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ON THE UNSTEADY FLOW OF TWO VISCO-ELASTIC FLUIDS BETWEEN TWO INCLINED POROUS PLATES[1]

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ABSTRACT

This study is concerned with both hydrodynamic and hydromagnetic unsteady slow flows of two immiscible visco-elastic fluids of Rivlin-Ericksen type between two porous parallel nonconducting plates inclined at a certain angle to the horizontal. The exact solutions for the velocity fields, skin frictions, and the interface velocity distributions are found for both fluid models. Numerical results are presented in graphs. A comparison is made between the hydrodynamic and hydromagnetic velocity profiles. It is shown that the velocity is diminished due to the presence of a transverse magnetic field.

Key words: Visco-elastic hydromagnetic and hydrodynamic fluids, skin frictions and interface velocity fields.

AMS (MOS) subject classifications: 76W05.

1. INTRODUCTION

The study of fluid flows in a porous medium plays an important role in the recovery of crude oil from the pores of reservoir rocks by displacement with immiscible water and forming polymetric adhesive joints between the solids. Various hydrodynamic and hydromagnetic flows in different fluid configurations have received considerable attention in recent years by several researchers including Kapur [1], Kapur and Sukhla [2], Bhattacharya [3], Gupta and Goyal [4], Gupta and Singh [5-6], and Sengupta and his associates [7-9]. In spite of this progress, some problems remained unsolved. Two such problems are considered in this paper.

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2. FORMULATION OF THE PROBLEM

We consider the unsteady flow of two incompressible, immiscible Rivlin-Ericksen fluids, each occupying a certain height between two porous parallel stationary plates inclined at an angle θ to the horizontal. We set a Cartesian coordinate system with the x-axis along the interface of the two fluids and parallel to the direction of the flow while the z-axis is chosen upward. Assuming that $u_j = u_j(x,z,t)$, $v_j = 0$, $w_j = 0$ and $\frac{\partial}{\partial y} = 0$ where $j = 1, 2$, the equation of continuity $\frac{\partial u_j}{\partial x} = 0$ leads to $u_j = u_j(z,t)$.

The unsteady equation of motion for the incompressible visco-elastic fluids in a porous medium is

$$\frac{du_j}{dt} = -\frac{1}{\rho_j}\frac{\partial p}{\partial x} + \left(\nu_j + \beta_j \frac{d}{dt}\right)\frac{d^2 u_j}{dz^2} - \frac{\alpha_j}{k_j}u_j + g sin\theta, \tag{2.1}$$

where ρ_j, ν_j, β_j, k_j and η_j are densities, coefficients of kinematic viscosity, kinematic visco-elasticity coefficients, permeabilities, and coefficients of viscosity of the fluids respectively, $j = 1$ refers to the first fluid $(0 \le z \le L)$, and $j = 2$ refers to the second fluid $(-L \le z \le 0)$.

The required boundary conditions are

$$\left.\begin{array}{ll} u_1 = 0 & \text{at } z = L, \\ u_1 = u_0 & \text{at } z = 0 \end{array}\right\} \quad \text{for the first fluid} \tag{2.1ab}$$

$$\left.\begin{array}{ll} u_2 = 0 & \text{at } z = -L, \\ u_2 = u_0 & \text{at } z = 0 \end{array}\right\} \quad \text{for the second fluid.} \tag{2.2ab}$$

3. SOLUTIONS OF THE HYDRODYNAMIC PROBLEM

With the usual initial conditions, it is convenient to introduce the Laplace transform with respect to time t

$$\bar{u}_j = \int_0^\infty e^{-st} u_j dt, \; Re(s) > 0 \tag{3.1}$$

Application of this transform reduces equation (2.1) to the form

$$\frac{d^2u_j}{dz^2} - \frac{R_j}{N_j}\left(u_j + \frac{p_j}{sR_j}\right) = 0, \tag{3.2}$$

where $R_j = s + \frac{\alpha_j}{k_j}$, $N_j = \nu_j + s\beta_j$ and $-p_j = -\frac{1}{\rho_j}\frac{\partial p}{\partial x} + g\sin\theta$ with $j = 1,2$.

The solution of the transformed velocity $\bar{u}_1$ for the first fluid is

$$\bar{u}_1(z,s) = \left(\frac{u_0}{s} + \frac{p_1}{sR_1}\right)\frac{\sinh(L-z)\sqrt{R_1/N_1}}{sR_1\sinh L\sqrt{R_1/N_1}} - \frac{p_1}{sR_1}. \tag{3.3}$$

The inverse Laplace transform gives the velocity in $0 \le z \le L$,

$$u_1(z,t) = u_0\frac{\sinh\left\{(L-z)\frac{1}{\sqrt{k_1}}\right\}}{\sinh\frac{L}{\sqrt{k_1}}} + \frac{p_1k_1}{\nu_1}\left\{\frac{\sinh(L-z)/\sqrt{k_1}}{\sinh L/\sqrt{k_1}} + \frac{\sinh z/\sqrt{k_1}}{\sinh L/\sqrt{k_1}} - 1\right\}$$

$$+\left\{2\pi u_0L^2\sum_{n=1}^{\infty}\frac{(-1)^n(k_1-\beta_1)n\,\sin(L-z)\frac{n\pi}{L}}{(L^2+n^2\pi^2\beta_1)(L^2+n^2\pi^2k_1)}\right\}exp\left\{-\frac{\nu_1(\pi^2n^2k_1+L^2)}{k_1(n^2\pi^2\beta_1+L^2)}\cdot t\right\}$$

$$-\frac{2p_1L^2k_1}{\nu_1\pi}\sum_{n=1}^{\infty}(-1)^n\frac{\{\sin\frac{n\pi}{L}(L-z)+\sin\frac{n\pi z}{L}\}}{n(L^2+n^2\pi^2k_1)}\cdot exp\left\{-\frac{\nu_1(\pi^2n^2k_1+L^2)}{k_1(n^2\pi^2\beta_1+L^2)}\cdot t\right\}. \tag{3.4}$$

The transformed velocity $\bar{u}_2$ in $-L \le z \le 0$ is

$$\bar{u}_2(z,s) = \left(\frac{u_0}{s} + \frac{p_2}{sR_2}\right)\frac{\sinh(L+z)\sqrt{R_2/N_2}}{\sinh L\sqrt{R_2/N_2}} - \frac{p_2}{sR_2}\left\{\frac{\sinh z\sqrt{R_2/N_2}}{\sinh L\sqrt{R_2/N_2}} + 1\right\}$$

$$+2\pi u_0L^2(k_2-\beta_2)\sum_{n=1}^{\infty}\frac{n(-1)^n\sin(L+z)\frac{n\pi}{L}}{(L^2+n^2\pi^2\beta_2)(L^2+\pi^2n^2k_2)}exp\left\{-\frac{\nu_2(n^2\pi^2k_2+L^2)}{k_2(n^2\pi^2\beta^2+L^2)}\cdot t\right\}$$

$$-\frac{2p_2L^2k_2}{\nu_2\pi}\sum_{n=1}^{\infty}(-1)^n\frac{\{\sin\frac{n\pi}{L}(L+z)-\sin\frac{n\pi z}{L}\}}{n(L^2+n^2\pi^2k_2)}exp\left\{-\frac{\nu_2(n^2\pi^2k_2+L^2)}{k_2(n^2\pi^2k^2+L^2)}\cdot t\right\}. \tag{3.6}$$

In the limit as $t \to \infty$, the exponential terms in (3.4) and (3.6) tend to zero and hence the steady state solutions are attained and are given by the first two terms in (3.4) for $u_1(z,t)$, and by the first two terms in (3.6) for $u_2(z,t)$.

Interface Velocity: The tangential stress is continuous at the interface of the liquids. Thus we get

$$\eta_1\frac{\partial u_1}{\partial z}\bigg|_{z=0} = \eta_2\frac{\partial u_2}{\partial z}\bigg|_{z=0}. \tag{3.7}$$

The interface velocity can be written as

$$u_0 = \frac{\frac{\eta_1 p_1 \sqrt{k_1}}{\nu_1}(cothL/\sqrt{k_1} - cosechL/\sqrt{k_1}) + \frac{p_2\eta_2\sqrt{k_2}}{\nu_2}(cothL/\sqrt{k_2} - cosechL/\sqrt{k_2})}{-\left(\frac{\eta_1}{\sqrt{k_1}}\, cothL/\sqrt{k_1} + \frac{\eta_2}{\sqrt{k_2}}\, cothL/\sqrt{k_2}\right)}. \tag{3.8}$$

On putting $q_j = -\dfrac{p_j}{2\nu_j u_0}$ in the steady-state solutions, we obtain

$$\frac{u_1}{u_0} = (1 - 2q_1k_1)\frac{sinh(L-z)/\sqrt{k_1}}{sinhL/\sqrt{k_1}} - 2q_1k_1\left(\frac{sinhz/\sqrt{k_1}}{sinhL/\sqrt{k_1}} - 1\right) \tag{3.9}$$

and

$$\frac{u_2}{u_0} = (1 - 2q_2k_2)\frac{sinh(L+z)/\sqrt{k_2}}{sinhL/\sqrt{k_2}} + 2q_2k_2\left(\frac{sinhz/\sqrt{k_2}}{sinhL/\sqrt{k_2}} + 1\right) \tag{3.10}$$

4. SOLUTIONS OF THE HYDRODYNAMIC PROBLEM

We consider the visco-elastic fluids electrically conducting in the presence of a constant transverse magnetic field B_0 along the z-axis. Thus, the equation of motion is

$$\frac{\partial u_j}{\partial t} = -\frac{1}{\rho_i}\frac{\partial p}{\partial x} + \left(\nu_j + \beta_j\frac{\partial}{\partial t}\right)\frac{\partial^2 u_j}{\partial z^2} - \frac{\nu_j u_j}{k_j} + gsin\theta - \frac{\sigma_j B_0^2}{\rho_j}u_j, \tag{4.2}$$

where σ_j is the electrical conductivity of the fluids. This equation is to be solved by the same initial and boundary conditions as for the hydrodynamic problem. The Laplace transformed equation of (4.2) is

$$\frac{d^2\bar{u}_j}{dz^2} - \frac{M_j}{N_j}\left(\bar{u}_j + \frac{p_j}{sM_j}\right) = 0, \tag{4.3}$$

where

$$M_j = s + \frac{\nu_j}{k_j} + \frac{\sigma_j B_0^2}{\rho_j} \text{ and } j = 1, 2. \tag{4.4}$$

The solution of (4.3) with the transformed boundary condition is

$$\bar{u}_1(z,s) = \left(\frac{u_0}{s} + \frac{p_1}{sM_1}\right)\frac{sinh(L-z)\sqrt{M_1/N_1}}{sinhL\sqrt{M_1/N_1}} + \frac{p_1}{sM_1}\frac{sinhz\sqrt{M_1/N_1}}{sinhL\sqrt{M_1/N_1}} - \frac{p_1}{sM_1}. \tag{4.5}$$

The inverse transform gives

$$u_1(z,t) = \left(u_0 + \frac{p_1}{\left(\frac{\nu_1}{k_1} + \frac{\sigma_1 B_0^2}{\rho_1}\right)}\right)\frac{sinh(L-z)\sqrt{\frac{1}{k_1} + \frac{\sigma_1 B_0^2}{\rho_1\nu_1}}}{sinhL\sqrt{\frac{1}{k_1} + \frac{\sigma_1 B_0^2}{\rho_1}}} + \frac{p_1}{\left(\frac{\nu_1}{k_1} + \frac{\sigma_1 B_0^2}{\rho_1}\right)}\left(\frac{sinhz\sqrt{\frac{1}{k_1} + \frac{\sigma_1 B_0^2}{\rho_1\nu_1}}}{sinhL\sqrt{\frac{1}{k_1} + \frac{\sigma_1 B_0^2}{\rho_1\nu_1}}} - 1\right)$$

$$+ 2\pi u_0 \sum_{n=1}^{\infty} (-1)^n \frac{nL^2(k_1\nu_1\rho_1 - \beta_1\rho_1\nu_1 - \sigma_1 B_0^2\beta_1 k_1) sin(L-z)\frac{n\pi}{L}}{(L^2 + n^2\pi\beta_1)(n^2\pi^2 k_1\nu_1\rho_1 + L^2\rho_1\nu_1 + \sigma_1 B_0^2 L^2 k_1)} \cdot$$

$$\cdot exp\left(- \frac{\nu_1(n^2\pi^2 k_1 + L^2)\rho_1 + \sigma_1 B_0^2 L^2 k_1}{k_1\rho_1(L^2 + n^2\pi^2\beta_1)} \cdot t \right)$$

$$- \frac{2p_1 L^2 k_1}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{\rho_1 (sin(L-z)\frac{n\pi}{L} + sin(\frac{n\pi}{L})z)}{n(N^2\pi^2 k_1\rho_1\nu_1 + L^2\rho_1\nu_1 + \sigma_1 B_0^2 k_1 L^2)} \cdot$$

$$\cdot exp\left(- \frac{\nu_1(n^2\pi^2 k_1 + L^2)\rho_1 + \sigma_1 B_0^2 L^2 k_1}{k_1\rho_1(L^2 + n^2\pi^2\beta_1)} \cdot t \right). \tag{4.6}$$

Similarly, we find

$$\bar{u}_2(z,s) = \left(\frac{u_0}{s} + \frac{p_2}{sM_2} \right) \frac{sinh(L+z)\sqrt{M_2/N_2}}{sinhL\sqrt{M_2/N_2}} - \frac{p_2}{sM_2} \left(\frac{sinhz\sqrt{M_2/N_2}}{sinhL\sqrt{M_2/N_2}} + 1 \right). \tag{4.7}$$

The inverse Laplace transform gives

$$u_2(z,t) = \left(u_0 + \frac{p_2}{\frac{\nu_2}{k_2} + \frac{\sigma_2^2 B_0^2}{\rho_2}} \right) \frac{sinh(L+z)\sqrt{\frac{1}{k_2} + \frac{\sigma_2 B_0^2}{\rho_2\nu_2}}}{sinhL\sqrt{\frac{1}{k_2} + \frac{\sigma_2 B_0^2}{\rho_2\nu_2}}} - \frac{p_2}{\left(\frac{\nu_2}{k_2} + \frac{\sigma_2 B_0^2}{\rho_2} \right)} \left(\frac{sinhz\sqrt{\frac{1}{k_2} + \frac{\sigma_2 B_0^2}{\rho_2\nu_2}}}{sinhL\sqrt{\frac{1}{k_2} + \frac{\sigma_2 B_0^2}{\rho_2\nu_2}}} - 1 \right)$$

$$+ 2\pi u_0 L^2 \sum_{n=1}^{\infty} \frac{n(k_2\nu_2\rho_2 - \beta_2\nu_2\rho_2 - \sigma_2 B_0^2\beta_2 k_2)(sin(L+z)\frac{n\pi}{L}) \cdot (-1)^n}{(L^2 + n^2\pi^2\beta_2)(N^2\pi^2 k_2\rho_2\nu_2 + L^2\rho_2\nu_2 + L^2\rho_2\nu_2 + \sigma^2 B_0^2 k_2 L^2)} \cdot$$

$$\cdot exp\left(- \frac{\nu_2(n^2\pi^2 k_2 + L^2)\rho_2 + \sigma_2 B_0^2 L^2 k_2}{(L^2 + n^2\pi^2\beta_2)k_2\rho_2} \cdot t \right)$$

$$- \frac{2p_2 L^2 k_2}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{\rho_2 (sin\frac{n\pi}{L}(L+z) - sin\frac{n\pi z}{L})}{n(n^2\pi^2 k_2\rho_2\nu_2 + L^2\rho_2\nu_2 + \sigma B_0^2 k_2 L^2)} \cdot$$

$$\cdot exp\left(- \frac{\nu_2(n^2\pi^2 k_2 + L^2)\rho_2 + \sigma_2 B_0^2 L^2 k_2}{(L^2 + n^2\pi^2\beta_2)k_2\rho_2} \cdot t \right). \tag{4.8}$$

In the limit as $t \to \infty$, the transient terms involved in (4.6) and (4.8) decay and the steady state solutions are attained and given by the first two terms in each of the results (4.6) and (4.8). These steady state solutions can be written as

$$u_1(z,t) = \left(u_0 + \frac{p_1}{\nu_1 T_1} \right) \frac{sinh(L-z)\sqrt{T_1}}{sinhL\sqrt{T_1}} + \frac{p_1}{\nu_1 T_1} \left(\frac{sinhz\sqrt{T_1}}{sinhL\sqrt{T_1}} - 1 \right), \tag{4.9}$$

and

$$u_2(z,t) = \left(u_0 + \frac{p_2}{\nu_2 T_2}\right)\frac{sinh(L+z)\sqrt{T_2}}{sinhL\sqrt{T_2}} - \frac{p_2}{\nu_2 T_1}\left(\frac{sinhz\sqrt{T_2}}{sinhL\sqrt{T_2}} + 1\right), \tag{4.10}$$

where

$$T_j = \frac{1}{k_j} + \frac{\sigma_j B_0^2}{\rho_j \nu_j} = \frac{1}{k_j} + \frac{H_j^2}{L^2} \tag{4.11}$$

and H_j are the Hartmann numbers.

Finally, the interface velocity is

$$u_0 = \frac{\frac{\eta_1 p_1}{\nu_i \sqrt{T_1}}(cothL\sqrt{T_1} - cosechL\sqrt{T_1}) + \frac{\eta_2 p_2}{\nu_2\sqrt{T_2}}(cothL\sqrt{T_2} - cosechL\sqrt{T_2})}{-(\eta_1\sqrt{T_1}cothL\sqrt{T_1} + \eta_2\sqrt{T_2}cothL\sqrt{T_2})}. \tag{4.12}$$

In terms of the notation $q_j = -\frac{p_j}{2\nu_j u_0}$, the steady state solution can be written in the form

$$\frac{u_1}{u_0} = \left(1 - \frac{2q_1}{T_1}\right)\frac{sinh(L-z)\sqrt{T_1}}{sinhL\sqrt{T_1}} - \frac{2q_1}{T_1}\left(\frac{sinhz\sqrt{T_1}}{sinhL\sqrt{T_1}} - 1\right), \tag{4.13}$$

and

$$\frac{u_2}{u_0} = \left(1 - \frac{2q_2}{T_2}\right)\frac{sinh(L+z)\sqrt{T_2}}{sinhL\sqrt{T_2}} + \frac{2q_2}{T_2}\left(\frac{sinhz\sqrt{T_2}}{sinhL\sqrt{T_2}} + 1\right). \tag{4.14}$$

5. DISCUSSIONS AND CONCLUSIONS

It follows from the numerical calculation that, for a conducting fluid in the presence of a transverse magnetic field, the velocity on the interface is at its maximum and then gradually decreases with continuous increase of distance in the upward or downward direction according to whether the upper or lower fluid is involved. For the upper fluid, the velocity attains the minimum value zero at the upper boundary plate, while for the lower fluid, the velocity reaches the minimum value zero at the lower boundary plate. In both cases, the velocity profile is very smooth and continuous. For nonconducting fluids (or in the absence of a magnetic fields), the maximum velocity does not occur at the interface; rather, it is attained for the upper fluid at a point a bit higher than the interface, while for the lower fluid it is attained at a point a bit lower than the interface. This shows a striking difference in the flow pattern between a conducting and a nonconducting fluid mode. However, the nature of the variation of the flow patterns of the conducting and nonconducting fluids remains very similar.

In both fluid models that u_1/u_0 and u_2/u_0 increase with the increase in q_1 and q_2 where q_1 and q_2 represents the pressure gradients. On the other hand, these velocity ratios increase with the increase in permeabilities k_1 and k_2 of the fluids. It is important to observe that when $\sigma_1 B_0^2/\rho_1\nu_1 = \sigma_2 B_0^2/\rho_2\nu_2 = 1$, then $\frac{u_1}{u_0} = 0.68$ and when $\sigma_1 B_0^2/\rho_1\nu_1 = \sigma_2 B_0^2/\rho_2\nu_2 = 0$, $\frac{u_1}{u_0} = 0.785$ for the same value of z/L. This means that the velocity decreases by 13% in the presence of a transverse magnetic field. Thus the decrease of the velocity for the hydromagnetic case is worth noting.

The velocity profiles are drawn in Figure 1 and Figure 2 for the hydrodynamic and hydromagnetic fluid models.

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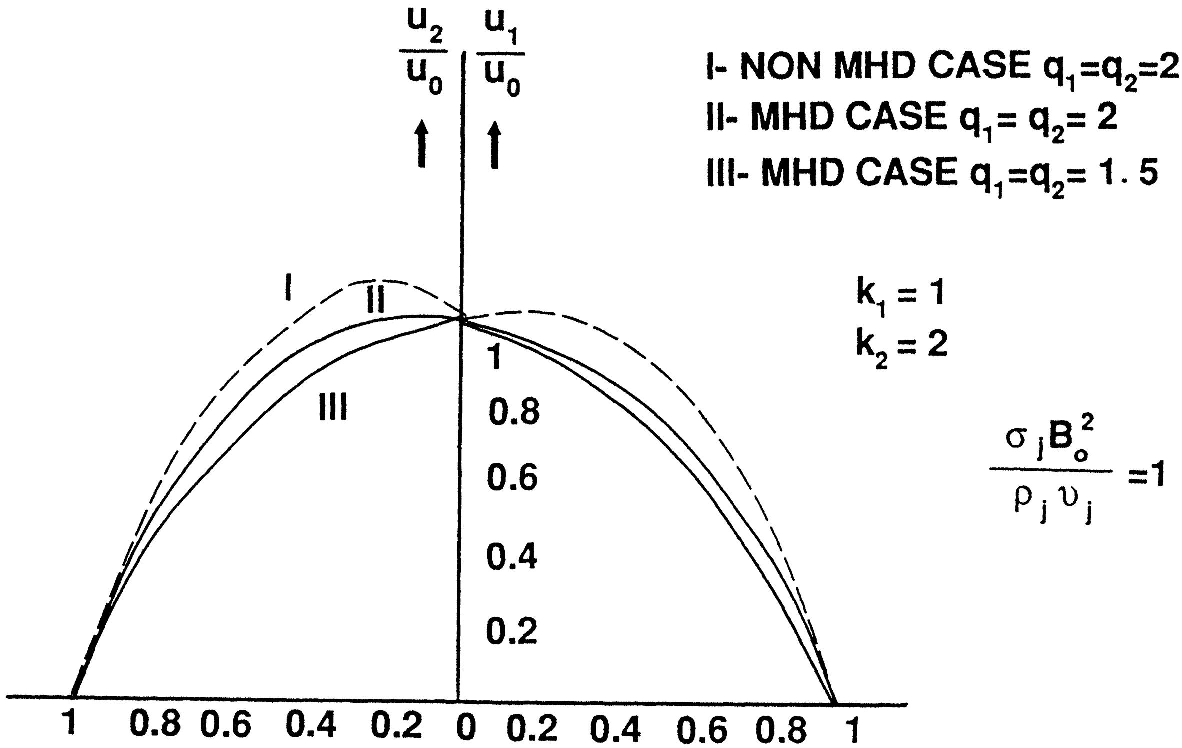

FIGURE 1

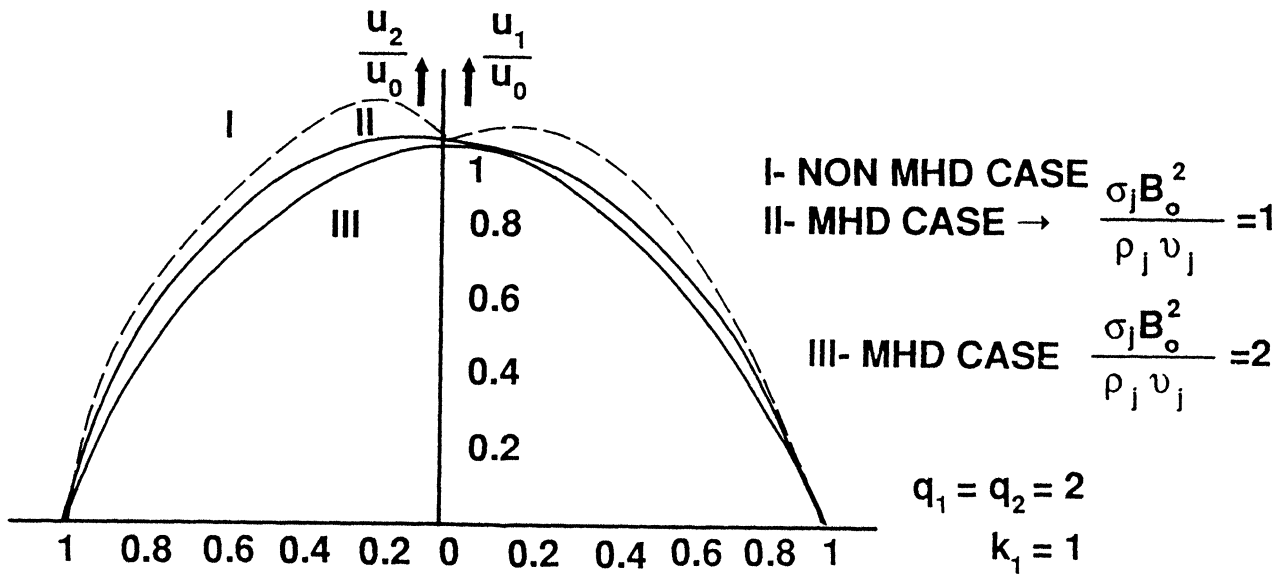

FIGURE 2