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ERROR ESTIMATES FOR THE SEMIDISCRETE FINITE ELEMENT APPROXIMATION OF LINEAR NONLOCAL PARABOLIC EQUATIONS[1]

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ABSTRACT

Existence and uniqueness are proved for nonlocal (in time) for solutions of linear parabolic partial differential equations. Instead of an initial condition, there is a relation connecting the initial value to values of the solution at other times. L^2 error estimates are obtained for the semidiscrete approximation of the problem using finite elements in the space variables.

Key words: Nonlocal parabolic equations, semidiscrete finite element approximations, error estimates.

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1. INTRODUCTION

Let Ω be a bounded open subset of R^n with a smooth boundary Γ. The following nonlocal problem will be considered:

$$\left.\begin{aligned} &u_t + Au = f(x,t) \text{ on } \Omega\times(0,T),\\ &u\,|_{\Gamma} = 0,\\ &u(x,0) + g(t_1,\ldots,t_N,u) = \psi(x), \end{aligned}\right| \tag{1.1}$$

where $0 < t_1 < t_2 < \ldots < t_N \le T$, $\psi(x) \in L^2(\Omega), f(x,t) \in L^\infty([0,T]; L^2(\Omega))$ and $g(t_1,\ldots,t_N,\cdot)$ maps $C^0([0,T]; L^2(\Omega))$ into $L^2(\Omega)$. Also assume A is a strongly elliptic operator defined by

$$Au = -\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}) + \sum_{i=1}^{n} a_i(x)\frac{\partial u}{\partial x_i} + a_0(x)u \tag{1.2}$$

with $a_{ij}(x), a_i(x) \in C^\infty(\overline{\Omega})$, with

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$$a(u,u) \geq \sigma \| u \|_1^2 - \lambda_0 \| u \|^2, \quad u \in H_0^1(\Omega), \tag{1.3}$$

where $\sigma > 0, \lambda_0 \in R$, $\| u \|^2 = \| u \|^2_{L^2(\Omega)} = (u,u)$,

$$a(u,v) = -\sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \sum_{i=1}^{n} \int_\Omega a_i(x) \frac{\partial u}{\partial x_i} v dx + \int_\Omega a_0(x) uv dx \tag{1.4}$$

and $H^s(\Omega)$ and $H_0^s(\Omega)$ are the usual Sobolev spaces with norms $\| \ \|_s$. See Adams [1] or Lions [10] for definitions.

Under the above conditions, A with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ generates an analytic semigroup $S(t) = e^{-At}$ such that for $a = \sigma - \lambda_0$

$$\| S(t) f \| \leq M e^{-at} \| f \|, \tag{1.5}$$

where $M \geq 1$ depends continuously on σ and λ_0 in (1.3). See Pazy [5].

The function $u \in C^0([0,T]; L^2(\Omega))$ is said to be a mild solution of (1.1) if

$$u(t) = S(t)\psi(x) - S(t) g(t_1, \quad , t_N, u) + \int_0^t S(t-\tau) f(x,\tau) d\tau. \tag{1.6}$$

We will assume for $u, v \in C^0([0,T]; L^2(\Omega))$ of the form $u, v = w$, where

$$w(t) = S(t) w(0) + \int_0^t S(t-\tau) f(x,\tau) d\tau, \tag{1.7}$$

we have the Lipschitz condition

$$\| g(t_1, \ldots, t_N, u) - g(t_1, \ldots, t_N, v) \| \leq \sum_{i=1}^{N} m_i \| u(t_i) - v(t_i) \|. \tag{1.8}$$

The following are some examples of $g(t_1, \ldots, t_N, u)$: If $h_i(x) \in C^\infty(\overline{\Omega})$, let

$$g(t_1, \ldots, t_N, u) = \sum_{i=1}^{N} h_i(x) u(t_i). \tag{1.9}$$

The m_i in (1.8) are $m_i = \max_{x \in \Omega} | h_i(x) |$.

Another useful example is

$$g(t_1, \ldots, t_N, u) = \sum_{i=1}^{N} \frac{1}{k_i} \int_{t_i}^{t_i + k_i} h_i(x,\tau) u(\tau) d\tau, \tag{1.10}$$

where $k_i > 0$ and $h_i(x,t) \in C^\infty(\overline{\Omega} \times [0,T])$. If u, v are as in (1.7) and $t_i \leq \tau \leq t_i + k_i$, then

$$\| u(\tau) - v(\tau) \| = \| S(\tau - t_i)(u(t_i) - v(t_i)) \| \leq Me^{-a(\tau - t_i)} \| u(t_i) - v(t_i) \| .$$

Thus the m_i in (1.8) are

$$m_i = \frac{M}{ak_i}(1 - e^{-ak_i}) \cdot (\max_{(x,t) \in \bar{\Omega} \times [t_i, t_i + k_i]} | h_i(x,t) |).$$

Nonlocal parabolic problems have been studied by several authors. See Byszewski [2-5], Chabrowski [6], Hess [7], Kerefov [8], and Vabishchewich [13].

2. EXISTENCE AND UNIQUENESS FOR NONLOCAL PROBLEMS

In this section we will prove under the conditions of section 1, (1.6) has a unique solution.

Let $W = C^0([0,T]; L^2(\Omega))$ with norm

$$\| u \|_W = \sup_{0 \leq t \leq T} e^{at} \| u(t) \| ,$$

where a satisfies (1.5). We have the following:

Theorem 2.1: *Assume* (1.5), (1.8) *hold,* $\psi(x) \in L^2(\Omega)$, *and* $\sum_{i=1}^{N} m_i e^{-at_i} < \frac{1}{M^2}$ *for* m_i *in* (1.8) *and* a, M *in* (1.5). *Then there is a unique* u *in* W *such that* $u(t)$ *satisfies* (1.6).

Proof: Let $\Phi: W \to W$ be defined by

$$\Phi v(t) = S(t)\psi(x) - S(t)g(t_1, \ldots, t_N, S(t)v(0) + \int_0^t S(t-\tau)f(x,\tau)d\tau) \tag{2.1}$$
$$+ \int_0^t S(t-\tau)f(x,\tau)d\tau$$

for $v \in W$.

We will show Φ is a contraction mapping on W. Let $u, v \in W$. Then

$$e^{at} \| \Phi u(t) - \Phi v(t) \|$$

$$\leq e^{at} M e^{-at} \sum_{i=1}^{N} m_i \| S(t_i)(u(0) - v(0)) \|$$

$$\leq M \sum_{i=1}^{N} m_i M e^{-at_i} \| u(0) - v(0) \|$$

$$\leq M^2 (\sum_{i=1}^{N} m_i e^{-at_i}) \| u - v \|_W .$$

Thus Φ is a contraction on W, which implies there is a unique $u \in W$ such that $u = \Phi(u)$. Since

$$u(0) = \Phi u(0) = \psi(x) - g(t_1,\ldots,t_n, S(t)u(0) + \int_0^t S(t-\tau)f(x,\tau)d\tau)$$

and

$$u(t) = S(t)u(0) + \int_0^t S(t-\tau)f(x,\tau)d\tau$$

it follows that $u(t)$ satisfies (1.6).

Since $S(t)$ has the smoothing property, $S(t)f \in D(A^{\frac{\alpha}{2}})$ for $t > 0$, $\alpha \geq 0$ and $f \in L^2(\Omega)$, we have the following regularity property:

Corollary 2.2: If the conditions of Theorem 2.1 are satisfied, $\psi(x) \in D(A^{\frac{\alpha}{2}})$, $\alpha \geq 0$; $f(x,t) \in L^\infty([0,T]; D(A^\mu)), \mu = max\{\frac{\alpha}{2} - 1 + \epsilon, 0\}$ for some $\epsilon > 0$; and $g(t_1,\ldots,t_N, \cdot)$ maps $C^0((0,T]; D(A^{\frac{\alpha}{2}}))$ into $D(A^{\frac{\alpha}{2}})$, then the solution $u(t)$ of (1.6) satisfies $u \in C^0([0,T]; D(A^{\frac{\alpha}{2}}))$.

Note: If $\sum_{i=1}^{N} m_i e^{-at} < \frac{1}{M^2}$ is not satisfied, there may not be a unique solution. For example, $u_t - u_{xx} + (a - \pi^2)u = 0$ on $(0,1), u(0,t) = 0 = u(1,t)$, and $u(x,0) - e^{-a}u(x,1) = 0$ has solutions $u(x,t) = 0$ and $u(x,t) = e^{-at} sin \pi x$.

3. THE SEMIDISCRETE APPROXIMATION

Let $\{V_h\}$ be a family of finite dimensional subspaces of $H^1(\Omega)$ such that for $f \in H^s(\Omega), 1 \leq s \leq r$,

$$\inf_{\chi \in V_h} \{ \| f - \chi \| + h \| f - \chi \|_1 \} \leq ch^s \| f \|_s, \tag{3.1}$$

where c is independent of h.

In this section we will assume (1.3) is satisfied with $\lambda_0 = 0$. If this is not the case, let $u = e^{\lambda_0 t} W$.

For fixed $\epsilon > 0$, assume $A_h : V_h \rightarrow V_h$ satisfies

$$(A_h f_h, f_h) \geq \sigma' \| f_h \|^2 \text{ if } f_h \in V_h, \tag{3.2}$$

where $0 < \sigma - \epsilon < \sigma' \leq \sigma$,

$$(A_h f_h, g_h) \leq c \| f_h \|_1 \| g_h \|_1 \text{ for all } f_h, g_h \in V_h \tag{3.3}$$

and

$$\| (P_h A^{-1} - A_h^{-1} P_h) f \| \leq c h^{\alpha+2} \| A^{\frac{\alpha}{2}} f \|, \quad 0 \leq \alpha \leq r-2, \tag{3.4}$$

where P_h is the L^2 projection of $L^2(\Omega)$ onto V_h.

Conditions (3.2), (3.3) and (3.4) are satisfied with $\sigma' = \sigma$ if the standard Galerkin method is used with $V_h \in H_0^1(\Omega)$ and A_h is defined by

$$(A_h f_h, g_h) = (A f_h, g_h), \quad f_n, v_n \in V_n.$$

The conditions are also satisfied if Nitsche's method is used, where $V_h \subseteq H^1(\Omega), V_h|_\Gamma \subseteq H^1(\Gamma)$, for $2 \leq s \leq r$,

$$\inf_{\chi \in V_h} \{ \| f - \chi \| + h \| f - \chi \|_1 + h^{\frac{1}{2}} \| f - \chi \|_{L^2(\Gamma)} + h^{\frac{3}{2}} \| f - \chi \|_{H^1(\Gamma)} \} \leq c h^s \| f \|_s$$

and $A_h: v_h \rightarrow v_n$ is defined by

$$(A_h f_h, g_h) = a(f_h, g_h) - (\frac{\partial f_h}{\partial n}, g_h)_{L^2(\Gamma)} - (f_h, \frac{\partial g_h}{\partial n})_{L^2(\Gamma)} + \beta h^{-1} (f_h, g_h)_{L^2(\Gamma)}$$

for β large enough such that (3.2) holds. See Lasiecka [9].

We will first show the following nonlocal system on V_h has a unique solution for $0 \leq t \leq T$:

$$\begin{aligned} u_h'(t) + A_h u_h &= P_h f(x,t), \\ u_h(0) + P_h g(t_1, \ldots, t_N, u_h) &= P_h \psi. \end{aligned} \tag{3.5}$$

Let $S_h(t) = e^{-A_h t}$, then (3.5) is equivalent to

$$u_h(t) = S_h(t) P_h \psi - S_h(t) P_n g(t_1, \ldots, t_N, u_h) + \int_0^t S_h(t-\tau) P_h f(x,\tau) d\tau. \tag{3.6}$$

Since $\| e^{-A_h t} f_h \| \leq M \sigma' e^{-\sigma' t} < \frac{1}{M_{\sigma'}^2}$, where $\lim_{\sigma' \rightarrow \sigma} M_{\sigma'} = M$, we can find $\epsilon > 0$ for (3.2) and $\delta > 0$ such that if $m_i' = m_i + \delta$ and $\sum_{i=1}^N m_i e^{-\sigma' t_i} < \frac{1}{M^2}$, then

$$\sum_{i=1}^N m_i' e^{-\sigma' t_i} < \frac{1}{M_{\sigma'}^2}. \tag{3.7}$$

Thus by a similar proof to that of Theorem 1.1, we can prove the following:

Theorem 3.1: *Assume the conditions in Theorem 1.1 are satisfied and V_h and A_h satisfy (3.1) – (3.4), where σ' from (3.2) is such that (3.7) holds and*

$$\| P_h(g(t_1,\ldots,t_N,u_h) - g(t_1,\ldots,t_N,v_h)) \| \le \sum_{i=1}^{N} m_i' \| u_h(t_i) - v_h(t_i) \| \tag{3.8}$$

for $u_h, v_h = w_n$ of the form $w_h(t) = S_h(t)w_h(0) + \int_0^t S_h(t-\tau)P_h f(x,\tau)d\tau$. Then there is a unique solution $u_h(t)$ of (3.6) such that $u_h \in C^0([0,T];V_h)$.

Since $\| P_h(h(x)f_h) \| \le (\sup_{x\in\Omega} |h(x)|) \| f_h \|$ for $f_h \in V_h$, if σ' is close enough to σ, then g defined in (1.9) and (1.10) satisfy (3.8).

Under the assumptions (3.1) – (3.4), we have for $a \le s \le r$ and $f \in D(A^{\frac{\alpha}{2}}), 0 \le \alpha \le s$ the condition

$$\| (S(t) - S_h(t)P_h)f \| \le \frac{Ch^s}{t^{\frac{s-\alpha}{2}}} \| A^{\frac{\alpha}{2}} f \| \tag{3.9}$$

and for $f(x,t) \in L^\infty(0,T;D(A^{\frac{\alpha'}{2}}))$, $\quad 0 \le \alpha' \le r-2$

$$\| \int_0^t (S(t-\tau) - S_h(t-\tau)P_h)f(x,\tau)d\tau \| \le Ch^{\alpha'+2} ln(\tfrac{1}{h}) \| f \|_{L^\infty(0,T;D(A^{\frac{\alpha'}{2}}))} . \tag{3.10}$$

See for example Lasiecka [9] or Thomée [12].

We can now prove similar error estimates for the semidiscrete approximation to the nonlocal problems.

Theorem 3.2: *Let the assumptions of Theorems 1.1 and 3.1 be satisfied, and let the hypotheses of Corollary 2.2 be satisfied for $\alpha \le r$, $f(x,t) \in L^\infty(0,T;D(A^{\frac{\theta}{2}})), \theta = max\{\mu, \alpha'\}$, $0 \le \alpha' \le r-2$, and for $u, v \in C^0([t_1,T], L^2(\Omega))$,*

$$\| g(t_1,\ldots,t_N,u) - g(t_1,\ldots,t_N,v) \| \le k \| u - v \|_{L^\infty(t_1,T;L^2(\Omega))} . \tag{3.11}$$

Also assume that $u(t)$ is the solution of (1.6) and $u_h(t)$ is the solution to (3.6) for $\alpha \le s \le r$. Then

$$\| u(t) - u_h(t) \| \le Ch^s(\frac{1}{t^{\frac{s-\alpha}{2}}} + 1) + Ch^{\alpha'+2} ln(\tfrac{1}{h}) \| f \|_{L^\infty(0,T;D(A^{\frac{\alpha'}{2}}))} . \tag{3.12}$$

Proof: We have

$$\| u(t) - u_h(t) \| \le \| (S(t) - S_h(t)P_h)\psi \| + \| (S(t) - S_h(t)P_h)g(t_1, \dots t_N, u) \|$$
$$+ \| S_h(t)P_h(g(t_1,\dots,t_N,u) - g(t_1,\dots,t_N,u_h)) \|$$
$$+ \| \int_0^t (S(t-\tau) - S_h(t-\tau)P_h)f(x,\tau)d\tau \| \tag{3.13}$$
$$\le \frac{Ch^s}{t^{\frac{s-\alpha}{2}}}(\| A^{\frac{\alpha}{2}}\psi \| + \| A^{\frac{\alpha}{2}}g(t_1,\dots,t_N,u) \|) + Ch^{\alpha'+2}ln\frac{1}{h} \| f \|_{L^\infty(0,T;D(A^{\frac{\alpha'}{2}}))}$$
$$+ M_{\sigma'}e^{-\sigma' t} \| g(t_1,\dots,t_N,u) - g(t_1,\dots,t_N,u_h) \| .$$

Since A_h is bounded, $S_h(-t) = e^{A_h t}$ exists. Let $t \ge t_1$, then

$$\| g(t_1,\dots,t_N,u) - g(t_1,\dots,t_N,u_h) \|$$
$$\le \| g(t_1,\dots,t_N,u) - g(t_1,\dots,t_N, S_h(t-t_1)P_hS(t_1)u(0) + \int_0^t S_h(t-\tau)P_hf(x,\tau)d\tau) \|$$
$$+ \| g(t_1,\dots,t_N, S_h(t)(S_h(-t_1)P_hS(t_1)u(0)) + \int_0^t S_h(t-\tau)P_hf(x,\tau)d\tau)$$
$$- g(t_1,\dots,t_N,u_h) \| \tag{3.14}$$
$$\le k \sup_{t_1 \le t \le T}(\| (S(t-t_1) - S_h(t-t_1)P_h)S(t_1)u(0) \| + \| \int_0^t (S(t-\tau) - S_h(t-\tau)P_h)f(x,\tau)d\tau \|$$
$$+ \sum_{i=1}^N m_i' \| S_h(t_i)(S_h(-t_1)P_hS(t_1)u(0)) - S_h(t_i-t_1)S_h(t_1)u_h \|$$
$$\le Ch^s \| A^{\frac{s}{2}}S(t_1)u(0) \| + Ch^{\alpha'+2}ln\frac{1}{h} \| f \|_{L^\infty(0,T;D(A^{\frac{\alpha'}{2}}))}$$
$$+ \sum_{i=1}^N m_i' M_{\sigma'}e^{-\sigma'(t_i-t_1)} \| S(t_1)u(0) - S_h(t_1)u_h(0) \|$$
$$\le Ch^s \| A^{\frac{s}{2}}S(t_1)u(0) \| + Ch^{\alpha'+2}ln\frac{1}{h} \| f \|_{L^\infty(0,T;D(A^{\frac{\alpha'}{2}}))}$$
$$+ \sum_{i=1}^N m_i M_{\sigma'}e^{-\sigma'(t_i-t_1)}(\| u(t_1) - u_h(t_1) \| + \| \int_0^{t_1} (S(t-\tau) - S_h(t-\tau)P_h)f(x,\tau)d\tau \|)$$

$$\leq Ch^s \| A^{\frac{s}{2}} S(t_1) u(0) \| + Ch^{\alpha' + 2} ln\frac{1}{h} \| f \|_{L^\infty(0,T;D(A^{\frac{\alpha'}{2}}))}$$
$$+ \sum_{i=1}^{N} m_i M_{\sigma'} e^{-\sigma'(t_i - t_1)} \| u(t_1) - u_h(t_1) \| .$$

Let $t = t_1$ in (3.13), then

$$\| u(t_1) - u_h(t_1) \| \leq C\left(\frac{h^s}{t_1^{\frac{s-\alpha}{2}}} + 1\right) + Ch^{\alpha' + 2} ln\left(\frac{1}{h}\right) \| f \|_{L^\infty(0,T;D(A^{\frac{\alpha'}{2}}))}$$
$$+ M_{\sigma'}^2 \sum_{i=1}^{N} m_i' e^{-\sigma' t_i} \| u(t_1) - u_h(t_1) \| . \tag{3.15}$$

Since $M_{\sigma'}^2 \sum_{i=1}^{N} m_i' e^{-\sigma' t_i} < 1$, (3.12) holds for $t = t_1$. Therefore the theorem follows from (3.13) and (3.15).

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