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CONSTRUCTIVE METHOD FOR SOLVING A NONLINEAR SINGULARLY PERTURBED SYSTEM OF DIFFERENTIAL EQUATIONS IN CRITICAL CASE[1]

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ABSTRACT

A singularly perturbed system of differential equations with degenerate matrix at the derivative is considered. The existence and construction of the periodic solutions are investigated in critical case. For the analysis of these algorithms the apparatus of Lyapunov's finite majorizing equations is used. An implementation of this method is given in two examples.

Key words: Singular perturbations, Lyapunov's majorant, finite majorizing equations.

AMS (MOS) subject classifications: 34C25, 34D15.

I. INTRODUCTION

In this paper the following system of differential equations is considered:

$$\epsilon B\dot{x} = Ax + \mu X(x,t), \tag{1}$$

where x is a n-dimensional vector, A is a constant $(n \times n)$ matrix, B is a constant $(n \times n)$ matrix for which $detB = 0$ and $rankB = n - r$, $X(x,t)$ is a 2π periodic vector-function, continuous on t, nonlinear and differentiable with respect to x in a certain domain $\| x \| \leq R$ and ϵ, μ are small positive parameters.

Systems of type (1) arise in solving many applied problems. For example, they are wide spread in problems of chemical and biological kinetics, economy, and the theory of electrical chains [1]. The method of expansion of solutions of such systems into asymptotic series in small parameter, first employed by A.N. Tikhonov is well known [4, 5, 6]. Our approach to this differs in that we propose iterative processes converging in the general (Cauchy's) sense for constructing the solutions, and consider, in this sense, the condition for

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convergence [2]. The general method of investigations is based on a transition from the initial system of differential equations to the operator system, followed by the application of simple iterations for constructing the periodic solutions, and their analysis with the help of finite majorizing equations.

2. PRELIMINARIES

The method of Lyapunov's finite majorizing equations can be applied for the construction of the solutions of the following operator system:

$$y = LF(y,t,\mu), \tag{2}$$

where $F(y,t,\mu)$ is the vector-function of the vector variable $y = (y_1, \ldots, y_n)$, t and small positive parameter μ. It belongs to $C[t]$ and $C[\mu]$ and it is differentiable (or Lipschitzian) in y in the domain

$$G_{n+2} = G_n \times I_T \times I_{\mu_0} (G_n\colon \| y \| \leq R; I_T\colon 0 \leq t \leq T; I_{\mu_0}\colon 0 \leq \mu \leq \mu_0). \tag{3}$$

Operator L is linear and bounded and hence continuous, defined in the space $G[I_T \times I_{\mu_0}]$. Suppose also that:

$$F(0,t,0) = 0, \frac{\partial F(0,t,0)}{\partial y} = 0. \tag{4}$$

The algorithm for the construction of the solutions of an operator system (2) is the following:

[A1] Inequalities expressing the boundedness of the operator L are constructed, for example, in the form:

$$(\| L\varphi(t) \|) \leq \Lambda q,$$

$$(\| \varphi(t) \|) \leq q, \; t \in I_T, \tag{5}$$

where the symbol $(\| \cdot \|)$ means a vector with components $\| \cdot \|$, and Λ is a constant matrix, chosen as accurately as possible;

[A2] Lyapunov's majorant $\Phi(w,\mu)$ is constructed for the function $F(y,t,\mu)$ in the domain G_{n+2};

[A3] write the system of equations

$$w = \Lambda\Phi(w,\mu) \tag{6}$$

representing Lyapunov's finite majorizing equations, w is a vector for which

$$(\| y(t,\mu) \|) \leq w(\mu);$$

[A4] construct the required solutions of the system (2) in any domain of μ with the help of the following convergent successive approximations

$$y_k = LF(y_{k-1}, t, \mu),\ k = 1, 2, \ldots,$$
$$y_0 \equiv 0. \tag{7}$$

If Lyapunov's majorizing functional equations (6) are constructed, it is possible to estimate the domain of values of μ, for which the required solution exists and the iterations (7) constructing this solution converge. The following basic theorem is then valid [2]:

Theorem 1: *If the system (6) has for $\mu \in [0, \mu_*]$ a solution $w(\mu) \in C[0, \mu_*]$, which is positive for $\mu > 0$ and is such that $w(0) = 0$, $\| w(\mu_*) \| \leq R$, then the sequence $\{x_k(t, \mu)\}$, defined in accordance with (7), converges for $t \in I_T$, $\mu \in I_{\mu_*}$ to the solution $x(t, \mu)$ of (2). This solution is unique in the class of functions $C(I_T \times I_{\mu_*})$, which vanish for $\mu = 0$.*

3. NECESSARY CONDITION FOR THE EXISTENCE OF PERIODIC SOLUTIONS IN CRITICAL CASE

Let us assume that $detA = 0$. We consider this as the fundamental case and call it the critical case. The above method of Lyapunov's functional majorizing equations can be applied for the system (1), if we reduce this system to the operator system of type (2). The task is to construct a 2π-periodic solution $x(t, \epsilon, \mu)$ of (1), which is continuous on ϵ and μ.

We replace in the system (1)

$$x = Sz, \tag{8}$$

where S is an arbitrary invertable $(n \times n)$ matrix, which can be chosen such that $S^{-1}B\ S$ is either diagonal or Jordan matrix. Then the following system is obtained for z:

$$\epsilon\widehat{\Lambda}\dot{z}^{(1)} = M_1 z^{(1)} + M_2 z^{(2)} + \mu Z^{(1)}(z, t),$$
$$0 = M_3 z^{(1)} + M_4 z^{(2)} + \mu Z^{(2)}(z, t) \tag{9}$$

or

$$\epsilon\dot{z} = \widehat{M} z + \mu\widehat{Z}(z, t), \tag{10}$$

where

$$\widehat{M} = \begin{bmatrix} \widehat{\Lambda}^{-1}M_1 & \widehat{\Lambda}^{-1}M_2 \\ M_3 & M_4 \end{bmatrix}, M = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix},$$

$M_i (i = 1,2,3,4)$ are $n-r$, $r\times(n-r)$, $(n-r)\times r$, $r\times r$ dimensional matrices, respectively; $z = col(z^{(1)}, z^{(2)})$, $z^{(i)}(i = 1,2,)$ are $n-r$, r dimensional vectors, respectively; $\widehat{Z}(z,t) = col(\widehat{\Lambda}^{-1}Z^{(1)}, Z^{(2)})$, $Z^{(i)}$ $(i = 1,2)$ are $n-r$, r dimensional vectors respectively. From the assumption that $detA = 0 \Rightarrow detM = 0$, $det\widehat{M} = 0$. Let us assume that $rank\widehat{M} = n-s$, $s > r$. When $\epsilon = 0$ and $\mu = 0$ we obtain, from (10), the reduced system

$$0 = \widehat{M}z_0, \tag{11}$$

which has a solution $z_0 = \Gamma c$, Γ is a $(s\times n)$ matrix of eigenvectors of $\widehat{M}$, and $c = col(c_1, c_2, \ldots, c_s)$ is an arbitrary constant vector.

Let us replace in system (10)

$$z = z_0 + Py, \tag{12}$$

where P is an invertable $(n\times n)$ matrix. Then the following is obtained for y:

$$\epsilon\dot{y} = P^{-1}\widehat{M}Py + \mu Y(t,y,c), \tag{13}$$

where $Y(t,y,c) = P^{-1}\widehat{Z}(z_0 + Py, t)$. The matrix P can be chosen such that $P^{-1}\widehat{M}P$ is either diagonal or Jordan matrix. Then system (13) can be written in the form:

$$\begin{aligned} \epsilon\dot{y}^{(1)} &= \Lambda y^{(1)} + \mu Y^{(1)}(t,y,c), \\ \epsilon\dot{y}^{(2)} &= \mu Y^{(2)}(t,y,c), \\ 0 &= \mu Y^{(3)}(t,y,c), \end{aligned} \tag{14}$$

where $y^{(1)}$, $y^{(2)}$, $y^{(3)}$ are $(n-r)$, $(s-r)$, r dimensional vectors, respectively; $Y^{(1)}$, $Y^{(2)}$, $Y^{(3)}$ are $(n-s)$, $(s-r)$, r dimensional vector-functions, respectively; $P^{-1}\widehat{M}P = \begin{bmatrix} \Lambda \\ 0 \end{bmatrix}$.

The conditions for the existence of a 2π-periodic solution of system (14) may be written in the form [3]:

$$\begin{aligned} P_2(c) &= \int_0^{2\pi} Y^{(2)}(s, z_0(c))ds = 0, \\ P_3(c) &= \int_0^{2\pi} y_0^*(s,\mu,\epsilon)[-Y_0^{(3)} - T(y,c)]ds = 0, \end{aligned} \tag{15}$$

where $y_0^*(s,\mu,\epsilon)$ is a solution of the conjugate system to the linearized third equation of system

(11) $0 = Y_0^{(3)}(t) + N(t)y_0 + T(y_0, t)$, $N(t) = \frac{\partial Y^{(3)}(y,t,c)}{\partial z}/z = z_0(c)$, and $Y^{(3)} = Y^{(3)}(z_0(c), t)$.

For the existence of a 2π-periodic solution of system (14), continuous in ϵ and μ, turning for $\epsilon = 0$, $\mu = 0$ into one of the solutions of (11), it is necessary that system (15) have at least one solution in c.

4. SUFFICIENT CONDITIONS FOR THE EXISTENCE OF A PERIODIC SOLUTION AND CONSTRUCTION OF THE EQUIVALENT OPERATOR SYSTEM

Suppose that $c = c_0$ is the solution of system (15). Putting $c = c_0$ in (13) we obtain

$$\epsilon \dot{y} = P^{-1}\widehat{M}Py + \mu F(y,t), \tag{16}$$

where $F(y,t) = Y(t, z_0(c_0) + Py)$.

The required periodic solution $y(t,\mu,\epsilon)$ of system (16) may be expressed in the following form:

$$y(t,\mu,\epsilon) = \Gamma c + \overline{y}\,(t,\mu,\epsilon).$$

An unknown 2π-periodic vector-function $\overline{y}\,(t,\mu,\epsilon)$ and an unknown constant vector $c = c(\mu,\epsilon)$ satisfy the operator relations:

$$\overline{y}^0 = \mu \int_0^{2\pi} G(t,s,\epsilon)\widetilde{F}\,(y(s,\mu,\epsilon),s)ds,$$

$$\int_0^{2\pi} Y^{(2)}(s, y(s,\mu,\epsilon))ds = 0,$$

$$\int_0^{2\pi} y_0^*(s,\mu,\epsilon)[-Y_0^{(3)}(s) - T(y(s,\mu,\epsilon),s)]ds = 0, \tag{17}$$

where

$$\overline{y}\,(t,\mu,\epsilon) = col(\overline{y}_1, \overline{y}_2, \overline{y}_0),$$

$$\overline{y}^0 = col(\overline{y}_1, \overline{y}_2),$$

$\widetilde{F} = col(F^{(1)}, F^{(2)})$, $c = col(c_2, c_3)$, c_2, c_3 are $s-r$, r dimensional vectors, respectively; $\overline{y}\,(t,\mu,\epsilon)$ is a particular 2π-periodic solution of (16).

Theorem 2: *Suppose that system* (15) *has a simple solution* $c = c_0$. *Suppose that the function* $\widehat{Z}$ *in* (10) *belongs to* $C^1[z]$ *in a certain neighborhood of the solution* $z_0(c_0)$. *Then by* *using* (17) *we can construct an operator system of type* (2), *which is equivalent to system* (1) *in the set of* 2π*-periodic functions, continuous in* μ *and* ϵ, *and vanishing for* $\mu = 0$, $\epsilon = 0$.

Proof: We shall transform $F(y,t) = col(F^{(1)}, F^{(2)}, F^{(3)})$, where $F^{(i)}$ $(i = 1,2,3)$ are $n-s$, $s-r$, r dimensional vector-functions, respectively, using the fact that the function $\widehat{Z}(z,t)$ is continuous in z in a certain neighborhood of $z_0(c_0)$, in the following form:

$$F(y,t) = F_0(t) + Q(t)y + R(y,t), \tag{18}$$

where

$$F_0(t) = F(t, z_0(c_0)) \in C[t], F_0(t) = col(F_0^{(1)}, F_0^{(2)}, F_0^{(3)}),$$

matrix $Q(t) = col(Q_1, Q_2, Q_3)$, $Q(t) = \frac{\partial Y}{\partial z} \backslash z = z_0(c_0)$ and it belongs to $C[t]$, $R(y,t) \in C^1[y]$ and $C[t]$ in a certain domain, for example, $\| y \| \leq l$, $R(0,t) = 0$, $\frac{\partial R(0,t)}{\partial y} = 0$,

$$R(y,t) = col(R^{(1)}, R^{(2)}, R^{(3)}).$$

Then system (17) can be written as the following operator system:

$$\begin{aligned} \overline{y}^0 &= \mu L^{(1)}[\widetilde{F}_0(t) + \widetilde{Q}(t)y + \widetilde{R}(y,t)], \\ c &= L^{(2)}[Q_2(t)\overline{y} + R^{(2)}(y,t)] = L_0^{(2)}\overline{y} + L_1^{(2)}R^{(2)}(y,t) \\ c_3^T &= L^{(3)}[\overline{y}_0^*(F_0^{(3)}(t) + R^{(3)}(y,t))] = L_0^{(3)}\overline{y}_0^*F_0^{(3)}(t) + L_1^{(3)}\overline{y}_0^*R^{(3)}(y,t). \end{aligned} \tag{19}$$

Obviously, the operators $L^{(1)}$, $L^{(2)}$, $L^{(3)}$ are linear and bounded, and they are defined by the formulas:

$$L^{(1)}g(t) = \int_0^{2\pi} G(t,s,\epsilon)g(s)ds,$$

$$L^{(2)}g(t) = -B_2^{-1}\int_0^{2\pi} g(s)ds,$$

$$L^{(3)}g(t) = -K^{-1}\int_0^{2\pi} g(s)ds,$$

where

$$detB_2(c_0) = det\left(\frac{\partial P_2}{\partial c}\right)\backslash c$$

$$= c_0 = det\left(\int_0^{2\pi} Q_2(t)\Gamma dt\right) \neq 0,$$

$$detK(t) = det\left(\int_0^{2\pi} H^T(t) \quad - F_0^{(3)}(t) - R^{(3)}(y,t) \right) \neq 0,$$

y_0^* is a conjugate to the solution $y_0(t,\mu,\epsilon)$, which can be found in the form:

$$y_0^*(t,\mu,\epsilon) = H(t)c_3 + \overline{y}_0,$$

$H(t)$-is a $(r \times r)$ matrix of eigenvectors of $N(t)$, c_3 is r-dimensional constant vector,

$$\widetilde{F}_0 = col(F_0^{(1)}, F_0^{(2)}), \widetilde{Q} = col(Q_1, Q_2),$$

$$\widetilde{R}(y,t) = col(R^{(1)}, R^{(2)}).$$

Thus, we obtain the operator system, equivalent to the initial system in the set of 2π-periodic functions, which are continuous in μ and ϵ and vanish for $\mu = 0$, $\epsilon = 0$. The theorem is proved.

Theorem 2 gives us sufficient conditions for the existence of a periodic solution $y(t,\mu,\epsilon)$ of system (14) in intervals $[0,\mu_*]$ and $[0,\epsilon_*]$. This solution is continuous inϵ and μ, and it vanishes for $\epsilon = 0$ and $\mu = 0$.

5. MAJORIZING EQUATIONS AND THE MAIN THEOREM FOR THE EXISTENCE OF A PERIODIC SOLUTION

According to the expressions of the righthands of (19), the following majorizing equations are obtained:

$$\begin{aligned} u_2 &= \rho_0^{(2)} v + \rho_1^{(2)} \Phi^{(2)}(v), \\ u_3 &= \rho_0^{(3)} dm + \rho_1^{(3)} d\Phi^{(3)}(v), \\ \overline{v}^0 &= \mu\rho[\widetilde{a} + \widetilde{b}\, v + \widetilde{\Phi}(v)], \end{aligned} \tag{20}$$

$$v = eu + \overline{v},$$

where $\rho_0^{(2)}$, $\rho_0^{(3)}$, $\rho_1^{(2)}$, $\rho_1^{(2)}$, ρ come from the estimates for the operators $L_0^{(2)}$, $L_0^{(3)}$, $L_1^{(3)}$, L, $\Phi(v) = col(\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)})$ is the Lyapunov's majorant for $R(y,t)$, $d = |||\overline{y}_0^*|||$, $m = |||F_0^{(3)}|||$, $a = |||F_0|||$, $b = |||Q(t)|||$, $e = \|\Gamma\|$, and $u = col(u_2, u_3)$, v, $\overline{v}$ major correspondingly $c = col(c_2, c_3)$, $y(t,\mu,\epsilon)$, $\overline{y}(t,\mu,\epsilon)$. In accordance with Theorem 1, the following theorem is valid:

Theorem 3: *If system* (20) *has positive solutions* $u(\mu,\epsilon)$, $v(\mu,\epsilon)$, $\overline{v}(\mu,\epsilon)$ *for* $\mu \in [0,\mu_*]$, *such that* $u(0,0)=v(0,0)=\overline{v}(0,0)=0$ *and* $v(\mu_*,\epsilon_*)\leq l$ *then system* (14) *has a unique* 2π-*periodic solution* $y(t,\mu,\epsilon)$, *which is continuous in* ϵ *and* μ *and vanishes for* $\epsilon=0$, $\mu=0$. *Moreover the successive approximations*

$$c_k = L_0^{(2)}\overline{y}_k + L_1^{(2)}R^{(2)}(\overline{y}_k,t),$$

$$c_k^{(3)} = L_0^{(3)}\overline{y}_0^* F_0^{(3)}(t) + L_1^{(3)}\overline{y}_0^* R^{(3)}(\overline{y}_k,t),$$

$$\overline{y}_{k+1}^0 = \mu L^{(1)}[\widetilde{F}_0(t)+\widetilde{Q}(t))(\Gamma c_k+\overline{y}_k)+\widetilde{R}(y_k,t)],$$

$$y_k = \Gamma c_{k-1}+\overline{y}_k,\quad k=1,2,\ldots$$

converge to $c(\mu,\epsilon)$, $y(t,\mu,\epsilon)$, *respectively, and the following estimates are true:*

$$|||\overline{y}(t,\mu,\epsilon)||| \leq \overline{v}(\mu,\epsilon),\quad \|c(\mu,\epsilon)\| \leq u(\mu,\epsilon),$$

$$|||y(t,\mu,\epsilon)||| \leq v(\mu,\epsilon).$$

As the variables y,z and x are connected by formulas (8) and (12), therefore in the intervals $\epsilon\in[0,\epsilon_*]$, $\mu\in[0,\mu_*]$, system (1) has a unique 2π-periodic solution $x(t,\mu,\epsilon)$, continuous in ϵ and μ.

6. EXAMPLES AND IMPLEMENTATION

In the next two examples we will show the applications of the iterative processes, described in the previous sections from the point of view of their numerical realization. For simplicity of an algorithm for construction of the solutions of singularly perturbed systems, it is more convenient to use iterative methods in comparison with the well known methods of the presentation of the solutions in the series of small parameter. Moreover, the use of these methods reduces the influence of accumulation of computational errors. It is important for an iterative process to be convergent (in Cauchy sense), because only in this case can the highest degree of precision be ensured while constructing solutions.

As the first example, we consider a nonlinear oscillation of the gyroscope systems in vector form:

$$\mu A_0\frac{d^2x}{dt^2}+A_1\frac{dx}{dt}=\mu F_0(t,x,\frac{dx}{dt})+F_2(t), \tag{21}$$

where A_0, A_1 are $(n\times n)$ constant real matrices, x is a n-dimensional vector, $\mu=\frac{1}{H}$, H is a kinetic moment of the quickly revolving rotors in the gyroscope systems, $F_0(t,x,\frac{dx}{dt})=$

$cos2t + x\frac{dx}{dt}$, $F_2(t) = cos\ t$- are n-dimensional known vector-functions, analytic and periodic with respect to their arguments.

Let us assume that $detA_0 = 0$, $rankA_0 = n - r$. As the task is to find periodic solutions of (21), we will seek them in the form of trigonometrical Fourier series. This allows us to make a numerical construction of the periodic solutions with the aid if an IBM, as in the calculations we reach to multiplication, raising to the power and addition of the Fourier polynomials, and it is possible to compose the standard programs for these processes.

After substituting $x = y_1$, $\frac{dx}{dt} = y_2$ we reduce system (21) to the system of first order similar to system (1)

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \mu A_0 \dot{y}_2 &= -A_1 y_2 + \mu F_0(t, y_1, y_2) + F_2(t). \end{aligned} \tag{22}$$

As $detA_1 \neq 0$, we have a noncritical case. Then the first approximation will be the solution of the system

$$\begin{aligned} \dot{y}_1^{(1)} &= y_2^{(1)}, \\ \mu A_0 \dot{y}_2^{(1)} &= -A_1 y_2^{(1)} + \mu F_{00}(t) + F_2. \end{aligned} \tag{23}$$

Solution of (23) will be sought in the following form:

$$\begin{aligned} y_1^{(1)} &= \sum_{n=1}^{N} A_{n1}^{(1)} cos\ nt + B_{n1}^{(1)} sin\ nt, \\ y_2^{(1)} &= \sum_{n=1}^{N} A_{n2}^{(1)} cos\ nt + B_{n2}^{(1)} sin\ nt. \end{aligned} \tag{24}$$

After replacing (24) in (23) for $A_{n1}^{(1)}$, $A_{n2}^{(1)}$, $B_{n1}^{(1)}$, $B_{n2}^{(1)}$ it is obtained:

$$A_{n1}^{(1)} = -\frac{1}{n} B_{n2}^{(1)},\ B_{n1}^{(1)} = \frac{1}{n} A_{n2}^{(1)},$$

$$A_{n2}^{(1)} = \frac{M_n^{(1)} - n\mu A_0 N_n^{(1)}}{n^2 \mu^2 A_0^2 + A_1^2},$$

$$B_{n2}^{(1)} = \frac{A_1 N_n^{(1)} + n\mu A_0 M_n^{(1)}}{n^2 \mu^2 A_0^2 + A_1^2},$$

where $M_n^{(1)}$ and $N_n^{(1)}$ are coefficients in the Fourier polynomials of $\mu F_{00}(t) + F_2(t) = \mu cos\ 2t + cos\ t = \sum_{n=1}^{n} M_n^{(1)} cos\ nt + N_n^{(1)} sin\ nt$.

For the next approximations we obtain the following system:

$$\dot{y}_1^{(k)} = y_2^{(k)},$$
$$\mu A_0 \dot{y}_2^{(k)} = -A_1 y_2^{(k)} + \mu F_0(t, y_1^{(k-1)}, y_2^{(k-1)}),\ k = 2,3,\ldots \tag{25}$$

Solutions of (25) can be presented in the form:

$$y_1^{(k)} = \sum_{n=1}^{N} A_{n1}^{(k)} cos\ nt + B_{n1}^{(k)} sin\ nt,$$
$$y_2^{(k)} = \sum_{n=1}^{N} A_{n2}^{(k)} cos\ nt + B_{n2}^{(k)} sin\ nt. \tag{26}$$

For the coefficients $A_{n1}^{(k)}$, $B_{n1}^{(k)}$, $A_{n2}^{(k)}$, $B_{n2}^{(k)}$ it is obtained

$$A_{n1}^{(k)} = -\tfrac{1}{n} B_{n2}^{(k)},\ \ B_{n1}^{(k)} = \tfrac{1}{n} A_{n2}^{(k)},$$

$$A_{n2}^{(k)} = \frac{M_n^{(k)} - n\mu A_0 N_n^{(k)}}{n^2\mu^2 A_0^2 + A_1^2},$$

$$B_{n2}^{(k)} = \frac{A_1 N_n^{(k)} + n\mu A_0 M_n^{(k)}}{n^2\mu^2 A_0^2 + A_1^2},$$

where $M_n^{(k)}$ and $N_n^{(k)}$ are, respectively, coefficients in the Fourier series of $\mu F_0(t, y_1^{(k-1)}, y_2^{(k-1)}) = \mu y_1^{(k-1)} y_2^{(k-1)} = \sum_{n=1}^{N} M_n^{(k)} cos\ nt + N_n^{(k)} sin\ nt.$

After running our computer programs, we obtain the following interval of changing of $\mu \in [0, 0.0001]$, in which there exist a 2π-periodic solution of (21). The algorithm was tested using VAX. The codes were written in FORTRAN language. The iterations were stopped, when $\| y^{(k)} - y^{(k-1)} \| \leq \delta$, δ is a given small number ($\delta = 10^{-5}$).

As our second example, we consider the problem which arises as a mathematical model of some presses in the theory of combustion [1]

$$x'' = x^2 - t^2 \equiv h(x,t),\ -1 < t < 1,$$
$$x(-1,\epsilon) = x(1,\epsilon) = 1, \tag{27}$$

where ϵ is a small parameter equal to the coefficient of diffusion to the flow rate, t-is a coordinate characterizing a distance, and it is chosen such that $t = 0$ is a place where oxidizer and fuel are mixed. Functions $(x - t)$ and $(x + t)$ are masses of a fuel and oxidizer, respectively.

When $\epsilon = 0$, a reduced system is obtained

$$0 = x_0^2 - t^2. \tag{28}$$

Its solution is $x_0 = \begin{pmatrix} t \\ -t \end{pmatrix} c$.

Let us replace in system (27) $y_1 = x$, $y_2 = x'$. The following system is obtained:

$$y_1' = y_2,$$
$$\epsilon y_2' = y_1^2 - t^2 \equiv h(y_1, t). \tag{29}$$

In this example we have a critical case, as $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Therefore $det A = 0$.

After substituting $y = x_0 + z$, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ we obtain:

$$z_1' = z_2,$$
$$\epsilon z_2' = 2x_{01} z_1 + z_1^2 + \epsilon c \equiv g(t, z_1, c). \tag{30}$$

A condition for existence of a periodic solution of (30) is

$$P(c) = \int_{-1}^{1} g(t, x_{01}, c) dt = 2c^2 + \epsilon c = 0.$$

So, we obtain $c^* = 0$, $c_* = -\epsilon$. Let us substitute $c_* = -\epsilon$ in system (30). Then the first approximation is considered as a solution of the following system:

$$\frac{dz_1^{(1)}}{dt} = z_2,$$
$$\epsilon \frac{dz_2^{(1)}}{dt} = -\epsilon^2. \tag{31}$$

We seek its solution in the form of the Fourier series:

$$z_1^{(1)} = \sum_{n=1}^{N} A_{n1}^{(1)} cos\ nt + B_{n1}^{(1)} sin\ nt,$$
$$z_2^{(1)} = \sum_{n=1}^{N} A_{n2}^{(1)} cos\ nt + B_{n2}^{(1)} sin\ nt. \tag{32}$$

After substituting (32) in (31), for the coefficients $A_{n1}^{(1)}$, $A_{n2}^{(1)}$, $B_{n1}^{(1)}$, $B_{n2}^{(1)}$ it is obtained

$$A_{n1}^{(1)} = -\tfrac{1}{n} M_n^{(1)}, \quad B_{n1}^{(1)} = -\tfrac{1}{n} N_n^{(1)},$$
$$A_{n2}^{(1)} = -\tfrac{1}{n} N_n^{(1)}, \quad B_{n2}^{(1)} = \tfrac{1}{n} M_n^{(1)},$$

where $M_n^{(1)}$ and $N_n^{(1)}$ are coefficients in Fourier polynomials of $g_0(t, c) = -\epsilon = \sum_{n=1}^{N} M_n^{(1)} cos\ nt + N_n^{(1)} sin\ nt$.

For $c_1(\epsilon)$ we have an equation

$$c_1(\epsilon) = L^{(1)}[g(t, z_1^{(1)}, c^*)] = -B_0^{-1}\int_{-1}^{1} g(t, z_1^{(1)}, c^*)dt, \tag{33}$$

$$B_0 = \frac{dP}{dc}/c = c^* = -2\epsilon \neq 0,$$

$$c_1(\epsilon) = -\frac{1}{2\epsilon}[(\sum_{n=1}^{N} A_{n1}^{(1)})^2 + (\sum_{n=1}^{N} B_{n1}^{(1)})^2] +$$

$$+\frac{1}{4\epsilon}\sum_{n=1}^{N}[(B_{n1}^{(1)})^2 - (A_{n1}^{(1)})^2]\,\frac{\sin 2n}{n} + 2\sum_{n=1}^{N} B_{n1}^{(1)}\left(\frac{\cos n}{n} - \frac{\sin n}{n^2}\right).$$

The next approximations can be written in the following form:

$$\frac{dz_1^{(k)}}{dt} = z_2^{(k)},$$

$$\epsilon\frac{dz_2^{(k)}}{dt} = g(t, z_1^{(k)}, \epsilon). \tag{34}$$

Its solution is similar to (32)

$$z_1^{(k)} = \sum_{n=1}^{N} A_{n1}^{(k)}\cos nt + B_{n1}^{(k)}\sin nt,$$

$$z_2^{(k)} = \sum_{n=1}^{N} A_{n2}^{(k)}\cos nt + B_{n2}^{(k)}\sin nt. \tag{35}$$

For the coefficients it is obtained respectively

$$A_{n1}^{(k)} = -\frac{1}{n^2\epsilon}M_n^{(k)},\ B_{n1}^{(k)} = -\frac{1}{n^2\epsilon}N_n^{(k)},$$

$$A_{n2}^{(k)} = -\frac{1}{n\epsilon}N_n^{(k)},\ B_{n2}^{(k)} = \frac{1}{n\epsilon}M_n^{(k)},$$

where $M_n^{(k)}$ and $N_n^{(k)}$ are coefficients in the Fourier polynomials of $g(t, z_1^{(k-1)}, c^*) = (z_1^{(k-1)})^2 - 2\epsilon t z_1^{(k-1)} = \sum_{n=1}^{N} M_n^{(k)}\cos nt + N_n^{(k)}\sin nt.$

An iterative process for finding c_k is the following:

$$c_k = L^{(1)}[g(t, z_1^{(k)}, c^*)]. \tag{36}$$

By specifying the (increasing) sequence of values of ϵ while constructing the solution, we can define the domain of values of ϵ for which the process of simple iterations practically converges. Moreover, we can establish the number of iterations required for obtaining a solution with a given precision. For our example we obtain the following results: $\epsilon \in [0, 0.01]$, $N = 40$, number of iterations is $k = 20$, $\| z^{(k)} - z^{(k-1)} \| \leq \delta$, $(\delta = 10^{-4})$.

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