NOTE ON THE INEQUALITIES OF J. KAZDAN¹

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ABSTRACT

In this note, we prove the Kazdan's inequalities without using what is called the Heisenberg uncertainty principle. Instead we prove it using Garofalo-Lin inequality among other things.

Key words: Heisenberg uncertainty principle, unique continuation theorem, Garofalo-Lin inequality, Schwarz inequality, Poincare inequality.

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1. INTRODUCTION

In [4], J. Kazdan has shown strong unique continuation theorem (Theorem 1.8 of [4]) whose proof is mainly based on his main lemma (Lemma 2.4 of [4]). Several analytic as well as geometric inequalities were used to prove the main lemma. Among them are the following inequalities:

There exist constants C_1, C_2, C_3, C_4, C_5 and r_0 such that for all $r \in (0, r_0)$

$$|I_j(r)| \leq C_j f(r)(H(r) + D(r)) \quad (j = 1, 2) \quad (1)_j$$

$$\frac{1}{r^{n-2}} \int_{\partial B_r} |\nabla u|^2 dS \leq rB(r) + C_3 H(r) + D(r) \quad (2)$$

$$|I_3(r)| \leq C_4 f(r)(H(r) + D(r) + \sqrt{rH(r)B(r)}) \quad (3)$$

$$|I_4(r)| \leq C_5 f(r)(H(r) + D(r) + rB(r)). \quad (4)$$

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Here $f(r), I_1(r), I_2, I_3(r), H(r), D(r), B(r)$ are defined as follows: let f be a smooth increasing function with $f(0) = 0$ satisfying $\int_0^{r_0} \frac{f(r)}{r} dr < \infty$ and let u satisfy for $n \geq 3$ the differential inequality with a and b constants:

$$|\Delta u(x)| \leq \frac{af(r)}{r^2} |u(x)| + \frac{bf(r)}{r} |\nabla u(x)| \quad (5)$$

$$I_1(r) = \frac{1}{r^{n-2}} \int_{B_r} u \Delta u dV \quad (6)$$

$$I_2(r) = \frac{2}{r^{n-2}} \int_{B_r} \rho u_\rho \Delta u dV \quad (7)$$

$$I_3(r) = \frac{1}{r^{n-3}} \int_{\partial B_r} u \Delta u dS \quad (8)$$

$$H(r) = \frac{1}{r^{n-1}} \int_{\partial B_r} |u|^2 dS \quad (9)$$

$$D(r) = \frac{1}{r^{n-2}} \int_{B_r} |\nabla u|^2 dV \quad (10)$$

$$B(r) = \frac{2}{r^{n-2}} \int_{\partial B_r} u_\rho^2 dS. \quad (11)$$

In his proof of inequalities (1)-(2), Kazdan relies on what is called the Heisenberg uncertainty principle (see [2], [3], & [4]):

$$\int_{B_r} \frac{w^2}{\rho^2} dV \leq \frac{C}{r} \int_{\partial B_r} w^2 dS + \tilde{C} \int_{B_r} |\nabla w|^2 dV, \quad n \geq 3 \quad (12)$$

$$\int_{B_r} \frac{2|w| |\nabla w|}{\rho} dV \leq \frac{C'}{r} \int_{\partial B_r} w^2 dS + \tilde{C}' \int_{B_r} |\nabla w|^2 dV, \quad n \geq 3 \quad (13)$$

where C and $\tilde{C}$ are dimensional constants. Inequality (13) is an easy consequence of (12). Indeed a straightforward computation shows that $C' = \frac{C\lambda}{r}$, $\tilde{C}' = \lambda(\frac{1}{\lambda^2} + \tilde{C})$ for any $\lambda > 0$.

Since there is nothing to comment on the proofs of inequalities (3) and (4), we prove inequalities (1)_j and (2) without using the Heisenberg uncertainty principle (12)-(13). Instead we use the following lemma which is the Garofalo-Lin inequality (see 4.11 of [2]) applied to the operator L where

$$Lu = -\Delta u + b(x) \cdot \nabla u + V(x)u = 0. \quad (14)$$

Here $b(x)$ and V are majorized by with constants a and b :

$$|b(x)| \leq \frac{bf(r)}{r}, \quad |V(x)| \leq \frac{af(r)}{r^2}. \quad (15)$$

Lemma: *Let $u \in w_{loc}^{1,2}$ satisfy equation (14). Then there exists a small constant $r_0 \in (0, 1)$ depending on n, b, V and u such that for all $r \in (0, r_0)$*

$$r \int_{\partial B_r} u^2 dS \geq \int_{B_r} u^2 dV. \quad (16)$$

Proof: First observe that

$$\begin{aligned} & \int_{B_r} u(b(x) \cdot u)(r^2 - |x|^2) dV \\ & \leq \int_{B_r} |u| |b(x)| |\nabla u| (r^2 - |x|^2) dV \\ & \leq \|b\|_{L^\infty} \left(\int_{B_r} u^2 (r^2 - |x|^2) dV \right)^{1/2} \left(\int_{B_r} |\nabla u|^2 (r^2 - |x|^2) dV \right)^{1/2} \\ & \quad \text{(Schwarz inequality)} \\ & \leq \|b\|_{L^\infty} r_0^2 \left(\int_{B_r} u^2 dV \right)^{1/2} \left(\int_{B_r} |\nabla u|^2 dV \right)^{1/2} \\ & \leq C \|b\|_{L^\infty} r_0^2 \left(\int_{B_r} |\nabla u|^2 dV \right) \quad \text{(Poincare inequality)} \end{aligned}$$

where C is a dimensional constant. Consequently we obtain

$$\int_{B_r} u(b(x) \cdot \nabla u)(r^2 - |x|^2) dV \geq -C \|b\|_{L^\infty} r_0^2 \int_{B_r} |\nabla u|^2 dV. \quad (17)$$

Choose r_0 so small that

$$r_0^2 \leq 1/(C \|b\|_{L^\infty} \int_{B_r} |\nabla u|^2 dV \int_{B_r} u^2 dV). \quad (18)$$

Inequalities (17)-(18) then reveal that

$$\int_{B_r} u(b(x) \cdot \nabla u)(r^2 - |x|^2) dV \geq - \int_{B_r} u^2 dV. \quad (19)$$

Secondly we have

$$\int_{B_r} V u^2 (r^2 - |x|^2) dV \geq - \|V\|_{L^\infty} r_0^2 \int_{B_r} u^2 dV. \quad (20)$$

Choose r_0 such that

$$r_0^2 \leq (n-2) / \|V\|_{L^\infty}. \quad (21)$$

Inequalities (20)-(21) then show that

$$\int_{B_r} V u^2 (r^2 - |x|^2) dV \geq -(n-2) \int_{B_r} u^2 dV. \quad (22)$$

Finally integration by parts and equation (14) give us the following identity:

$$\int_{B_r} (|\nabla u|^2 + u(b(x) \cdot \nabla u + V u^2)(r^2 - |x|^2) dV = r \int_{\partial B_r} u^2 dS - n \int_{B_r} u^2 dV. \quad (23)$$

Equation (23) combined with inequalities (19) and (22) shows that

$$\begin{aligned} r \int_{\partial B_r} u^2 dS &\geq \int_{B_r} |\nabla u|^2 (r^2 - |x|^2) dV - \int_{B_r} \nabla u^2 dV - (n-2) \int_{B_r} u^2 dV \\ &\quad + n \int_{B_r} u^2 dV \\ &\geq - \int_{B_r} u^2 dV - (n-2) \int_{B_r} u^2 dV + n \int_{B_r} u^2 dV \\ &= \int_{B_r} u^2 dV \end{aligned}$$

for all $r \in (0, r_0)$ where r_0 is chosen to be the minimum of the right hand sides of inequalities (18) and (21). This completes the proof.

We give the proof of $(I)_1$ only as the proofs of $(I)_2$ and (2) are essentially the same.

Proof of $(I)_1$:

$$\begin{aligned} |I_1(r)| &\leq \frac{1}{r^{n-2}} \int_{B_r} |u| |\Delta u| dV \\ &\leq \frac{1}{r^{n-2}} \int_{B_r} |u| \left(\frac{af(r)}{r^2} |u| + \frac{bf(r)}{r} |\nabla u| \right) dV \end{aligned} \quad (\text{by (5)})$$

$$\begin{aligned}
 &\leq \frac{af(r)}{r^{n-1}} \int_{\partial B_r} u^2 dS + \frac{bf(r)}{r^{n-1}} \left(\int_{B_r} u^2 dV \right)^{1/2} \left(\int_{B_r} |\nabla u|^2 dV \right)^{1/2} && \text{(Lemma and Schwarz)} \\
 &\leq \frac{af(r)}{r^{n-1}} \int_{\partial B_r} u^2 dS + \frac{bf(r)}{r^{n-1}} \left(\frac{1}{2r} \int_{B_r} u^2 dV + \frac{r}{2} \int_{B_r} |\nabla u|^2 dV \right) \\
 &\leq af(r)H(r) + \frac{b}{2}f(r)H(r) + \frac{b}{2}f(r)D(r) && \text{(Lemma, (9) \& (10))} \\
 &\leq C_1 f(r)(H(r) + D(r))
 \end{aligned}$$

where $C_1 = a + b/2$. This complete the proof of $(I)_1$.

A simple computation shows inequalities $(I)_2$, (2), (3) and (4) are satisfied with $C_2 = a + 2b$, $C_3 = (n-2) + (n+2)\gamma + C_2 f(r)$, $C_4 = a + (3b/2)\sqrt{C_3}$, $C_5 = (3/2)C_4$, where γ satisfies $0(r) < (n+2)\gamma$.

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