Journal of Applied Mathematics and Stochastic Analysis 6, Number 1, Spring 1993, 49-54

UNIQUENESS CRITERION FOR SOLUTION OF ABSTRACT NONLOCAL CAUCHY PROBLEM[1]

L. BYSZEWSKI[2]

Florida Institute of Technology
Department of Applied Mathematics
150 West University Blvd.
Melbourne, Florida 32901-6988, U.S.A.

ABSTRACT

The aim of the paper is to prove an uniqueness criterion for a solution of an abstract nonlocal Cauchy problem. A dissipative operator in the nonlocal problem and an arbitrary functional in the nonlocal condition are considered. The paper is a continuation of papers [1]-[3] and generalizes some results from [4].

Key words: Abstract nonlinear nonlocal Cauchy problem, dissipative operator, arbitrary functional in nonlocal condition, uniqueness criterion for a solution.

AMS (MOS) subject classifications: 34A10, 34A34, 34A40, 34A60, 34A99, 34G20, 34G99, 49K24.

1. INTRODUCTION

In this paper, a theorem about an uniqueness criterion for a strong solution of an abstract nonlocal Cauchy problem in a Banach space is proved. A multivalued dissipative operator is considered in the nonlocal Cauchy problem. Moreover, an arbitrary functional is established in the nonlocal condition. This functional is more general than linear combinations of the values of the strong solution at some points and linear combinations of integrals which are functionals of the strong solution. To prove the theorem about an uniqueness criterion, a modification of a method used by Lakshmikantham and Leela (see [4], Chapter 3) is applied. The paper is a continuation of papers [1]-[3] and, analogously as those papers, can be applied in physics.

[1]Received: July, 1992. Revised: October, 1992.

[2]Permanent address: Cracow University of Technology, Institute of Mathematics, Warszawska 24, 31-155 Cracow, Poland.

2. PRELIMINARIES

Let X be a real Banach space with norm $\|\cdot\|$ and let 2^X denote the set of subsets of X. We shall call a mapping $A: X \to 2^X$ an operator in X. Let

$$D(A) := \{x \in X : Ax \neq \emptyset\}$$

and

$$R(A) := \bigcup_{x \in D(A)} Ax$$

be the domain and the range of A, respectively.

We write

$$[x, y] \in A \text{ if } y \in Ax$$

and we use $A^{-1}y$ in the sense

$$A^{-1}y := \{x \in X : y \in Ax\}.$$

An operator $A: X \to 2^X$ is said to be dissipative if

$$\| x_1 - x_2 \| \leq \| x_1 - \lambda y_1 - (x_2 - \lambda y_2) \|$$

for any $\lambda > 0$, $x_1, x_2 \in D(A)$, $y_1 \in Ax_1$ and $y_2 \in Ax_2$.

Let $x, y \in X$, and let

$$[x, y]_h := \frac{1}{h}(\| x + hy \| - \| x \|) \text{ for any } h \in \mathbb{R}$$

and

$$[x, y]_- := \lim_{h \to 0^-} [x, y]_h, \; [x, y]_+ := \lim_{h \to 0^+} [x, y]_h.$$

We will use the following two lemmas:

Lemma 1: (see [4], Chapter 3). An operator $A: X \to 2^X$ is dissipative if and only if

$$[x_1 - x_2, y_1 - y_2]_- \leq 0$$

for any $x_i \in D(A)$, $y_i \in Ax_i$ $(i = 1, 2)$.

Lemma 2: (see [4], Chapter 1). If a function $\xi: [a, b] \to X$ is such that the right-hand and left-hand derivatives $\xi'_\pm(t)$ exist for some $t \in (a, b)$, then

$$\frac{d}{dt}^{(\pm)} \| \xi(t) \| = [\xi(t), \xi'_\pm(t)]_\pm \tag{1}$$

for the above some $t \in (a, b)$, where the left-hand side of (1) is meant in the sense of the right-

hand and left-hand derivatives of $\| \xi(t) \|$, respectively.

For a given dissipative operator A, for a given positive constant T, for a given function

$$g: C([0,T],X) \to X \tag{2}$$

and for a given $x_0 \in X$ we shall consider the nonlocal Cauchy problem

$$u'(t) \in Au(t), \quad u(0) + g(u) = x_0, \quad t \in (0,T]. \tag{3}$$

A function $u \in C([0,T],X)$ is said to be a strong solution of problem (3) on $[0,T]$ if it satisfies the conditions:

(C_1) $u(t)$ is Lipschitz continuous for $t \in [0,T]$,

(C_2) $u(0) + g(u) = x_0$,

and

(C_3) the derivative $u'(t)$ exists almost everywhere for $t \in (0,T]$ and $u'(t) \in Au(t)$ almost everywhere for $t \in (0,T]$.

3. THEOREM ABOUT AN UNIQUENESS CRITERION

Now, we shall prove a theorem about an uniqueness criterion for the strong solution of the nonlocal Cauchy problem (3).

Theorem: *Assume that:*

(i) *X is a real Banach space with norm $\| \cdot \|$,*

(ii) *$A: X \to 2^X$ is a dissipative operator,*

(iii) *T is a positive constant and x_0 is a given element of X,*

(iv) *$g: C([0,T],X) \to X$ is a given function such that there exists a constant $0 < L < 1$ so that*

$$\| g(w_1) - g(w_2) \| \leq L \| w_1 - w_2 \|_{C([0,T],X)} \tag{4}$$

for all $w_1, w_2 \in C([0,T],X)$.

Then there exists, at most, one strong solution of the nonlocal Cauchy problem (3) *on* $[0,T]$.

Proof: Suppose that u and v are two strong solutions of problem (3) on $[0,T]$. Then the function $u(t) - v(t)$ is Lipschitz continuous for $t \in [0,T]$ and, hence, $\| u(t) - v(t) \|$ is also Lipschitz continuous for $t \in [0,T]$. Consequently, there exists a derivative $\frac{d}{dt} \| u(t) - v(t) \|$ for almost all $t \in (0,T]$. From this, from the assumption that the derivative

$\frac{d}{dt}[u(t)-v(t)]$ exists almost everywhere for $t \in (0,T]$ (see condition (C_3)) and from Lemma 2,

$$\frac{d}{dt}\| u(t)-v(t) \| = [u(t)-v(t), \frac{d}{dt}(u(t)-v(t))]_- \tag{5}$$

for almost all $t \in (0,T]$.

Since $u'(t) \in Au(t)$ and $v'(t) \in Av(t)$, and since A is dissipative then (5) implies, according to Lemma 1, that

$$\frac{d}{dt}\| u(t)-v(t) \| \leq 0 \tag{6}$$

for almost all $t \in (0,T]$.

By (6),

$$\| u(t)-v(t) \| \leq \| u(0)-v(0) \| \text{ for all } t \in [0,T]. \tag{7}$$

Since u and v are two strong solutions of problem (3) on $[0,T]$, then

$$u(0)-v(0) = g(v)-g(u). \tag{8}$$

From (8) and from assumption (iv),

$$\| u(0)-v(0) \| \leq L \| u-v \|_{C([0,T],X)}. \tag{9}$$

Inequalities (7) and (9) imply that

$$\| u(t)-v(t) \| \leq L \| u-v \|_{C([0,T],X)} \text{ for all } t \in [0,T].$$

Hence,

$$\| u-v \|_{C([0,T],X)} \leq L \| u-v \|_{C([0,T],X)}$$

and, therefore

$$(1-L) \| u-v \|_{C([0,T],X)} \leq 0. \tag{10}$$

By (10) and by assumption (iv),

$$\| u-v \|_{C([0,T],X)} = 0$$

and, consequently,

$$u(t) = v(t) \text{ for } t \in [0,T].$$

The proof of the theorem is complete.

4. EXAMPLES

We will give two examples of the function g from the nonlocal condition (C_2).

Example 1: Let $T_1, T_2, \ldots, T_n$ be real numbers satisfying the inequalities

$$0 < T_1 < T_2 < \ldots < T_n \leq T \tag{11}$$

and let $k_1, k_2, \ldots, k_n$ be real constants such that

$$0 < \sum_{i=1}^{n} |k_i| < 1. \tag{12}$$

A function g given by the formula

$$g(w) := \sum_{i=1}^{n} k_i w(T_i) \text{ for all } w \in C([0,T], X)$$

satisfies (2) and satisfies condition (4), where

$$L := \sum_{i=1}^{n} |k_i|. \tag{13}$$

Similarly, if $\{T_n\}_{n=1}^{\infty}$ is a real sequence such that

$$0 < T_1 < T_2 < \ldots < T_n < \ldots < T \tag{14}$$

and if $\{k_n\}_{n=1}^{\infty}$ is a real sequence such that

$$0 < \sum_{n=1}^{\infty} |k_n| < 1, \tag{15}$$

then a function g given by the formula

$$g(w) := \sum_{n=1}^{\infty} k_n w(T_n) \text{ for all } w \in C([0,T], X)$$

satisfies (2) and satisfies condition (4), where

$$L := \sum_{n=1}^{\infty} |k_n|. \tag{16}$$

Example 2: Let $T_1, T_2, \ldots, T_n$ be real numbers satisfying inequalities (11) and let $k_1, k_2, \ldots, k_n$ be real constants satisfying condition (12). A function g defined by the formula

$$g(w) := \sum_{i=1}^{n} \frac{k_i}{\epsilon_i} \int_{T_i - \epsilon_i}^{T_i} w(\tau) d\tau \text{ for all } w \in C([0,T], X),$$

where $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are given positive constants such that

$$0 < T_1 - \epsilon_1 \text{ and } T_{i-1} < T_i - \epsilon_i \quad (i = 2, 3, \ldots, n),$$

satisfies (2) and satisfies condition (4) together with constant L defined by (13).

Moreover, if $\{T_n\}_{n=1}^{\infty}$ and $\{k_n\}_{n=1}^{\infty}$ are real sequences satisfying conditions (14) and (15), respectively, and if $\{\epsilon_n\}_{n=1}^{\infty}$ is a sequence of positive numbers such that

$$0 < T_1 - \epsilon_1 \text{ and } T_{n-1} < T_n - \epsilon_n \quad (n = 2, 3, \ldots),$$

then a function g given by the formula

$$g(w) := \sum_{n=1}^{\infty} \frac{k_n}{\epsilon_n} \int_{T_n - \epsilon_n}^{T_n} w(\tau) d\tau \text{ for all } w \in C([0,T], X),$$

satisfies (2) and satisfies condition (4) together with constant L defined by (16).

REFERENCES

[1] Byszewski, L. and Lakshmikantham, V., Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space, *Applicable Analysis* **40**, (1990), 11-19.

[2] Byszewski, L., Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, *Journal of Mathematical Analysis and Applications* **162.2**, (1991), 494-505.

[3] Byszewski, L., Existence of approximate solution to abstract nonlocal Cauchy problem, *Journa of Applied Mathematics and Stochastic Analysis* **5.4**, (1992), 363-374.

[4] Lakshmikantham, V. and Leela, S., *Nonlinear Differential Equations in Abstract Spaces*, Pergamon Press, Oxford, New York, Toronto, Sydney, 1981.