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A GENERALIZED UPPER AND LOWER SOLUTIONS METHOD FOR NONLINEAR SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS[1]

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ABSTRACT

The purpose of this paper is to study a nonlinear boundary value problem of second order when the nonlinearity is a Carathéodory function. It is shown that a generalized upper and lower solutions method is valid, and the monotone iterative technique for finding the minimal and maximal solutions is developed.

Key words: Periodic boundary value problem, upper and lower solutions, monotone method.

AMS (MOS) subject classification: 34B15.

1. INTRODUCTION

We shall, in this paper, develop the method of upper and lower solutions and the monotone iterative technique for second order boundary value problems of the form

$$\begin{aligned} -u''(t) &= f(t,u(t)),\ t \in I = [0,\pi] \\ Bu(0) &= c_0 \\ Bu(\pi) &= c_1 \end{aligned} \qquad (P)$$

where f is a Carathéodory function, $Bu(0) = a_0u(0) - b_0u'(0)$, and $Bu(\pi) = a_1u(\pi) + b_1u'(\pi)$,

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a_0, $a_1 \geq 0$, b_0, $b_1 > 0$.

We first note that the classical arguments of [2] for f continuous are no longer valid since if u is a solution of (P), then u'' needs not to be continuous but only $u'' \in L^1(0,\pi)$. Here we extend classical and well-known results when f is continuous (see [2]) to the case when f is a Carathéodory function.

If we choose $a_0 = a_1 = c_0 = c_1 = 0$, then the boundary conditions read $u'(0) = u'(\pi) = 0$. Thus, we have the Neumann boundary value problem

$$-u'' = f(t,u),\ u'(0) = u'(\pi) = 0. \tag{N}$$

We shall consider in Sections 2 and 3 this simpler boundary value problem so as to clearly bring out the ideas involved. On the other hand, there is no additional complication in studying (P) instead of (N). We list the corresponding results for (P) in Section 4.

Finally, in Section 5 and following the ideas developed in previous sections, we present the method of upper and lower solutions for the boundary value problem (P) when a_0, $a_1 > 0$ and b_0, $b_1 \geq 0$. In particular, we do so for the Dirichlet problem

$$-u' = f(t,u), u(0) = u(\pi) = 0. \tag{D}$$

2. GENERALIZED UPPER AND LOWER SOLUTIONS

Let us assume that $f: I \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, that is, $f(\cdot, u)$ is measurable for every $u \in \mathbb{R}$ and $f(t, \cdot)$ is continuous for *a.e.* $t \in I$. Moreover, we suppose that for every $R > 0$ there exists a function $h = h_R \in L^1(I)$ with

$$|f(t,u)| \leq h(t) \text{ for } a.e.\ t \in I \text{ and every } |u| \leq R. \tag{2.1}$$

Let $E = \{u \in W^{2,1}(I)\colon\ u'(0) = u'(\pi) = 0\}$ with the norm of $W^{2,1}(I)$ and $F = L^1(I)$ with the usual one. We shall denote by $\|\cdot\|_E$ and $\|\cdot\|$ the norms in E and F, respectively. By a solution of (N) we mean a function $u \in E$ satisfying the equation for *a.e.* $t \in I$.

Now, suppose that α, $\beta \in W^{2,1}(I)$ are such that $\alpha(t) \leq \beta(t)$, $t \in I$. Then, relative to (N) we shall consider the following modified problem

$$-u''(t) = g(t,u(t)) - u(t) + p(t,u(t)), u'(0) = u'(\pi) = 0, \tag{2.2}$$

where

$$g(t,u) = f(t, p(t,u)) \text{ and } \qquad p(t,u) = \begin{cases} \alpha(t) \text{ for } u < \alpha(t) \\ u \text{ for } \alpha(t) \le u \le \beta(t) \\ \beta(t) \text{ for } u > \beta(t). \end{cases}$$

We note that g is a Carathéodory function and that the Neumann problem (N) is equivalent to the integral equation

$$u(t) = u(0) - \int_0^t (t-s) f(s, u(s)) ds \tag{2.3}$$

with

$$\int_0^\pi f(s, u(s)) ds = 0. \tag{2.4}$$

We say that $\alpha \in W^{2,1}(I)$ is a lower solution for (N) if

$$-\alpha''(t) \le f(t, \alpha(t)) \text{ for } a.e.\ t \in I \tag{2.5}$$

and

$$\alpha'(0) \ge 0 \ge \alpha'(\pi). \tag{2.6}$$

Similarly, $\beta \in W^{2,1}(I)$ is an upper solution for (N) if

$$-\beta'(t) \ge f(t, \beta(t)) \text{ for } a.e.\ t \in I \tag{2.7}$$

and

$$\beta'(0) \le 0 \le \beta'(\pi). \tag{2.8}$$

We are now in a position to prove the following result which shows that the method of upper and lower solutions is still valid when f is a Carathéodory function.

Theorem 2.1: *Suppose that α, $\beta \in W^{2,1}(I)$ are lower and upper solutions for (N), respectively, such that $\alpha(t) \le \beta(t)$ for every $t \in I$. Then there exists at least one solution u of (N) such that $\alpha(t) \le u(t) \le \beta(t)$ for every $t \in I$.*

Proof: We first note that any solution u of (N) such that $\alpha \le u \le \beta$ is also a solution of (2.2). On the other hand, any solution u of (2.2) with $\alpha \le u \le \beta$ is a solution of (N). We shall show that any solution u of (2.2) is such that $\alpha \le u \le \beta$ on I and that (2.2) has at least one solution.

Now, let u be a solution of (2.2). We first show that $\alpha(t) \le u(t)$, for every $t \in I$. If $\alpha(t) > u(t)$ for every $t \in I$, then $-u''(t) = f(t, \alpha(t)) - u(t) + \alpha(t)$ for $a.e.$ $t \in I$. Thus we obtain the following contradiction

$$0 = \int_0^\pi u''(t)dt = \int_0^\pi [f(t,\alpha(t)) + \alpha(t) - u(t)]dt > -\int_0^\pi \alpha''(t)dt = \alpha'(0) - \alpha'(\pi) \geq 0.$$

Thus, there exists $t_1 \in I$ with $\alpha(t_1) \leq u(t_1)$. Now, suppose that there exists $t' \in I$ such that $\alpha(t') > u(t')$. Set $\varphi = \alpha - u$ and let $t_0 \in I$, $\varphi(t_0) = max\{\varphi(t): t \in I\}$. We first suppose that $t_0 \in (0,\pi)$ and $t_0 < t_1$ (the case $t_0 > t_1$ is similar). Then $\varphi'(t_0) = 0$ and there exists $t_2 \in (t_0, t_1)$ with $\varphi(t_2) = 0$ and $\varphi(t) > 0$ for every $t \in [t_0, t_2)$. On the other hand, we have that $\varphi''(t) \geq \varphi(t) > 0$ for *a.e.* $t \in [t_0, t_2)$. This implies that φ' is increasing on $[t_0, t_2)$ and, in consequence, $\varphi'(t) \geq 0$, $t \in [t_0, t_2)$ since $\varphi'(t_0) = 0$. Therefore, φ is increasing on $[t_0, t_2)$ which is not possible.

Now, if $t_0 = 0$, then $\varphi'(0) \leq 0$ and we get that $\varphi'(0) = \alpha'(0) \geq 0$ and $\varphi'(0) = 0$. As before, there exists $t_2 > 0$ such that $\varphi(t_2) = 0$ and $\varphi(t) > 0$ for every $t \in [0, t_2)$ and φ' is increasing on $[0, t_2)$ which contradicts that $\varphi(t_2) = 0$. The case $t_0 = \pi$ is analogous.

This shows that $\alpha(t) \leq u(t)$ for every $t \in I$ and by the same reasoning we obtain that $u(t) \leq \beta(t)$ for every $t \in I$.

We next prove that (2.2) has at least one solution. For $\lambda \in [0,1]$, consider the following Neumann boundary value problem

$$-u''(t) + u(t) = \lambda[g(t,u(t)) + p(t,u(t))],\ u'(0) = u'(\pi) = 0. \tag{2.9}$$

In order to apply the well-known theorem of Leray-Schauder, define the operators $L: E \to F$ and $N: F \to F$ by $Lu = -u'' + u$ and $Nu = g(\cdot, u(\cdot)) + p(\cdot, u(\cdot))$ respectively. Note that L is continuous, one-to-one, and onto. Thus, the Neumann problem (2.9) is equivalent to the abstract equations

$$Lu = \lambda Nu,\ \lambda \in [0,1],\ u \in E \tag{2.10}$$

or

$$u = \lambda HNu,\ \lambda \in [0,1],\ u \in F, \tag{2.11}$$

where $H = i \cdot L^{-1}: F \to F$ and $i: E \to F$ is the canonical injection. H is continuous and compact since $W^{2,1}(I)$ is compactly imbedded into $L^1(I)$.

Let $\gamma = min\{\alpha(t): t \in I\}$ and $\delta = max\{\beta(t): t \in I\}$. If u is a solution of (2.2), then $|u(t)| \leq R = max\{\gamma, \delta\}$ for every $t \in I$. Taking into account this, condition (2.1), and that $\alpha(t) \leq p(t,u(t)) \leq \beta(t)$ for every $t \in I$, we have

$$\| \lambda N u \| \leq \| h_R \| + 2\pi R = C.$$

In consequence, if u is a solution of (2.9) we have that $\| u \|_E \leq C \cdot \| H \|$, where C is a constant independent of $\lambda \in [0,1]$ and $u \in F$. Thus, we have proved that all the solutions of (2.11) are bounded independent of $\lambda \in [0,1]$ and we can conclude that (2.11) with $\lambda = 1$, that is (2.2), is solvable. This concludes the proof of the theorem.

3. THE MONOTONE METHOD

When f is a continuous function the following comparison result is fundamental in the development of the monotone iterative technique.

Lemma 3.1: *Let $\varphi \in C^2(I)$ and $\varphi'(0) \geq 0 \geq \varphi'(\pi)$. Suppose that there exists $M > 0$ with $\varphi''(t) \geq M\varphi(t)$ for a.e. $t \in I$. Then $\varphi(t) \leq 0$ for every $t \in I$.*

We now extend this result in order to cover the case when f is a Carathéodory function.

Lemma 3.2: *Let $\varphi \in W^{2,1}(I)$, $\varphi'(0) \geq 0 \geq \varphi'(\pi)$. If there exists $M \in L^1(I)$ with $M(t) > 0$ for a.e. $t \in I$ such that $\varphi''(t) \geq M(t)\varphi(t)$ for a.e. $t \in I$, then $\varphi(t) \leq 0$ for every $t \in I$.*

Proof: If $\varphi(t) > 0$ for every $t \in I$, then $\varphi''(t) > 0$ for *a.e.* $t \in I$. Thus, φ' is strictly increasing on I and $\varphi'(0) < \varphi'(\pi)$ which is a contradiction. Now, if there exists some $t \in I$ with $\varphi(t) > 0$, then choose $s \in I$ such that $\varphi(s) = max\{\varphi(t): t \in I\}$. If $s \in (0,\pi)$, then $\varphi'(s) = 0$ and there exists $t_1 \in [0,s)$ (or $t_1 \in (s,\pi]$ and the reasoning is analogous) with $\varphi(t_1) = 0$ and $\varphi(t) > 0$ for every $t \in (t_1, s)$. However, $\varphi''(t) > 0$ for *a.e.* $t \in (t_1, s)$ and φ' is increasing on (t_1, s). Hence, $\varphi'(t) \leq 0$ for $t \in (t_1, s)$ and φ is decreasing on (t_1, s) which is not possible. If $s = 0$ or $s = \pi$ the argument is similar. This completes the proof.

For $M \in F$ with $M(t) > 0$ for *a.e.* $t \in I$ and $\eta \in F$ we shall consider the following Neumann boundary value problem

$$-u''(t) = f(t, \eta(t)) - M(t)[u(t) - \eta(t)],\ u'(0) = 0 = u'(\pi) \tag{3.1}$$

or equivalently

$$-u''(t) + M(t)u(t) = f(t, \eta(t)) + M(t)\eta(t),\ u'(0) = 0 = u'(\pi). \tag{3.2}$$

The operator L (defined in the proof of Theorem 2.1) is continuous, one-to-one and

onto. Thus, by the open mapping theorem, its inverse L^{-1} is continuous. For $\sigma \in F$, let $L^{-1}\sigma = u$ be the unique solution of the linear problem $-u'' + Mu = \sigma$, $u'(0) = 0 = u'(\pi)$.

If α, β are lower and upper solutions for (N) respectively, let us introduce the following condition in order to develop the monotone method: There exists $M \in F$ with $M(t) \geq 0$ for *a.e.* $t \in I$ and we have that

$$f(t,u) - f(t,v) \geq -M(t)(u-v) \tag{3.3}$$

for *a.e.* $t \in I$ and for every u, $v \in \mathbb{R}$ such that $\alpha(t) \leq u \leq v \leq \beta(t)$.

For $\eta \in F$ with $\alpha \leq \eta \leq \beta$, that is, $\eta \in [\alpha,\beta] = \{u \in F: \alpha \leq u \leq \beta$ for *a.e.* $t \in I\}$, let us define the (nonlinear) operator $K:[\alpha,\beta] \to E$ by $K\eta = L^{-1}\sigma$ where $\sigma(t) = f(t,\eta(t)) + M(t)\eta(t)$, $t \in I$. The operator K is monotone and its properties are summarized in the following result.

Lemma 3.3: *Assume that* (3.3) *holds. Then the operator K has the following properties.*

$$\textit{If } \alpha \leq \eta \leq \beta \textit{ on } I, \textit{ then } \alpha \leq K\eta \leq \beta \textit{ on } I \tag{3.4}$$

and

$$\textit{if } \alpha \leq \eta_1 \leq \eta_2 \leq \beta \textit{ on } I, \textit{ then } \alpha \leq K\eta_1 \leq K\eta_2 \leq \beta \textit{ on } I. \tag{3.5}$$

Proof: Let $\alpha \leq \eta \leq \beta$ on I. We shall prove that $u \leq \beta$ on I. Indeed, let $\varphi = u - \beta$. Thus, for *a.e.* $t \in I$ we have that

$$\varphi''(t) = u''(t) \geq -f(t,\eta(t)) - M(t)\eta(t) + M(t)u(t) + f(t,\beta(t)) \geq$$
$$-M(t)[\beta(t) - \eta(t)] - M(t)\eta(t) + M(t)u(t) = M(t)\varphi(t).$$

By Lemma 3.1 we can conclude that $\varphi(t) \leq 0$ for every $t \in I$, that is, $u \leq \beta$ on I. The proof that $\alpha \leq u$ is similar.

To show that validity of (3.5), let $\varphi = K\eta_1 - K\eta_2$. Thus, $\varphi''(t) \geq M(t)\varphi(t)$ for *a.e.* $t \in I$ and, in consequence, we obtain that $K\eta_1 \leq K\eta_2$ on I.

Theorem 3.4: *Suppose that α and β are lower and upper solutions, respectively, of (N) such that $\alpha \leq \beta$ on I and* (3.3) *holds. Then, there exists monotone sequences $\{\alpha_n\}\uparrow$ and $\{\beta_n\}\downarrow$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$, $\alpha_n \leq \beta_m$ for every n, $m \in \mathbb{N}$ and $\lim_{n\to\infty} \alpha_n = r$, $\lim_{n\to\infty} \beta_n = \rho$ uniformly on I. Here, r and ρ are respectively the minimal and maximal solutions of (N) between α and β in the sense that if u is a solution with $\alpha \leq u \leq \beta$ on I, then $r \leq u \leq \rho$ on I.*

Proof: Let $\alpha_0 = \alpha$ and $\alpha_n = K\alpha_{n-1} (n = 1, 2, \ldots)$. We first prove that $\alpha_0 \leq \alpha_1$. Indeed, let $\varphi = \alpha_0 - \alpha_1$. Thus, $\varphi''(t) \geq -f(t, \alpha(t)) - M(t)\alpha_1(t) + f(t, \alpha(t)) + M(t)\alpha(t) = M(t)\varphi(t)$ for *a.e.* $t \in I$. This implies that $\varphi \leq 0$ on I in view of Lemma 3.2. Taking into account property (3.5) we see that $\alpha_1 = K\alpha_0 \leq K\alpha_1 = \alpha_2$ and, by induction, that $\alpha_n \leq \alpha_{n+1}$ for every $n \in \mathbf{N}$. Similarly, defining $\beta_0 = \beta$ and $\beta_n = K\beta_{n-1}$ we have that $\beta_{n+1} \leq \beta_n$, $n \in \mathbf{N}$. Combining properties (3.4) and (3.5) we see that $\alpha \leq \alpha_n \leq \beta_m \leq \beta$ for every n, $m \in \mathbf{N}$.

Therefore, the sequence $\{\alpha_n\}$ is uniformly bounded and increasing and it has a pointwise limit, say $r(t)$, $t \in I$. We now prove that r is a solution of (N). Choose $R > 0$ such that $|\alpha_n(t)| \leq R$ for every $n \in \mathbf{N}$, $t \in I$. The sequence $\{\alpha_n''\}$ is bounded in F since

$$-\alpha_n''(t) = -M(t)\alpha_n(t) + f(t, \alpha_{n-1}(t)) + M(t)\alpha_{n-1}(t) \tag{3.6}$$

and hence, $\|\alpha_n''\| \leq \|M\| \cdot \|\alpha_n\| + \|h_R\| + \|M\| \cdot \|\alpha_{n-1}\| \leq 2\|M\| \cdot R \cdot 2\pi + \|h_R\| = C$. Here, C is a constant independent of $n \in \mathbf{N}$.

On the other hand, $\alpha_n'(t) = \int_0^t \alpha_n''(s)ds$, which implies that the sequence $\{\alpha_n'\}$ is bounded in $L^\infty(I)$.

Therefore, $\{\alpha_n\}$ is bounded in E. This together with the montonicity of $\{\alpha_n\}$ implies that $\{\alpha_n\}$ is uniformly convergent to r. From (3.6) we obtain that

$$\alpha_n(t) = \alpha_n(0) = \int_0^t (t-s)[M(s)\alpha_n(s) - f(s, \alpha_{n-1}(s)) - M(s)\alpha_{n-1}(s)]ds$$

and

$$\int_0^\pi f(s, \alpha_{n-1}(s))ds = \int_0^\pi M(s)[\alpha_n(s) - \alpha_{n-1}(s)]ds.$$

Letting $n \to \infty$ and using the uniform convergence of $\{\alpha_n\}$, we see that r satisfies the integral equation (2.3) and (2.4), that is, r is a solution of (N).

Using the same integral representation for the solutions of (N) we get that $\{\beta_n\}$ converges uniformly to a solution ρ of (N) and it is obvious that $\alpha \leq r \leq \rho \leq \beta$.

Finally, if u is a solution of (N) with $\alpha \leq u \leq \beta$ on I, then $\alpha_1 \leq Ku = u \leq \beta_1$. By induction we get that $\alpha_n \leq u \leq \beta_n$ for every $n \in \mathbf{N}$ which implies that $r \leq u \leq \rho$ and concludes the proof of the theorem.

4. NONLINEAR SECOND ORDER BOUNDARY VALUE PROBLEMS

A function $v \in W^{2,1}(I)$ is said to be a lower solution of (P) if

$$-v''(t) \leq f(t, v(t)) \text{ for } a.e.\ t \in I \tag{4.1}$$

and

$$Bv(0) \leq c_0,\ Bv(\pi) \leq c_1, \tag{4.2}$$

and an upper solution of (P) if the reversed inequalities hold in (4.1) and (4.2).

If we know the existence of upper and lower solutions for (P), then we can guarantee the existence of a solution for (P).

Theorem 4.1: *Assume that α and β are lower and upper solutions of (P) respectively, such that $\alpha(t) \leq \beta(t)$ for every $t \in I$. Then there exists at least one solution u for the problem (P) such that $\alpha(t) \leq u(t) \leq \beta(t)$, $t \in I$.*

In order to develop the monotone method, we need the following result which is analogous to Lemma 3.2.

Lemma 4.2: *Let $\varphi \in W^{2,1}(I)$ be such that $B\varphi(0) \leq 0$ and $B\varphi(\pi) \leq 0$. Assume that there exists $M \in L^1(I)$ such that $M(t) > 0$ for a.e. $t \in I$, and $\varphi''(t) \geq M(t)\varphi(t)$ for a.e. $t \in I$. Then $\varphi(t) \leq 0$ for every $t \in I$.*

Proof: If $\varphi(t) > 0$ for every $t \in I$, then $\varphi''(t) > 0$ for *a.e.* $t \in I$ and φ' is strictly increasing on I and $\varphi'(0) < \varphi'(\pi)$. However, $B\varphi(0) \leq 0$ and $B\varphi(\pi) \leq 0$ implies that $\varphi'(0) \geq 0$ and $\varphi'(\pi) \leq 0$ which is a contradiction. Now, reasoning as in the proof of Lemma 3.2 we see that there is no $t \in I$ with $\varphi(t) > 0$.

This allows us to show the validity of the monotone iterative technique for the boundary value problem (P).

Theorem 4.3: *Let the assumptions of Theorem 4.1 hold. In addition, suppose that there exists $M \in L^1(I)$ and $M(t) > 0$ for a.e. $t \in I$, such that for a.e. $t \in I$ and every u, $v \in \mathbb{R}$ with $\alpha(t) \leq u \leq v \leq \beta(t)$ we have*

$$f(t,u) - f(t,v) \geq -M(t)(u-v). \tag{4.3}$$

Then there exist monotone sequences $\{\alpha_n\}\uparrow r$ and $\{\beta_n\}\downarrow \rho$ uniformly on I. Here, r and ρ are the minimal and maximal solutions respectively, of (P) between α and β.

Proof: For $\alpha \leq \eta \leq \beta$, we solve the boundary value problem

$$-u''(t) + M(t)u(t) = f(t, \eta(t)) + M(t)\eta(t),\ Bu(0) = c_0,\ Bu(\pi) = c_1$$

which has a unique solution $u = K\eta$.

The operator K has the properties (3.4) and (3.5) and then one can generate the monotone iterates.

5. DIRICHLET PROBLEM

We say that $\alpha \in W^{2,1}(I)$ is a lower solution of (D) if $-\alpha''(t) \leq f(t, \alpha(t))$ for *a.e.* $t \in I$, $\alpha(0) \leq 0$, and $\alpha(\pi) \leq 0$. Similarly, β is an upper solution if the reversed inequalities hold.

Now, using the following result it is easy to show that the monotone method for the Dirichlet problem (D) is also valid.

Lemma 5.1: *Let $\varphi \in W^{2,1}(I)$ and suppose that there exists $M \in L^1(I)$, $M(t) > 0$ for a.e. $t \in I$, such that $\varphi''(t) \geq M(t)\varphi(t)$ for a.e. $t \in I$. If $\varphi(0) \leq 0$ and $\varphi(\pi) \leq 0$, then $\varphi(t) \leq 0$ for every $t \in I$.*

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