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APPLICATION OF LAKSHMIKANTHAM'S MONOTONE-ITERATIVE TECHNIQUE TO THE SOLUTION OF THE INITIAL VALUE PROBLEM FOR IMPULSIVE INTEGRO-DIFFERENTIAL EQUATIONS[1]

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ABSTRACT

In the present paper, a technique of V. Lakshmikantham is applied to approximate finding of extremal quasisolutions of an initial value problem for a system of impulsive integro-differential equations of Volterra type.

Key words: Monotone-iterative technique, impulsive integro-differential equations.

AMS (MOS) subject classifications: 34A37.

1. INTRODUCTION

The monotone-iterative technique of V. Lakshmikantham is one of the most effective methods for finding approximate solutions of initial value and periodic problems for differential equations. This technique is a fruitful combination of the method of upper and lower solutions and a suitably chosen monotone method [1]-[8].

In the present paper, by means of this monotone-iterative technique, minimal and maximal quasisolutions of the initial value problem for a system of impulsive integro-differential equations of Volterra type are obtained.

2. STATEMENT OF THE PROBLEM, PRELIMINARY NOTES

Consider the initial value problem for the system of impulsive integro-differential equations

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$$\begin{aligned} \dot{x} &= f(t,x,Qx(t)) && \text{for } t \neq t_i, t \in [0,T] \\ \Delta x \mid_{t=t_i} &= I_i(x(t_i - 0)) && \qquad (1) \\ x(t) &= \varphi(t) && \text{for } t \in [-h, 0], \end{aligned}$$

where $x = (x_1, x_2, \ldots, x_n)$, $f:[0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $f = (f_1, f_2, \ldots, f_n)$, $Qx = (Q_1x, Q_2x, \ldots, Q_nx)$, $Q_jx(t) = \int_{t-h}^{t} k_j(t,s)x_j(s)ds$, $k_j:[0,T] \times [-h,T] \to [0,\infty)$, $\varphi:[-h,0] \to \mathbb{R}^n$, $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$, $h = const > 0$, $0 < t_1 < t_2 < \ldots < t_p < T$, $\Delta x \mid_{t=t_i} = x(t_i + 0) - x(t_i - 0)$, $I_i:\mathbb{R}^n \to \mathbb{R}^n$, $I_i = (I_{i1}, I_{i2}, \ldots, I_{in})$.

With any integer $j = 1, \ldots, n$, we associate two nonnegative integers p_j and q_j such that $p_j + q_j = n - 1$ and introduce the notation

$$(x_j, [x]_{p_j}, [y]_{q_j}) = \begin{cases} (x_1, x_2, \ldots, x_{p_j+1}, y_{p_j+2}, \ldots, y_n) & \text{for } p_j \geq j \\ (x_1, \ldots, x_{p_j}, y_{p_j+1}, \ldots, y_{j-1}, x_j, y_{j+1}, \ldots, y_n) & p_j < j. \end{cases}$$

With the notation introduced, the initial value problem (1) can be written down in the form

$$\dot{x}_j = f_j(t, x_j, [x]_{p_j}, [x]_{q_j}, Q_jx(t), [Qx(t)]_{p_j}, [Qx(t)]_{q_j}) \text{ for } t \neq t_i,\ t \in [0,T]$$

$$\Delta x_j \mid_{t=t_i} = I_{ij}(x_j(t_i), [x(t_i)]_{p_j}, [x(t_i)]_{q_j}),$$

$$x_j(t) = \varphi_j(t) \text{ for } t \in [-h, 0], j = 1, \ldots, n.$$

Let $x, y \in \mathbb{R}^n$, $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$. We shall say that $x \geq (\leq) y$ if for any $i = 1, \ldots, n$, the inequality $x_i \geq (\leq) y_i$ holds.

Consider the set $G([a,b], \mathbb{R}^n)$ of all functions $u:[a,b] \to \mathbb{R}^n$ which are piecewise continuous with points of discontinuity of the first kind at the points $t_i \in (a,b)$, $u(t_i) = u(t_i - 0)$ and the set $G^1([a,b], \mathbb{R}^n)$ of all functions $u \in G([a,b], \mathbb{R}^n)$ which are continuously differentiable for $t \neq t_i$, $t \in [a,b]$ and have continuous left derivatives at the points $t_i \in (a,b)$.

Definition 1: The couple of functions $v, w \in G([-h,T], \mathbb{R}^n)$, $v, w \in G^1([0,T], \mathbb{R}^n)$, $v = (v_1, v_2, \ldots, v_n)$, $w = (w_1, w_2, \ldots, w_n)$ is said to be a couple of lower and upper quasisolutions of the initial value problem (1) if the following inequalities hold.

$$\dot{v}_j \leq f_j(t, v_j, [v]_{p_j}, [w]_{q_j}, Q_jv, [Qv]_{p_j}, [Qw]_{q_j}) \text{ for } t \neq t_i, t \in [0,T] \qquad (2)$$

$$\dot{w}_j \geq f_j(t, w_j, [w]_{p_j}, [v]_{q_j}, Q_jw, [Qw]_{p_j}, [Qv]_{q_j})$$

$$\Delta v_j \mid_{t=t_i} \leq I_{ij}(v_j(t_i), [v(t_i)]_{p_j}, [w(t_i)]_{q_j}) \qquad (3)$$

$$\Delta w_j \mid_{t=t_i} \geq I_{ij}(w_j(t_i), [w(t_i)]_{p_j}, [v(t_i)]_{q_j})$$

$$v_j(t) \leq \varphi_j(t) \leq w_j(t) \text{ for } t \in [-h, 0],\ j = 1, \ldots, n. \tag{4}$$

Definition 2: In the case when (1) is an initial value problem for a scalar impulsive integro-differential equation, i.e. $n = 1$ and $p_1 = q_1 = 0$, the couple of upper and lower quasisolutions of (1) are said to be upper and lower solutions of the same problem.

Definition 3: The couple of functions $v, w \in G([-h, T], \mathbb{R}^n)$, $v, w \in G^1([0, T], \mathbb{R}^n)$ is said to be a couple of quasisolutions of the initial value problem (1) if (2), (3) and (4) hold only as equalities.

Definition 4: The couple of functions $v, w \in G([-h, T], \mathbb{R}^n)$, $v, w \in G^1([0, T], \mathbb{R}^n)$ is said to be a couple of minimal and maximal quasisolutions of the initial value problem (1) if they are a couple of quasisolutions of the same problem and for any couple of quasisolutions of (1) (u, z) the inequalities $v(t) \leq u(t) \leq w(t)$ and $v(t) \leq z(t) \leq w(t)$ hold for $t \in [-h, T]$.

Remark 1: Note that for the couple of minimal and maximal quasisolutions (v, w) of (1) the inequality $v(t) \leq w(t)$ holds for $t \in [-h, T]$, while for an arbitrary couple of quasisolutions (u, z) of (1) an analogous inequality may not be valid.

Remark 2: If for any $j = 1, \ldots, n$, the equalities $p_j = n - 1$ and $q_j = 0$ hold and the couple of functions (v, w) is a couple of quasisolutions of the initial value problem (1), then the functions $v(t)$ and $w(t)$ are two solutions of the same problem. If, in this case, problem (1) has a unique solution $u(t)$, then the couple of functions (u, u) is a couple of minimal and maximal quasisolutions of (1).

For any couple of functions $v, w \in G([-h, T], \mathbb{R}^n)$, $v, w \in G^1([0, T], \mathbb{R}^n)$ such that $v(t) \leq w(t)$ for $t \in [-h, T]$ define the set of functions

$$S(v, w) = \{u \in G([-h, T], \mathbb{R}^n),\ u \in G^1([0, T], \mathbb{R}^n):\ v(t) \leq u(t) \leq w(t) \text{ for } t \in [-h, T]\}.$$

3. MAIN RESULTS

Lemma 1: *Let the following conditions hold:*

1. *The function* $k \in C([0, T] \times [-h, T], [0, \infty))$.
2. *The function* $g \in G([-h, T], \mathbb{R})$, $g \in G^1([0, T], \mathbb{R}^n)$ *satisfies the inequalities*

$$\dot{g}(t) \leq -Mg(t) - N \int_{t-h}^{t} k(t, s) g(s) ds \text{ for } t \neq t_i, t \in [0, T] \tag{5}$$

$$\Delta g\mid_{t=t_i} \leq -L_i g(t_i) \tag{6}$$

$$g(0) \leq g(t) \leq 0 \text{ for } t \in [-h, 0], \tag{7}$$

where $M, N, L_i (i = 1,\ldots, p)$ are constants such that $M, N > 0$, $0 \leq L_i < 1$.

3. *The inequality*

$$(M + N\kappa_0 h)p\tau < (1 - L)^p \tag{8}$$

holds, where

$$\kappa_0 = max\{\kappa(t,s): t \in [0,T], s \in [-h,T]\},$$

$$\tau = max\{t_1, T - t_p, max[t_{i+1} - t_i: i = 1, 2,\ldots, p-1]\},$$

$$L = max\{L_i: i = 1, 2,\ldots, p\}.$$

Then $g(t) \leq 0$ for $t \in [-h, T]$.

Proof: Suppose that this is not true, i.e. that there exists a point $\xi \in [0,T]$ such that $g(\xi) > 0$. The following three cases are possible:

Case 1: Let $g(0) = 0$ and $g(t) \geq 0$, $g(t) \not\equiv 0$ for $t \in [0, b)$ where $b > 0$ is a sufficiently small number. From inequality (7), it follows that $g(t) \equiv 0$ for $t \in [h, 0]$. Then by assumption there exist points $\xi_1, \xi_2 \in [0,T]$, $\xi_1 < \xi_2$, such that $g(t) = 0$ for $t \in [-h, \xi_1]$ and $g(t) > 0$ for $t \in (\xi_1, \xi_2]$. From inequality (5), it follows that $\dot{g}(t) \leq 0$ for $t \in [\xi_1, \xi_2] \cap [\xi_1, \xi_1 + h]$, $t \neq t_i$, which together with inequality (6) shows that the function $g(t)$ is monotone nonincreasing in the interval $[\xi_1, \xi_2] \cap [\xi_1, \xi_1 + h]$, i.e. $g(t) \leq g(\xi_1) = 0$ for $t \in [\xi_1, \xi_2] \cap [\xi_1, \xi_1 + h]$. The last inequality contradicts the choice of points ξ_1 and ξ_2.

Case 2: Let $g(0) < 0$. By assumption and inequality (7) there exists a point $\eta \in (0,T]$, $\eta \neq t_i$ $(i = 1,\ldots, p)$, such that $g(t) \leq 0$ for $t \in [-h, \eta)$, $g(\eta) = 0$ and $g(t) > 0$ for $t \in (\eta, \eta + \epsilon)$ where $\epsilon > 0$ is a sufficiently small number. Introduce the notation $inf\{g(t): t \in [-h, \eta]\} = -\lambda$, $\lambda = const > 0$. Then there are two possibilities:

Case 2.1: Let a point $\rho \in [0, \eta]$ exist, $\rho \neq t_i$ $(i = 1,\ldots, p)$ such that $g(\rho) = -\lambda$. For the sake of definiteness, let $\rho \in (t_k, t_{k+1}]$ and $\eta \in (t_{k+m}, t_{k+m+1}]$, $m \geq 0$. Choose a point $\eta_1 \in (t_{k+m}, t_{k+m+1}]$, $\eta_1 < \eta$ such that $g(\eta_1) > 0$. By the mean value theorem, the following equations are valid.

$$g(\eta_1) - g(t_{k+m} + 0) = \dot{g}(\xi_m)(\eta_1 - t_{k+m})$$

$$g(t_{k+m}-0)-g(t_{k+m-1}+0)=\dot{g}(\xi_{m-1})(t_{k+m}-t_{k+m-1})$$
$$\cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots \tag{9}$$
$$g(t_{k+1}-0)-g(\rho)=\dot{g}(\xi_0)(t_{k+1}-\rho)$$

where $\xi_0 \in (\rho, t_{k+1}), \xi_m \in (t_{k+m}, \eta_1), \xi_i \in (t_{k+i}, t_{k+i+1}), i=1,\ldots,m-1$.

From (6) and (9) we obtain the inequalities

$$g(\eta_1)-(1-L_{k+m})g(t_{k+m}) \le \dot{g}(\xi_m)\tau,$$
$$g(t_{k+m})-(1-L_{k+m-1})g(t_{k+m-1}) \le \dot{g}(\xi_{m-1})\tau,$$
$$\cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots \tag{10}$$
$$g(t_{k+1})-g(\rho) \le \dot{g}(\xi_0)\tau.$$

From inequalities (10), by means of elementary transformations, we obtain the inequalities

$$g(\eta_1)-(1-L_{k+1})(1-L_{k+2})\ldots(1-L_{k+m})g(\rho)$$
$$\le [\dot{g}(\xi_m)-(1-L_{k+m})\dot{g}(\xi_{m-1})+\ldots+ \tag{11}$$
$$(1-L_{\kappa+m})(1-L_{\kappa+m-1})\ldots(1-L_{\kappa+1})\dot{g}(\xi_0)]\tau.$$

Inequalities (6) and (11) and the choice of the points ρ and η_1 imply the inequality

$$(1-L)^m\lambda < [1+(1-L_{\kappa+m})+\ldots+(1-L_{\kappa+m})(1-L_{\kappa+m-1})\ldots(1-L_{\kappa+1})](M+N\kappa_0 h)\tau\lambda$$

or

$$1 < \frac{(M+N\kappa_0 h)}{(1-L)^p} p\tau. \tag{12}$$

Inequality (12) contradicts inequality (8).

Case 2.2: Let a point $t_\kappa \in [0,\eta)$ exist such that $g(t_\kappa+0) < g(t)$ for $t \in [0,\eta)$, i.e. $g(t_\kappa+0) = -\lambda$. By arguments analogous to those in Case 2.1, where $\rho = t_\kappa + 0$, we again obtain a contradiction.

Case 3: Let $g(0)=0$ and $g(t) \le 0$, $g(t) \not\equiv 0$ for $t \in (0,b]$ where $b>0$ is a sufficiently small number. By arguments analogous to those in Case 2 we obtain a contradiction.

This completes the proof of Lemma 1.

Theorem 1: *Let the following conditions hold:*

1. *The couple of functions* $v, w \in G([-h,T],\mathbb{R}^n)$, $v, w \in G^1([0,T],\mathbb{R}^n)$ *is a couple of lower and upper quasisolutions of the initial value problem* (1) *and satisfies the inequalities* $v(t) \leq w(t)$ *for* $t \in [-h,T]$ *and* $v(0) - \varphi(0) \leq v(t) - \varphi(t)$, $w(0) - \varphi(0) \geq w(t) - \varphi(t)$ *for* $t \in [h,0]$.
2. *The functions* $\kappa_j \in C([0,T] \times [-h,T], [0,\infty))$, $j = 1,\ldots,n$.
3. *The function* $f \in C([0,T] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $f = (f_1, f_2, \ldots, f_n)$, $f_j(t,x,y) = f_j(t, x_j, [x]_{p_j}, [x]_{q_j}, y_j, [y]_{p_j}, [y]_{q_j})$ *is monotone nondecreasing with respect to* $[x]_{p_j}$ *and* $[y]_{p_j}$ *and monotone nonincreasing with respect to* $[x]_{q_j}$ *and* $[y]_{q_j}$ *and for* $x, y \in S(v,w)$, $y(t) \leq x(t)$ *satisfies the inequalities*

$$f_j(t, x_j, [x]_{p_j}, [x]_{q_j}, Q_j x, [Qx]_{p_j}, [Qx]_{q_j})$$
$$- f_j(t, y_j, [x]_{p_j}, [x]_{q_j}, Q_j y, [Qx]_{p_j}, [Qx]_{q_j})$$
$$\geq -M_j(x_j - y_j) - N_j(Q_j x - Q_j y), \; j = 1,\ldots,n,$$

where M_j, N_j $(j = 1,\ldots,n)$ *are positive constants.*

4. *The functions* $I_i \in C(\mathbb{R}^n, \mathbb{R}^n)$, $I_i = (I_{i1}, I_{i2}, \ldots, I_{in})$, $(i = 1,\ldots,p)$, $I_{ij}(x) = I_{ij}(x_j, [x]_{p_j}, [x]_{q_j})$ *are monotone nondecreasing with respect to* $[x]_{p_j}$ *and monotone nonincreasing with respect to* $[x]_{q_j}$ *and for* $x, y \in S(v,w), y(t_i) \leq x(t_i)$ *satisfy the inequalities*

$$I_{ij}(x_j(t_i), [x(t_i)]_{p_j}, [x(t_i)]_{q_j}) - I_{ij}(y_j(t_i), [x(t_i)]_{p_j}, [x(t_i)]_{q_j})$$
$$\geq -L_{ij}(x_j(t_i) - y_j(t_i)), j = 1,\ldots,n, i = 1,\ldots,p,$$

where $L_{ij} (i = 1,\ldots,p, j = 1,\ldots,n)$ *are nonnegative constants,* $L_{ij} < 1$.

5. *The inequalities*

$$(M_j + N_j \kappa_{0j} h)\tau p \leq (1 - L_i)^p, j = 1,\ldots,n$$

hold, where

$$\kappa_{0j} = max(\kappa_j(t,s): t \in [0,T], s \in [-h,T]\},$$

$$\tau = max\{t_1, T - t_p, max[t_{i+1} - t_i: i = 1,2,\ldots,p-1]\},$$

$$L_i = max\{L_{ij}: i = 1,2,\ldots,p\}.$$

Then there exist two monotone sequences of functions $\{v^{(\kappa)}(t)\}_0^\infty$ *and* $\{w^{(\kappa)}(t)\}_0^\infty$, $v^{(0)}(t) \equiv v(t)$, $w^{(0)}(t) \equiv w(t)$ *which are uniformly convergent in the interval* $[-h,T]$ *and their limits* $\bar{v}(t) = \lim_{\kappa \to \infty} v^{(\kappa)}(t)$ *and* $\bar{w}(t) = \lim_{\kappa \to \infty} w^{(\kappa)}(t)$ *are a couple of minimal and maximal* *quasisolutions of the initial value problem* (1). *Moreover, if* $u(t)$ *is any solution of the initial value problem* (1) *such that* $u \in S(v,w)$, *then the inequalities* $\overline{v}(t) \le u(t) \le \overline{w}(t)$ *hold for* $t \in [-h, T]$.

Proof: Fix two functions $\eta, \mu \in S(v,w)$, $\eta(\eta_1, \eta_2, \ldots, \eta_n)$, $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$. Consider the initial value problems for the linear impulsive integro-differential equations

$$\dot{x}_j + M_j x_j(t) + N_j \int_{t-h}^{t} \kappa_j(t,s) x_j(s) ds = \sigma_j(t, \eta, \mu) \text{ for } t \neq t_i, t \in [0,T] \tag{13}$$

$$\Delta x_j \mid_{t = t_i} = -L_{ij} x_j(t_i) + \gamma_{ij}(\eta, \mu) \tag{14}$$

and

$$x_j(t) = \varphi_j(t) \text{ for } t \in [-h, 0] \tag{15}$$

where

$$\sigma_j(t, \eta, \mu) = f_j(t, \eta_j, [\eta(t)]_{p_j}, [\mu(t)]_{q_j}, Q_j \eta(t), [Q\eta(t)]_{p_j}, [q\mu(t)]_{q_j}) + M_j \eta_j(t) + N_j Q_j \eta(t),$$

$$\gamma_{ij}(\eta, \mu) = I_{ij}(\eta_j(t_i), [\eta(t_i)]_{p_j}, [\mu(t_i)]_{q_j}) + L_{ij} \eta_j(t_i), \ j = 1, \ldots, n.$$

The initial value problem (13)-(15) has a unique solution for any fixed couple of functions $\eta, \mu \in S(v,w)$.

Define the map $A: S(v,w) \times S(v,w) \to S(v,w)$ by the equality $A(\eta, \mu) = x$, where $x = (x_1, x_2, \ldots, x_n)$ and $x_j(t)$ is the unique solution of the initial value problem (13)-(15) for the couple of functions $\eta, \mu \in S(v,w)$.

We shall prove that $v \le A(v,w)$. Introduce the notations $x^{(1)} = A(v,w)$, $g = v - x^{(1)}$, $g = (g_1, g_2, \ldots, g_n)$. Then the following inequalities hold:

$$\dot{g}_j(t) = \dot{v} - x^{(1)} \le f_j(t, v_j, [v]_{p_j}, [w]_{q_j}, Q_j v, [Qv]_{p_j}, [Qw]_{q_j}) + M_j x_j^{(1)} + N_j Q_j x^{(1)} - \sigma_j(t, v, w)$$

$$= -M_j g_j(t) - N_j \int_{t-h}^{t} \kappa_j(t,s) g_j(s) ds \text{ for } t \neq t_i, t \in [0,T],$$

$$\Delta g_j \mid_{t = t_i} \le I_i(v_j(t_i), [v(t_i)]_{p_j}, [w(t_i)]_{q_j}) + L_{ij} x_j^{(1)}(t_i) - \gamma_{ij}(v,w) = -L_{ij} g_j(t_i), \tag{16}$$

$$g_j(0) \le g_j(t) \le 0 \text{ for } t \in [-h, 0], j = 1, \ldots, n.$$

By Lemma 1, the functions $g_j(t)$, $j = 1,\ldots,n$ are nonpositive, i.e. $v \leq A(v,w)$. In an analogous way it is proved that $w \geq A(v,w)$.

Let $\eta, \mu \in S(v,w)$ be such that $\eta(t) \leq \mu(t)$ for $t \in [-h,T]$. Set $x^{(1)} = A(\eta,\mu)$, $x^{(2)} = A(\mu,\eta)$, $g = x^{(1)} - x^{(2)}, g = (g_1, t_2, \ldots, g_n)$. By Lemma 1 the functions $g_j(t)$, $j = 1,\ldots,n$, are nonpositive, i.e. $A(\eta,\mu) \leq A(\mu,\eta)$.

Define the sequences of functions $\{v^{(\kappa)}(t)\}_0^\infty$ and $\{w^{(\kappa)}(t)\}_0^\infty$ by the equations

$$v^{(0)}(t) \equiv v(t), \quad w^{(0)}(t) \equiv w(t),$$
$$v^{(\kappa+1)}(t) = A(v^{(\kappa)}, w^{(\kappa)}), \quad w^{(\kappa+1)}(t) = A(w^{(\kappa)}, v^{(\kappa)}).$$

The functions $v^{(\kappa)}(t)$ and $w^{(\kappa)}(t)$ for $t \in [-h,T]$ and $\kappa \geq 0$ satisfy the inequalities

$$v^{(0)}(t) \leq v^{(1)}(t) \leq \ldots \leq v^{(\kappa)}(t) \leq \ldots \leq w^{(\kappa)}(t) \leq \ldots \leq w^{(1)}(t) \leq w^{(0)}(t). \tag{17}$$

Hence the sequences of functions $\{v^{(\kappa)}(t)\}_0^\infty$ and $\{w^{(\kappa)}(t)\}_0^\infty$ are uniformly convergent for $t \in [-h,T]$. Introduce the notation $\bar{v}(t) = \lim_{\kappa\to\infty} v^{(\kappa)}t$ and $\bar{w}(t) = \lim_{\kappa\to\infty} w^{(\kappa)}(t)$. We shall show that the couple of functions $(\bar{v}, \bar{w})$ is a couple of minimal and maximal quasisolutions of the initial value problem (1). From the definitions of the functions $v^{(\kappa)}(t)$ and $w^{(\kappa)}(t)$, it follows that these functions satisfy the initial value problem

$$\dot{v}_j^{(\kappa+1)} + M_j v_j^{(\kappa+1)} + N_j Q_j v^{(\kappa+1)} = \sigma_j(t, v^{(\kappa)}, w^{(\kappa)}) \text{ for } t \neq t_i,\ t \in [0,T]$$
$$\dot{w}_j^{(\kappa+1)} + M_j w_j^{(\kappa+1)} + N_j Q_j w^{(\kappa+1)} = \sigma_j(t, w^{(\kappa)}, v^{(\kappa)}), \tag{18}$$
$$\Delta v_j^{(\kappa+1)} |_{t=t_i} = -L_{ij} v_j^{(\kappa+1)}(t_i) + \gamma_{ij}(v^{(\kappa)}, w^{(\kappa)})$$
$$\Delta w_j^{(\kappa+1)} |_{t=t_i} = -L_{ij} w_j^{(\kappa+1)}(t_i) + \gamma_{ij}(w^{(\kappa)}, v^{(\kappa)}), \tag{19}$$
$$v_j^{(\kappa+1)}(t) = w_j^{(\kappa+1)}(t) = \varphi_j(t) \text{ for } t \in [-h,0],\ j = 1,\ldots,n. \tag{20}$$

We pass to the limit in equations (18)-(20) and obtain that the functions $\bar{v}(t)$ and $\bar{w}(t)$ are a couple of quasisolutions of the initial value problem (1). From inequalities (17) it follows that the inequality $\bar{v}(t) \leq \bar{w}(t)$ holds for $t \in [-h,T]$.

Let $\zeta, z \in S(v,w)$ be a couple of quasisolutions of problem (1). From inequalities (17) it follows that there exists an integer $\kappa \geq 1$ such that $v^{(\kappa-1)}(t) \leq \zeta(t) \leq w^{(\kappa-1)}(t)$ and $v^{(\kappa-1)}(t) \leq z(t) \leq w^{(\kappa-1)}(t)$ for $t \in [-h,T]$. Introduce the notation $g(t) = v^{(\kappa)}(t) - \zeta(t)$, $g = (g_1, g_2, \ldots, g_n)$. By Lemma 1, the inequality $g_j(t) \leq 0$ holds for $t \in [-h,T]$, $j = 1,\ldots,n$, i.e. $v^{(\kappa)}(t) \leq \zeta(t)$.

In an analogous way, it is proved that the inequalities $\zeta(t) \leq w^{(\kappa)}(t)$ and $v^{(\kappa)}(t) \leq z(t) \leq w^{(\kappa)}(t)$ hold for $t \in [-h, T]$, which shows that the couple of functions $(\overline{v}, \overline{w})$ is a couple of minimal and maximal quasisolutions of the initial value problem (1).

Let $u(t)$ be a solution of (1) such that $u \in S(v, w)$. Consider the couple of functions (u, u) which is a couple of quasisolutions of problem (1). By what was proved above, the inequalities $\overline{v}(t) \leq u(t) \leq \overline{w}(t)$ hold for $t \in [-h, T]$.

This completes the proof of Theorem 1.

In the case when (1) is an initial value problem for a scalar impulsive integro-differential equation, the following theorem is valid.

Theorem 2: *Let the following conditions hold:*

(1) *The functions $v, w \in G([-h, T], \mathbb{R})$, $v, w \in G^1([0, T], \mathbb{R})$ are a couple of lower and upper solutions of the initial value problem* (1) *and satisfy the inequalities $v(t) \leq w(t)$ for $t \in [-h, T]$ and $v(0) - \varphi(0) \leq v(t) - \varphi(t), w(0) - \varphi(0) \geq w(t) - \varphi(t)$ for $t \in [-h, 0]$.*

(2) *The function $\kappa(t, s) \in C([0, T] \times [-h, T], [0, \infty))$.*

(3) *The function $f \in C([0, T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ satisfies for $x, y \in S(v, w)$, $y(t) \leq x(t)$ the inequality*

$$f\left(t, x(t), \int_{t-h}^{t} \kappa(t,s)x(s)ds\right) - f\left(t, y(t), \int_{t-h}^{t} \kappa(t,s)y(s)ds\right)$$
$$\geq -M(x(t) - y(t)) - N \int_{t-h}^{t} \kappa(t,s)(x(s) - y(s))ds,$$

where M and N are positive constants.

(4) *The function $I_i \in C(\mathbb{R}, \mathbb{R})$ $(i = 1, \ldots, p)$ satisfies for $x, y \in S(v, w), y(t_i) \leq x(t_i)$ the inequality $I_i(x(t_i)) - I_i(y(t_i)) \geq -L_i(x(t_i) - y(t_i)), i = 1, \ldots, p$ where L_i $(i = 1, \ldots, p)$ are nonnegative constants such that $L_i < 1$.*

(5) *The inequality*

$$(M + N\kappa_0 h)p\tau < (1 - L)^p$$

holds, where

$$\kappa_0 = max\{\kappa(t,s): t \in [0,T], s \in [-h, T]\},$$

$$\tau = max\{t_1, T - t_p, max[t_{i+1} - t_i: i = 1, 2, \ldots, p-1]\},$$

$$L = max\{L_i: i = 1, 2, \ldots, n\}.$$

Then there exist two sequences of functions $\{v^{(\kappa)}(t)\}_0^\infty$ and $\{w^{(\kappa)}(t)\}_0^\infty$ which are uniformly *convergent in the interval* $[-h, T]$ *and their limits* $\bar{v}(t) = \lim_{\kappa \to \infty} v^{(\kappa)}(t)$ *and* $\bar{w}(t) = \lim_{\kappa \to \infty} w^{(\kappa)}(t)$ *are a couple of minimal and maximal solutions of the initial value problem* (1).

The proof of Theorem 2 is analogous to the proof of Theorem 1.

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