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EXISTENCE OF PERIODIC SOLUTIONS OF IMPULSIVE DIFFERENTIAL SYSTEMS[1]

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ABSTRACT

In this paper, the existence of periodic solutions of impulsive differential systems is considered. Since the solutions of such a system are peicewise continuous, it is necessary to introduce piecewise continuous Lyapunov functions. By means of such functions, together with the comparison principle, some sufficient conditions for the existence of periodic solutions of impulsive differential systems are established.

Key words: periodic solution, impulsive differential system, comparison principle, Lyapunov function, Brouwer fixed point theorem.

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1. Introduction

It is now recognized that the concept of Lyapunov functions and the theory of differential inequalities provide a very general comparison principle under relatively unrestrictive assumptions which can be utilized to study various qualitative and quantitative properties of nonlinear differential equations. In this set-up, the Lyapunov function technique serves as a vehicle to transform a given complicated differential system into a relatively simpler scalar differential equation. The original idea of the comparison method is to determine the stability properties of a vector differential equation from the stability properties of a scalar equation through the choice of a suitable Lyapunov function which satisfies a certain differential inequality. For an excellent exposition of this method we refer the reader to [3]. In this paper, we apply the comparison principle to the problem of existence of periodic solutions of impulsive differential systems. Since the solutions of impulsive differential systems are piecewise continuous functions, it is necessary to introduce certain analogues of Lyapunov functions which possess discontinuities of the first kind. By means of such functions, together with the comparison principle, we apply the Brouwer fixed point theorem to the map $x(t_0) \to x(t_0 + T)$ and obtain some sufficient conditions for the existence of periodic solutions of nonlinear impulsive differential systems. Stability properties of impulsive differential systems were also considered in [2] and [4].

2. Preliminaries

We shall consider the impulsive differential system with fixed moments of impulse effects

$$\begin{aligned} &x' = f(t,x), \quad t \neq t_k, \\ &\Delta x|_{t=t_k} = I_k(x), \quad k = 1,2,\ldots, \end{aligned} \tag{2.1}$$

under the assumptions

(i) $f : R_+ \times R^n \to R^n$ is continuous on $(t_{k-1}, t_k] \times R^n$, $\lim\limits_{\substack{(t,y)\to(t_k,x)\\ t>t_k}} f(t,y) = f(t_k^+, x)$ exists and $f(t,x)$ is T-periodic with respect to its first argument;

(ii) $0 < t_1 < t_2 < \cdots < t_k, \ldots, t_k \to \infty$ as $k \to \infty$ and there exists a positive integer N such that $t_{k+N} = t_k + T$, for all $k = 1, 2, \ldots,$

(iii) $I_k : R^n \to R^n$ is continuous on R^n and $I_{k+N}(x) = I_k(x)$ for all $x \in R^n$ and $k = 1, 2, \ldots$.

A solution $x(t, t_0, x_0)$ of (2.1) with $x(t_0^+, t_0, x_0) = x_0$ existing on some interval $[t_0, t_0 + \alpha)$ and undergoing impulses at the points $\{t_k\}$, $t_0 < t_k < t_0 + \alpha$, is described as follows:

$$x(t, t_0, x_0) = \begin{cases} x(t, t_0, x_0), & t_0 \le t \le t_1, \\ x(t, t_1, x_1^+), & t_1 < t \le t_2, \\ \cdots \\ x(t, t_k, x_k^+), & t_k < t \le t_{k+1}, \\ \cdots \\ \cdots \end{cases} \tag{2.2}$$

where $x_k^+ = x_k + I_k(x_k)$ and $x_k = x(t_k)$.

For simplicity, we assume that for any initial value $(t_0, x_0) \in R_+ \times R^n$ the solution $x(t) = x(t, t_0, x_0)$ of (2.1) is unique. Conditions which guarantee this may be found, for example, in [2]. We shall use the following notation:

$$K = \{\sigma \in C[R_+, R_+],\ \sigma(t) \text{ is strictly increasing and } \sigma(0) = 0\};$$

$$\mathcal{V}_0 = \Big\{V : R_+ \times R^n \to R_+,\ V(t, x) \text{ is continuous on } (t_{k-1}, t_k] \times R^n \text{ and}$$

$$\lim_{\substack{(t,y)\to(t_k,x)\\ t>t_k}} V(t, y) = V(t_k^+, x) \text{ exists}\Big\}.$$

DEFINITION 2.1 Let $V \in \mathcal{V}_0$. Then for $(t, x) \in (t_{k-1}, t_k) \times R^n$, the upper right derivative of $V(t, x)$ with respect to the impulsive differential system (2.1) is defined as

$$D^+V(t, x) = \limsup_{h \to 0^+} \frac{1}{h}[V(t + h, x + hf(t, x)) - V(t, x)].$$

DEFINITION 2.2 Let $V \in \mathcal{V}_0$. Then $V(t, x)$ is said to be convex if for each positive number θ, the set $E = \{x \in R^n : V(t, x) \le \theta\}$ is convex in R^n for each fixed t.

In applying the comparison technique, we shall make use of the scalar impulsive differential equation

$$\begin{aligned} u' &= g(t, u), \quad t \ne t_k, \\ u(t_k^+) &= J_k(u(t_k)), \quad k = 1, 2, \ldots \end{aligned} \tag{2.3}$$

where $g : R_+ \times R_+ \to R$ is continuous on $(t_{k-1}, t_k] \times R_+$, $k = 1, 2, \ldots$, $\lim_{\substack{(t,v)\to(t_k,u)\\ t>t_k}} g(t,v) = g(t_k^+, u)$ exists and $J_k : R_+ \to R_+$ is nondecreasing.

DEFINITION 2.3 Let $r(t) = r(t, t_0, u_0)$ be a solution of (2.3) on $[t_0, t_0 + \alpha)$. Then $r(t)$ is said to be the maximal solution of (2.3) if for any solution $u(t) = u(t, t_0, u_0)$ of (2.3) existing on $[t_0, t_0 + \alpha)$, the inequality

$$u(t) \le r(t), \ t \in [t_0, t_0 + \alpha) \tag{2.4}$$

holds. A minimal solution of (2.3) may be defined similarly by reversing the inequality in (2.4).

3. Main results

Let us begin by stating the following comparison result.

LEMMA 3.1 Assume that

(i) $m : [t_0, t_0+\alpha) \to R$, $0 < \alpha \le \infty$, is continuous for $t \ne t_k$, $k = 1, 2, \ldots$, $\lim_{t\to t_k^-} m(t) = m(t_k)$ and $\lim_{t\to t_k^+} m(t) = m(t_k^+)$ exists for all $t_k \in [t_0, t_0 + \alpha)$. Furthermore, the following inequalities hold:

$$\begin{aligned} D^+m(t) &\le g(t, m(t)), \ t \ne t_k, \\ m(t_k^+) &\le J_k(m(t_k)), \ t_k \in [t_0, t_0 + \alpha), \\ m(t_0^+) &\le u_0, \end{aligned}$$

where g, J_k are as defined in (2.3);

(ii) $r(t) = r(t, t_0, u_0)$ is the maximal solution of the impulsive differential equation (2.3) existing on $[t_0, t_0 + \alpha)$ such that $r(t_0) = u_0$.

Then we have $m(t) \le r(t)$, $t \in [t_0, t_0 + \alpha)$.

The proof of Lemma 3.1 is similar to that of Theorem 1.4.4 in [2]. We omit it here.

THEOREM 3.1 Assume that

(i) $V \in \mathcal{V}_0$, $V(t,x)$ is convex, locally Lipschitzian in x, periodic in t with period T. Moreover, there exists $b \in K$ such that $b(u) \to \infty$ as $u \to \infty$ and

$$b(\| x \|) \leq V(t,x), \ (t,x) \in R_+ \times R^n;$$

(ii) there exists $t_0 \geq 0$ such that the equation (2.3) has a unique solution $u(t)$ existing on $[t_0, t_0+T]$ such that $u(t_0) \geq u(t_0+T)$ and $u(t) \geq M$ for $t \in [t_0, t_0+T]$, where

$$M > \sup\{V(t,0); t \in [0,T]\};$$

(iii) $D^+V(t,x) \leq g(t, V(t,x)), \ (t,x) \in R_+ \times R^n, \ V(t,x) \geq M, \ t \neq t_k;$

(iv) $V(t_k^+, \ x + I_k(x)) \leq J_k(V(t_k,x)), \ (t_k,x) \in R_+ \times R^n, \ V(t_k,x) \geq M.$

Then the differential system (2.1) has a T-periodic solution.

PROOF: Define the following set E by

$$E = \{x \in R^n : V(t_0,x) \leq u(t_0)\}.$$

Let $x_0 \in E$ and $x(t) = x(t,t_0,x_0)$ be a solution of (2.1) existing on $[t_0, t_0+\alpha)$, $\alpha > 0$. Set $m(t) = V(t,x(t))$, $t \in [t_0, t_0+\alpha)$. Then $m(t_0) \leq u(t_0)$. We claim that

$$m(t) \leq u(t), \ t \in [t_0, t_0+\alpha) \cap [t_0, t_0+T]. \tag{3.1}$$

If this is not true, then there exist $p, q \in [t_0, t_0+\alpha) \cap [t_0, t_0+T]$, $p < q$ such that

$$m(p) = u(p), \ m(q) > u(q) \quad \text{and} \quad m(t) \geq u(t), \ t \in [p,q]. \tag{3.2}$$

It then follows from conditions (ii), (iii), (iv) and (3.2) that

$$\begin{aligned} D^+m(t) &\leq g(t,m(t)), \ t \in [p,q] \ t \neq t_k, \\ m(t_k^+) &\leq J_k(m(t_k)), \ t_k \in [p,q], \ k = 1,2,\ldots . \end{aligned} \tag{3.3}$$

Since $m(p) = u(p)$, Lemma 3,1 implies that

$$m(t) \leq u(t), \ t \in [p,q],$$

which is a contradiction. We shall show next $\alpha > T$. If, on the contrary, $\alpha \leq T$, then we must have

$$\sup \| x(t) \| \to \infty \quad \text{as} \quad t \to \alpha.$$

This, together with condition (i) and (3.1) implies that

$$u(t) \to \infty \quad \text{as} \quad t \to \alpha,$$

which is absurd. Since $V(t,x)$ is T-periodic, it then follows from (3.1) and condition (ii) that

$$V(t_0, x(t_0+T, t_0, x_0)) = V(t_0+T, x(t_0+T, t_0, x_0)) = m(t_0+T) \le u(t_0), \tag{3.4}$$

which implies that the operator

$$Q : x_0 \to x(t_0+T, t_0, x_0)$$

maps the set E into itself. Clearly E is nonempty, bounded, closed and convex in R^n. By the theorem of Brouwer the operator $Q : E \to E$ has a fixed point in E, (cf. [1] for details). Thus the proof is complete.

COROLLARY 3.1 Assume that

(i) $V \in \mathcal{V}_0, V(t,x)$ is convex, locally Lipschitzian in x, periodic in t with period T and $V(t,0) \equiv 0$ for $t \in [0,T]$. Moreover, there exists $b \in K$ such that $b(u) \to \infty$ as $u \to \infty$ and

$$b(\| x \|) \le V(t,x), \ (t,x) \in R_+ \times R^n;$$

(ii) for some $M > 0$, $D^+V(t,x) \le \lambda(t)V(t,x)$, $(t,x) \in [0,T] \times R^n$, $t \ne t_k$, $V(t,x) \ge M$ where $\lambda \in C[R_+, R_+]$ and $\lambda(t)$ is periodic in t with period T;

(iii) $V(t_k^+, x + I_k(x)) \le d_k V(t_k, x)$, $(t_k, x) \in R_+ \times R^n$, $V(t_k, t) \ge M$, where $d_k > 0$;

(iv) there exists $t_0 \ge 0$ such that

$$\left(\prod_{t_0 < t_k < t_0+T} d_k \right) e^{\int_{t_0}^{t_0+T} \lambda(s)ds} \le 1.$$

Then the differential system (2.1) has a T-periodic solution.

PROOF: Consider the impulsive differential equation

$$\begin{aligned} u' &= \lambda(t)u, \ t \ne t_k, \\ u_k^+ = u(t_k^+) &= d_k u(t_k), \ k = 1, 2, \dots . \end{aligned} \tag{3.5}$$

It is sufficient to show that (3.5) has a unique solution $u(t)$ existing on $[t_0, t_0+T]$ such that $u(t_0) \geq u(t_0+T)$ and $u(t) \geq M$ for some constant $M > 0$ for $t \in [t_0, t_0+T]$. Choose $u_0 > 0$ and let $u(t_0) = u_0$. We may assume that $t_k > t_0$ for $k = 1, 2, \ldots$. Clearly (3.5) has a unique solution

$$u(t, t_0, u_0) = u_0 e^{\int_{t_0}^{t} \lambda(s)ds} \quad \text{for} \quad t \in [t_0, t_1]. \tag{3.6}$$

From (3.6), we obtain

$$u_1^+ = d_1 u(t_1, t_0, u_0) = u_0 d_1 e^{\int_{t_0}^{t_1} \lambda(s)ds}. \tag{3.7}$$

Then (3.5) has a unique solution

$$\begin{aligned} u(t, t_1, u_1^+) &= u_1^+ e^{\int_{t_1}^{t} \lambda(s)ds} \\ &= u_0 d_1 e^{\int_{t_0}^{t} \lambda(s)ds} \quad \text{for} \quad t \in [t_1, t_2]. \end{aligned} \tag{3.8}$$

Suppose that for $i \geq 1$

$$u_i^+ = u_0 \left(\prod_{j=1}^{i} d_j \right) e^{\int_{t_0}^{t_i} \lambda(s)ds}. \tag{3.9}$$

Then (3.5) admits a unique solution

$$\begin{aligned} u(t, t_i, u_i^+) &= u_i^+ e^{\int_{t_i}^{t} \lambda(s)ds} \\ &= u_0 \left(\prod_{j=1}^{i} d_j \right) e^{\int_{t_0}^{t} \lambda(s)ds} \quad \text{for} \quad t \in [t_i, t_{i+1}]. \end{aligned} \tag{3.10}$$

Let

$$u(t) = \begin{cases} u(t, t_0, u_0), & t_i \leq t \leq t_1 \\ u(t, t_1, u_1^+), & t_1 < t \leq t_2 \\ \cdots \\ u(t, t_i, u_i^+), & t_i < t \leq t_{i+1} \\ \cdots \\ u(t, t_p, t_p^+), & t_p < t \leq T. \end{cases}$$

Then $u(t)$ is a unique solution of (3.5) such that

$$u(t_0) = u_0 \quad \text{and} \quad u(t_0+T) = u_0 \left(\prod_{t_0 < t_k < t_0+T} d_k \right) e^{\int_{t_0}^{t_0+T} \lambda(s)ds}. \tag{3.11}$$

Let $M = \min_{t_0 < t_i < t_0 + T} u_0 \prod_{j=1}^{i} d_j$. It then follows from condition (iv) that

$$u(t_0 + T) \leq u(t_0) \quad \text{and} \quad u(t) \geq M \quad \text{for} \quad t \in [t_0, t_0 + T].$$

Thus the proof of Corollary 3.1 is complete.

If $\lambda(t) \equiv 0$ for $t \in [0, T]$ and $d_k \equiv 1$ for $k = 1, 2, \ldots$, then we have the following result.

COROLLARY 3.2 Assume that

(i) $V \in \mathcal{V}_0$, $V(t, x)$ is convex, locally Lipschitzian in x, periodic in t with period T and $V(t, 0) \equiv 0$ for $t \in [0, T]$. Moreover, there exists $b \in k$ such that $b(u) \to \infty$ as $u \to \infty$ and

$$b(\| x \|) \leq V(t, x), \ (t, x) \in R_+ \times R^n;$$

(ii) for some $M > 0, D^+V(t, x) \leq 0, \ (t, x) \in [0, T] \times R^n, \ t \neq t_k, V(t, x) \geq M$;

(iii) $V(t_k^+, x + I_k(x)) \leq V(t_k, x), \ (t_k, x) \in R_+ \times R^n, \ V(t_k, x) \geq M$.

Then the differential system (2.1) has a T-periodic solution.

REMARK: It is easy to see from the proof of Corollary 3.1 that we can choose the constant M to be greater than 1 in both Corollary 3.1 and 3.2.

As an application, we consider the following examples.

EXAMPLE 1: Consider the differential system

$$\begin{aligned} x' &= x(1 + y)\sin^2 t - y\cos t + \tfrac{\sin^2 t}{2\sqrt{2\alpha}}, \quad t \neq \tfrac{n\pi}{2}, \ n = 1, 2, \ldots \\ y' &= (-2\alpha x^2 + y)\sin^2 t + 2\alpha x \cos t - \tfrac{\sin^2 t}{2}, \quad t \neq \tfrac{n\pi}{2}, \ n = 1, 2, \ldots \\ \Delta x|_{t=\frac{n\pi}{2}} &= -\tfrac{2}{3}x, \quad \Delta y|_{t=\frac{n\pi}{2}} = -\tfrac{2}{3}y, \quad n = 1, 2, \ldots, \end{aligned} \tag{3.12}$$

where $\alpha > 0$ is a constant. Clearly, the right hand side of (3.12) is 2π-periodic in t.

Let $V(t,x,y) = 2\alpha x^2 + y^2$. Then, for $t \neq \frac{n\pi}{2}$,

$$D^+V(t,x,y) = 4\alpha x\left[x(1+y)\sin^2 t - y\cos t + \frac{\sin^2 t}{2\sqrt{2\alpha}}\right]$$

$$+2y\left[(-2\alpha x^2 + y)\sigma n^2 t + 2\alpha x\cos t - \frac{\sin^2 t}{2}\right]$$

$$\leq 4V(t,x,y)\sin^2 t, \quad \text{provided} \quad V(t,x,y) \geq 1.$$

$$V\left(\frac{n\pi^+}{2}, x, y\right) = 2\alpha\left(1-\frac{2}{3}\right)^2 x^2 + \left(1-\frac{2}{3}\right)^2 y^2 = \frac{1}{9}V\left(\frac{n\pi}{2}, x, y\right).$$

For any $t_0 \neq \frac{n\pi}{2}$, we have

$$\left(\prod_{t_0 < \frac{n\pi}{2} < t_0 + 2\pi} \frac{1}{9}\right) e^{\int_{t_0}^{t_0+2\pi} 4\sin^2 \sigma d\sigma} = \frac{1}{9^4}e^{4\pi} < 1.$$

It is easy to check that the rest of the conditions in Corollary 3.1 are satisfied. Thus (3.12) has a nontrivial 2π-periodic solution.

EXAMPLE 2: Consider the impulsive differential system

$$x' = -x(2+\sin^2 x) - 2y + \tfrac{1}{2}\cos^2 t, \quad t \neq \tfrac{n\pi}{4},\ n = 1,2,\ldots$$

$$y' = 2x - y - y(2+\cos^2 x) - \tfrac{1}{2}\sin^2 t, \quad t \neq \tfrac{n\pi}{4},\ n = 1,2,\ldots$$

$$\Delta x|_{t=\frac{n\pi}{4}} = -\tfrac{1}{2}x,\ \Delta y|_{t=\frac{n\pi}{4}} = -\tfrac{1}{8}y,\ n = 1,2,\ldots .$$

The right hand side of (3.13) is 2π-periodic in t. Let $V(t,x,y) = x^2 + y^2$. Then for $t \neq \frac{n\pi}{4}$,

$$D^+V(t,x,y) = 2x\left[-x(2+\sigma n^2 x) - 2y + \cos^2 t\right] + 2y\left[2x - y(2+\cos^2 x) - \sin^2 t\right]$$
$$\leq 0, \quad \text{provided} \quad V(t,x,y) \geq 1,$$

and

$$V\left(\tfrac{n\pi}{4}, x, y\right) = \left(x - \tfrac{1}{4}x\right)^2 + \left(y - \tfrac{1}{8}y\right)^2 < V\left(\tfrac{n\pi}{4}, x, y\right).$$

It then follows from Corollary 3.2 that (3.13) admits a 2π-periodic solution.

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REFERENCES

1. Krasnosel'skiĭ, M.A., *Translation Along Trajectories of Differential Equations*, AMS Providence, Rhode Island, 1968.

2. Lakshmikantham, V., Bainov, D.D. and Simeonov, P.S., *Theory of Impulsive Differential Equations*, World Scientific Publications of Singapore, 1989.

3. Lakshmikantham, V. and Leela, S., *Differential and Integral Inequalities*, Vol. I, Academic Press, 1969.

4. Liu, Xinzhi and Pirapakaran, R., Asymptotic stability for impulsive differential systems, *J. Appl. Math. Comp.* **30** (1989), 149–163.