $u(t) = U(t)$. Therefore, it suffices to show

$$Lu(t) = v(t) \text{ iff } p(t) = \Psi(t, v'(t) - \int_0^t LF_2(s, u(s), p(t-s))ds - LAu(t))$$

for each t in $[0, T]$.

First, we differentiate $Lu(t) = v(t)$ to obtain

$$Lu'(t) = L\{Au(t) + F_1(t, u(t), p(t)) + \int_0^t F_2(s, u(s), p(t-s))ds\} = v'(t).$$

Hence

$$\begin{aligned} H_1(t, v(t), p(t)) &= LF_1(t, u(t), p(t)) \\ &= v'(t) - \int_0^t LF_2(s, u(s), p(t-s))ds - LAu(t). \end{aligned}$$

Using (H7, 2) we get

$$p(t) = \Psi(t, v'(t) - \int_0^t LF_2(s, u(s), p(t-s))ds - LAu(t)).$$

Conversely, this last equality implies that

$$\begin{aligned}
H_1(t, v(t), p(t)) &= v'(t) - L\left\{\int_0^t F_2(s, u(s), p(t-s))ds - Au(t)\right\} \\
&\quad - v'(t) - L\{u'(t) - F_1(t, u(t), p(t))\} \\
&= v'(t) - Lu'(t) + H_1(t, Lu(t), p(t)).
\end{aligned}$$

Thus

$$\frac{d}{dt}(v(t)) = H_1(t, v(t), p(t)) - H_1(t, Lu(t), p(t)).$$

Integrating and using the fact that $v(0) = Lu(0) = Lx$, we obtain

$$v(t) - Lu(t) = \int_0^t (H_1(s, v(s), p(s)) - H_1(s, Lu(s), p(s)))ds.$$

But, (H7, 1) leads to

$$\|v(t) - Lu(t)\|_Y = \int_0^t C(R, s) \|v(s) - Lu(s)\|_Y ds,$$

where

$$R = \max\left(\sup_{0 \leq t < T} (\|Lu(t)\|_Y + \|p(t)\|_Y), \sup_{0 \leq t \leq T} (\|v(t)\|_Y + \|p(t)\|_Y)\right).$$

Hence, by using Gronwall's inequality, it follows that

$$v(t) - Lu(t) = 0, \quad 0 \leq t \leq T.$$

Now, we combine propositions 1 and 2 to deduce the following local existence and uniqueness theorem for the inverse problem (1) – (3).

Theorem 1. *Under the assumptions (H1) – (H7), there exist T_0 in $[0, T]$ and (u_0, p_0) in $C([0, T_0]: D(A) \times Y)$ which is the unique solution of the inverse problem (1) – (3) in $[0, T_0]$.*

Remark. Theorem 1 is still valued if we add to the right side of equality (1) a function $f: [0, T] \rightarrow X$ such that f and Af are continuous.

3. GLOBAL SOLUTION

We begin this section by showing that any solution (u_0, p_0) in $C([0, T_0]: D(A) \times Y)$ of the inverse problem (1) – (3) in $[0, T_0]$ can be uniquely extended to a solution in $[0, T_0 + T_1]$ for some $T_1 > 0$, whenever $0 < T_0 < T$.

If $\bar{T}$ is in $[0, \min(T_0, T - T_0)]$, we consider the following inverse problem:

$$\begin{aligned}
(4) \quad & u'(t) = Au(t) + K_1(t, u(t), p(t)) + \int_0^t K_2(s, u(s), p(t-s))ds + f(t), \quad 0 \leq t \leq \bar{T} \\
(5) \quad & u(0) = x_1 = u_0(T_0) \\
(6) \quad & Lu(t) = w(t), \quad 0 \leq t \leq \bar{T}
\end{aligned}$$

where

$$\begin{aligned}
K_1(t, u(t), p(t)) &= F_1(t + T_0, u(t), p(t)), \quad 0 \leq t \leq \bar{T}, \\
K_2(s, u(s), p(t-s)) &= F_2(s, u_0(s), p(t-s)) + F_2(s + T_0, u(s), p_0(t-s)), \quad 0 \leq t \leq \bar{T},
\end{aligned}$$

$$f(t) = \int_t^{T_0} F_2(s, u_0(s), p_0(t + T_0 - s))ds, \quad 0 \leq t \leq \bar{T}, \text{ and}$$

$$w(t) = v(t + T_0), \quad 0 \leq t \leq \bar{T}.$$

Proposition 3. If (u_0, p_0) in $C([0, T_0]: D(A) \times Y)$ denotes any solution of the inverse problem (1)–(3) in $[0, T_0]$, then there exist T_1 in $[0, \min(T_0, T - T_0)]$ and (u, p) in $C([0, T_0 + T_1]: D(A) \times Y)$ such that $(u, p) = (u_0, p_0)$ in $[0, T_0]$, and (u, p) satisfies (1)–(3) in $[0, T_0 + T_1]$.

Proof. It is not difficult to see that K_1, K_2, w have the same properties as F_1, F_2 , and v , and that f and Af are continuous. It follows from Theorem 1 that there exist $T_1 \in]0, \bar{T}]$ and $(u_1, p_1) \in C([0, T_1]: D(A) \times Y)$, which is the unique solution of the inverse problem (4)–(6) given by

$$\begin{aligned}
u_1(t) &= e^{At}x + \int_0^t e^{A(t-s)}K_1(s, u(s), p(s))ds + \int_0^t e^{A(t-s)}f(s)ds \\
&\quad + \int_0^t e^{A(t-s)} \int_0^s K_2(\sigma, u(\sigma), p(s-\sigma))d\sigma ds, \quad 0 \leq t \leq T_1, \\
p_1(t) &= \Psi(t + T_0, w'(t) - LAu_1(t) - \int_0^t LK_2(s, u_1(s), p_1(t-s))ds - Lf(t)), \quad 0 \leq t \leq T_1.
\end{aligned}$$

We have

$$\begin{aligned}
p_1(0) &= \Psi(T_0, w'(0) - LAu_1(0) - Lf(0)) \\
&= \Psi(T_0, v'(T_0) - LAu(T_0) - \int_0^{T_0} LF_2(s, u_0(s), p_0(T_0 - s))ds \\
&= p(T_0).
\end{aligned}$$

One can easily check that

$$(u(t), p(t)) = \begin{cases} (u_0(t), p_0(t)), & 0 \leq t \leq T_0, \\ (u_1(t), p_1(t)), & T_0 < t \leq T_1, \end{cases}$$

belongs to $C([0, T_0 + T_1]; D(A) \times Y)$. It remains to show that (u, p) is a solution of the inverse problem (1) – (3) in $[0, T_0 + T_1]$. Since u_1 satisfies (4), we can deduce that

$$\begin{aligned} u'(t + T_1) &= u'_1(t) \\ &= Au_1(t) + F_1(t + T_0, u_1(t), p_1(t)) + \int_0^t F_2(s, u_0(s), p_1(t - s))ds \\ &\quad + \int_0^t F_2(s + T_0, u_1(s), p_0(t - s))ds + \int_t^{T_0} F_2(s, u_0(s), p_0(t + T_0 - s))ds \\ &= Au(t + T_0) + F_1(t + T_0, u(t + T_0), p(t + T_0)) + \int_0^t F_2(s, u(s), p(t + T_0 - s))ds \\ &\quad + \int_{T_0}^{t + T_0} F_2(s, u(s), p(t + T_0 - s))ds + \int_t^{T_0} F_2(s, u(s), p(t + T_0 - s))ds \\ &= Au(t + T_0) + F_1(t + T_0, u(t + T_0), p(t + T_0)) \\ &\quad + \int_0^{t + T_0} F_2(s, u(s), p(t + T_0 - s))ds, 0 \leq t \leq T_1. \end{aligned}$$

On the other hand

$$Lu(t + T_0) = Lu_1(t) = w(t) = v(t + T_0), 0 \leq t \leq T_1.$$

Therefore we may conclude that (u, p) is a solution of the inverse problem (1) – (3) in $[0, T_0 + T_1]$.

Proposition 4. Let $(u, p) \in C([0, T_{max}]; D(A) \times Y)$ be the maximal solution of the inverse problem (1) – (3), where $0 < T_{max} \leq T$. If

$$(7) \quad \max_{0 < t < T_{max}} \left(\sup_{0 \leq s \leq t} (\|u(s)\|_{D(A)} + \|p(s)\|_Y) \right) < +\infty,$$

then $T_{max} = T$.

Proof. Clearly, from Proposition 2 (u, p) can be continuously extended to a solution in

$[0, T_{max}]$. If $T_{max} < T$, then, following the previous proposition, the solution in $[0, T_{max}]$ can be extended to a solution in $[0, T_{max} + \epsilon]$, for some $\epsilon > 0$. This contradicts the maximality of T_{max} .

Now, we will give a sufficient conditions to realize (7). For this purpose, we recall the following comparison theorem.

Theorem 2 [2]. *Let I be a real interval, and let $G: I \times I \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be continuous such that $G(t, s, r)$ is monotone nondecreasing in r for each (t, s) in $I \times I$. Let b in $C(I)$, and let f in $C(I)$ denote the maximal solution of the integral equation*

$$f(t) = b(t) + \int_{t_0}^t G(t, s, f(s)) ds, t \geq t_0.$$

If $g \in C(I)$ is such that

$$g(t) \leq b(t) + \int_{t_0}^t G(t, s, g(s)) ds, t \geq t_0,$$

then $g(t) \leq f(t), t \geq t_0$.

Here, by a maximal solution we mean that any other solution $h \in C(I)$ must satisfy $h(t) \leq f(t), t \geq t_0$.

Before stating a global existence and uniqueness result for our inverse problem, we need to modify some assumptions on F_1 , and F_2 .

Instead of (H5,2) and (H6,2) we suppose that there exist $G_i(t, r): [0, T] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ continuous and monotone nondecreasing in r for each t in $[0, T], i = 1, 2$, such that

$$(H5, 2') \quad \|F_1(t, u, p)\|_{D(A)} \leq G_1(t, \|u\|_{D(A)} + \|p\|_Y),$$

$$(H6, 2') \quad \left\| \int_0^t F_2(s, u(s), p(t-s)) ds \right\|_{D(A)} \leq \int_0^t G_2(s, \|u(s)\|_{D(A)} + \|p(s)\|_Y) ds.$$

Set

$$G(t, s, r) = M(1 + k(t) \|L\|)(G_1(s, r) + (t-s)G_2(s, r)) + k(t) \|L\| G_2(s, r), 0 \leq s \leq t \leq T, i = 0, 1.$$

Clearly, $G(t, s, r)$ is monotone nondecreasing in $r, 0 \leq s \leq t \leq T$.

Theorem 3. *Assume that (H1)–(H7) are satisfied, where (H5,2) and (H6,2) are changed by (H5,2') and (H6,2'). If the nonlinear Volterra integral equation:*

$$(8) \quad r(t) = a(t) + \int_0^t G(t, s, r(s)) ds, 0 \leq t \leq T,$$

has a continuous maximal solution in $[0, T]$, then the inverse problem (1) – (3) has a unique solution in $[0, T]$.

Proof. Let r denote the continuous maximal solution of the integral equation (8). Proceeding in the manner of the proof of Proposition 1, we obtain

$$\|u(t)\|_{D(A)} + \|p(t)\|_Y \leq a(t) + \int_0^t G(t, s, \|u(s)\|_{D(A)} + \|p(s)\|_Y) ds, 0 \leq t \leq T.$$

Thus, the condition (7) is satisfied.

The uniqueness of the global solution is just a consequence of the fact that the unique local solution allows a unique extension.

4. STABILITY RESULT

First of all, we give the exact assumptions under which the stability result will hold.

We assume that $(H1) - (H5, 1), (H6, 1), (H7)$ are satisfied, and there exist $G_i(t, r): [0, T] \times R^+ \rightarrow R^+$ continuous and monotone nondecreasing in r for each t in $[0, T], i = 1, 2$, such that

$$(H8, 1) \quad \|F_1(t, u_1, p_1) - F_1(t, u_2, p_2)\|_{D(A)} \leq G_1(t, \|u_1 - u_2\|_{D(A)} + \|p_1 - p_2\|_Y), \text{ for each } (u_i, p_i) \text{ in } D(A) \times Y, i = 1, 2, \text{ and } 0 \leq t \leq T.$$

$$(H8, 2) \quad \left\| \int_0^t (F_2(s, u_1(s), p_1(t-s)) - F_2(s, u_2(s), p_2(t-s))) ds \right\|_{D(A)} \leq \int_0^t G_2(s, \|u_1(s) - u_2(s)\|_{D(A)} + \|p_1(s) - p_2(s)\|_Y) ds$$

for each (u_i, p_i) in $C([0, T]: D(A) \times Y), i = 1, 2$, and $0 \leq t \leq T$.

$(v(t), H_1(t, v(t), p)) \rightarrow \Phi(t, K(p))$ has the following property: there exist continuous $g: [0, T] \times R^+ \rightarrow R^+$, such that

$$\|\Phi_1(t, w_1) - \Phi_2(t, w_2)\|_Y \leq g(t)(\|v_1(t) - v_2(t)\|_Y + \|w_1 - w_2\|_Y),$$

for each v_i in $C([0, T]: Y), w_i \in Y, i = 1, 2$, and $0 \leq t \leq T$.

Here, $\Phi_i(t, \cdot)$ denotes the inverse of the mapping $K_i: p \rightarrow H_1(t, v_i(t), p)$, $(i = 1, 2)$. We set

$$G(t, s, r) = M(1 + g(t) \|L\|)(G_1(s, r) + (t - s)G_2(s, r)) + g(t) \|L\| G_2(s, r),$$

$$0 \leq s \leq t \leq T, \quad i = 0, 1.$$

Theorem 4. Suppose that the assumptions listed below are satisfied for $x = x_i$, $v = v_i$, $i = 1, 2$. Let (u_i, p_i) in $C([0, T]: D(A) \times Y)$ denote any solution of the inverse problem (1)–(3) corresponding to $x = x_i$, $v = v_i$, $i = 1, 2$, and let

$$r_0(t) = M(1 + g(t) \|L\|) \|x_1 - x_2\|_{D(A)} + g(t) (\|v_1(t) - v_2(t)\|_Y + \|\dot{v}_1'(t) - \dot{v}_2'(t)\|_Y).$$

If the maximal continuous solution, given its existence, of the Volterra integral equation

$$(9) \quad m(t) = r_0(t) + \int_0^t G(t, s, m(s)) ds, \quad 0 \leq t \leq T,$$

satisfies the condition that there exists a constant $C > 0$, not depending on m , such that

$$(10) \quad m(t) \leq Cr_0(t), \quad 0 \leq t \leq T,$$

then

$$(11) \quad \|u_1(t) - u_2(t)\|_{D(A)} + \|p_1(t)\|_Y \leq Cr_0(t), \quad 0 \leq t \leq T.$$

Proof. Let m denote the maximal solution of the integral equation (9), and let

$$r(t) = \|u_1(t) - u_2(t)\|_{D(A)} + \|p_1(t) - p_2(t)\|_Y, \quad 0 \leq t \leq T.$$

It is easy to see that

$$r(t) \leq r_0(t) + \int_0^t G(t, s, r(s)) ds, \quad 0 \leq t \leq T.$$

Using the comparison Theorem 2, we deduce that $r(t) \leq m(t)$. Hence, (11) follows from (10).

Remark. We have $G(t, s, r) \leq G(T, s, r)$. Then if $G(T, s, r)$ takes the form $G(T, s, r) = G(s)r$, the conclusion of Theorem 4.1 follows from Gronwall's inequality.

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