EXISTENCE OF SOLUTIONS FOR SECOND-ORDER EVOLUTION INCLUSIONS

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ABSTRACT

In this paper we examine second-order nonlinear evolution inclusions and prove two existence theorems; one with a convex-valued orientor field and the other with a nonconvex-valued field. An example of a hyperbolic partial differential inclusion is also presented.

Key words: Evolution Triple, Monotone Operator, Hemicontinuous Operator, Symmetric Operator, Fixed Point, Sobolev Space, Program, Average Turnpike Property, Separation Theorem.

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1. Introduction

In this paper we study the existence of solutions for second order nonlinear evolution inclusions. Our work here complements the existence results of [7], where we considered first order nonlinear evolution inclusions. We present two existence results. One in which the multi-valued term (orientor field) is convex valued and the other with a nonconvex valued orientor field. At the end of the paper, we work in detail an example of a hyperbolic partial differential inclusion, illustrating the applicability of our result.

2. Mathematical Preliminaries

Let $T = [0, r]$ and Y a separable Banach space. Throughout this paper we will be using the following notation: $P_{f(c)}(Y) = \{A \subseteq Y: \text{nonempty, closed (and convex)}\}$. A multifunction (set-valued function), $F: T \rightarrow P_f(Y)$ is said to be measurable if for all $x \in Y$, the $\mathbb{R}_+$ -valued function $t \rightarrow d(x, F(t)) = \inf\{\|x - y\| : y \in F(t)\}$ is measurable. By S_F^p ($1 \leq p \leq \infty$), we will denote the set of selectors of $F(\cdot)$ that belong to the Lebesgue-Bochner space $L^p(Y)$; i.e. $S_F^p = \{f \in L^p(Y) : f(t) \in F(t) \text{ a.e.}\}$. It is easy to check using Aumann's selection theorem (see for example Wagner [8], theorem 5.10), that S_F^p is nonempty if and only if the $\mathbb{R}_+$ -valued function $t \rightarrow \inf\{\|x\| : x \in F(t)\}$ belongs to L^p_+ .

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Let H be a separable Banach space and X a dense subspace of H , carrying the structure of a separable, reflexive Banach space, which embeds in H continuously. Identifying H with its dual (pivot space), we have $X \rightarrow H \rightarrow X^*$, with all embeddings being continuous and dense. Such a triple of spaces is known in the literature as “evolution triple” (or “Gelfand triple” or “spaces in normal position”). We will also assume that the above embeddings are compact, a condition that is very often satisfied in applications. By $\|\cdot\|$ (resp. $|\cdot|$, $\|\cdot\|_*$), we will denote the norm of X (resp. of H, X^*). Also by $\langle \cdot, \cdot \rangle$ we will denote the duality brackets for the pair (X, X^*) and by $(\cdot, \cdot)$ the inner product of H . The two are compatible in the sense that $\langle \cdot, \cdot \rangle|_{X \times H} = (\cdot, \cdot)$. To have a concrete example in mind let $Z \subseteq \mathbb{R}^N$ be a bounded domain, $X = W_0^{m,p}(Z)$, $H = L^2(Z)$ and $X^* = W_0^{m,p}(Z)^* = W^{-m,q}(Z)$, $2 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. From the well-known Sobolev’s embedding theorem we know that (X, H, X^*) is an evolution triple and furthermore all embeddings are compact. Let $W(T) = \{x \in L^2(X) : \dot{x} \in L^2(X^*)\}$. The derivative in this definition is taken in the sense of vector valued distributions. Equipped with the norm $\|x\|_{W(T)} = [\|x\|_{L^2(X)}^2 + \|\dot{x}\|_{L^2(X^*)}^2]^{1/2}$, $W(T)$ becomes a separable reflexive Banach space. Furthermore if X is a Hilbert space, then $W(T)$ is too, with inner product $(x, y)_{W(T)} = (x, y)_{L^2(X)} + (\dot{x}, \dot{y})_{L^2(X^*)}$, $x, y \in W(T)$. Note that the elements in $W(T)$ are up to a Lebesgue-null subset of T , equal to an X^* -valued absolutely continuous function, and, therefore the derivative $\dot{x}(\cdot)$, is also the strong derivative of the function $x: T \rightarrow X^*$. Also, it is well-known that $W(T)$ embeds continuously into $C(T, H)$. Thus, every equivalence class in $W(T)$, has a unique representative in $C(T, H)$. Furthermore, since we have assumed that $X \rightarrow H$ compactly, we have that $W(T) \rightarrow L^2(H)$ compactly. Recently, Nagy [3] proved that if X is a Hilbert space too, then $W(T) \rightarrow C(T, H)$ compactly. For further details on evolution triples and the abstract Sobolev space $W(T)$ we refer to the book of Zeidler [9] and, in particular, chapter 23.

Let Z and V be Hausdorff topological spaces. A multifunction $G: Z \rightarrow 2^V \setminus \{\emptyset\}$ is said to be *upper semicontinuous (u.s.c.)* (resp. *lower semicontinuous (l.s.c.)*), if for every open set $U \subseteq V$, the set $G^+(U) = \{z \in Z : G(z) \subseteq U\}$ (resp. the set $G^-(U) = \{z \in Z : G(z) \cap U \neq \emptyset\}$) is open in Z . Other equivalent definitions and further properties of such multifunctions can be found in the book of Klein-Thompson [2].

3. Existence Theorems

Let $T = [0, r]$ and (X, H, X^*) be an evolution triple of spaces with all embeddings assumed to be compact. We will be considering the following second order nonlinear evolution inclusion:

$$\left\{ \begin{array}{l} \ddot{x}(t) + A(t, \dot{x}(t)) + Bx(t) \in F(t, x(t)) \text{ a.e.} \\ x(0) = x_0 \in X, \dot{x}(0) = x_1 \in H. \end{array} \right\} \quad (*)$$

By a solution of $(*)$, we understand a function $x \in C(T, X)$ such that $\dot{x} \in W(T)$ and an $f \in S_{F(\cdot, x(\cdot))}^2$ such that $\ddot{x}(t) + A(t, \dot{x}(t)) + Bx(t) = f(t)$ a.e. with $x(0) = x_0$ and $\dot{x}(0) = x_1$. Recall (see Section 2), that $W(T) \rightarrow C(T, H)$ and so the initial condition $\dot{x}(0) = x_1 \in H$ makes sense.

First we prove an existence theorem for $(*)$, for the case where the multivalued perturbation term $F(t, x)$ is convex-valued. To this end, we will need the following hypotheses on the data of $(*)$.

- $H(A)$: $A: T \times X \rightarrow X^*$ is a map such that
- (1) $t \mapsto A(t, v)$ is measurable,
  - (2) $v \mapsto A(t, v)$ is monotone, hemicontinuous (i.e. for all $v, v' \in X$, $\langle A(t, v) - A(t, v'), v - v' \rangle \geq 0$ (monotonicity) and for all vectors $v, y, x \in X$, the map $\lambda \mapsto \langle A(t, v + \lambda y), x \rangle$ is continuous on $[0, 1]$ (demicontinuity)),
  - (3) $\langle A(t, v), v \rangle \geq c \|v\|^2$ a.e. with $c > 0$,
  - (4) $\|A(t, v)\|_* \leq a(t) + b \|v\|$ a.e. with $a(\cdot) \in L^2_+$, $b > 0$.
- $H(B)$: $B \in \mathcal{L}(X, X^*)$, $\langle Bx, y \rangle = \langle x, By \rangle$ for all $x, y \in X$ (i.e. B is symmetric) and $\langle Bx, x \rangle \geq c' \|x\|^2$ with $c' > 0$.
- $H(F)_1$: $F: T \times H \rightarrow P_{fc}(H)$ is a multifunction such that
- (1) $t \mapsto F(t, x)$ is measurable,
  - (2) $x \mapsto F(t, x)$ is u.s.c. from H into H_w ,
  - (3) $|F(t, x)| = \sup\{|v| : v \in F(t, x)\} \leq a_1(t) + b_1 |x|$ a.e. with $a_1(\cdot) \in L^2_+$, $b_1 > 0$.
- We will denote the solution set of (*) by $S(x_0, x_1) \subseteq C(T, X)$.

Theorem 3.1: *If hypotheses $H(A)$, $H(B)$, $H(F)_1$ hold and $x_0 \in X$, $x_1 \in H$, then $S(x_0, x_1)$ is a nonempty and compact subset of $C(T, X)$.*

Proof: First we will derive some *a priori* bounds for the solutions of (*). Let $x(\cdot) \in C(T, X)$ be such a solution. Then, by the definition, for some $f \in S^2_{F(\cdot, x(\cdot))}$, we have

$$\ddot{x}(t) + A(t, \dot{x}(t)) + B(x(t)) = f(t) \text{ a.e.}$$

it yields
$$\langle \ddot{x}(t), \dot{x}(t) \rangle + \langle A(t, \dot{x}(t)), \dot{x}(t) \rangle + \langle Bx(t), \dot{x}(t) \rangle = (f(t), \dot{x}(t)) \text{ a.e.} \quad (1)$$

Since $\dot{x} \in W(T)$, from proposition 23.23 (iv), p. 422 of Zeidler [9], we know that

$$\langle \ddot{x}(t), \dot{x}(t) \rangle = \frac{1}{2} \frac{d}{dt} |\dot{x}(t)|^2. \quad (2)$$

Also because of hypothesis $H(A)$ (3), we have that

$$\langle A(t, \dot{x}(t)), \dot{x}(t) \rangle \geq c \|\dot{x}(t)\|^2 \text{ a.e.} \quad (3)$$

Using the product rule and the symmetry hypothesis on B , we get

$$\begin{aligned} \frac{d}{dt} \langle Bx(t), x(t) \rangle &= \langle B\dot{x}(t), x(t) \rangle + \langle Bx(t), \dot{x}(t) \rangle \\ &= 2\langle Bx(t), \dot{x}(t) \rangle. \end{aligned} \quad (4)$$

Substituting (2), (3) and (4) into (1) above, we finally have

$$\frac{1}{2} \frac{d}{dt} |\dot{x}(t)|^2 + c \|\dot{x}(t)\|^2 + \frac{1}{2} \frac{d}{dt} \langle Bx(t), x(t) \rangle \leq (f(t), \dot{x}(t)) \text{ a.e.}$$

Integrating the above inequality, we get that

$$\frac{1}{2} |\dot{x}(t)|^2 - \frac{1}{2} |x_1|^2 + c \int_0^t \|\dot{x}(s)\|^2 ds + \frac{1}{2} \langle Bx(t), x(t) \rangle - \frac{1}{2} \langle Bx_0, x_0 \rangle \leq \int_0^t (f(s), \dot{x}(s)) ds$$

it yields

$$|\dot{x}(t)|^2 + 2c \int_0^t \|\dot{x}(s)\|^2 ds + c' \|x(t)\|^2 \leq M + 2 \int_0^t (f(s), \dot{x}(s)) ds \quad (5)$$

where $M = |x_1|^2 + \|B\|_{\mathcal{L}} \|x_0\|^2$.

Applying Cauchy's inequality with $\epsilon > 0$, we get

$$\begin{aligned} \int_0^t (f(s), \dot{x}(s)) ds &\leq \int_0^t |f(s)| \cdot |\dot{x}(s)| ds \\ &\leq \frac{\epsilon}{2} \int_0^t |f(s)|^2 ds + \frac{1}{2\epsilon} \int_0^t |\dot{x}(s)|^2 ds \\ &\leq \frac{\epsilon}{2} \int_0^t (2a_1(s)^2 + 2b_1^2 |x(s)|^2) ds + \frac{1}{2\epsilon} \int_0^t |\dot{x}(s)|^2 ds \\ &\leq \epsilon \int_0^t (a_1(s)^2 + b_1^2 |x(s)|^2) ds + \frac{1}{2\epsilon} \int_0^t \beta^2 \|\dot{x}(s)\|^2 ds \end{aligned}$$

where $\beta > 0$ is such that $|\cdot| \leq \beta \|\cdot\|$. It exists since by hypothesis $X \rightarrow H$ continuously. So, we have

$$\begin{aligned} |\dot{x}(t)|^2 + 2c \int_0^t \|\dot{x}(s)\|^2 ds + c' \|x(t)\|^2 \\ \leq M + \epsilon \|a_1\|_2^2 + \epsilon b_1^2 \int_0^t |x(s)|^2 ds + \frac{\beta^2}{2\epsilon} \int_0^t \|\dot{x}(s)\|^2 ds. \end{aligned}$$

Let $\frac{\beta^2}{2\epsilon} = 2c$ implies that $\epsilon = \frac{\beta^2}{4c}$. Then we have:

$$|\dot{x}(t)|^2 + \frac{c'}{\beta^2} |x(t)|^2 \leq M + \frac{\beta^2}{4c} \|a_1\|_2^2 + \frac{\beta^2}{4c} b_1^2 \int_0^t |x(s)|^2 ds. \quad (*)$$

From (*) by neglecting $|\dot{x}(t)|^2$ and using Gronwall's inequality, we get

$$|x(t)|^2 \leq \left(\frac{\beta^2}{c'} M + \frac{\beta^4}{4cc'} \|a_1\|_2^2 \right) \exp\left(\frac{\beta^2 b_1^2}{4cc'} t \right) = M_2^2, \quad t \in T. \quad (6)$$

Using (6) and neglecting $\frac{c'}{\beta^2} |x(t)|^2$ in (*), we obtain

$$\|\dot{x}(t)\|^2 \leq M + \frac{\beta^2}{4c} \|a_1\|_2^2 + \frac{\beta^2}{4c} b_1^2 M_2^2 r = M_1^2, \quad t \in T. \quad (7)$$

Coming back to (5) and using estimates (6) and (7) above, we get

$$\|\dot{x}\|_{L^2(X)} \leq \frac{1}{2c} (M + 2\|a_1\|_2^2 + M_2^2 r + M_1^2 r) = M_3^2. \quad (8)$$

Finally, from (5) and (8), we deduce that

$$\|x(t)\|^2 \leq \frac{1}{c} (M + 2\|a_1\|_2^2 + 2b_1^2 M_2^2 r + M_1^2 r) = M_4^2. \quad (9)$$

Finally, let $p \in L^2(X)$ and denote by $((\cdot, \cdot))_0$ the duality brackets for the pair $(L^2(X), L^2(X^*) = L^2(X)^*)$. Also let $\hat{A}: L^2(X) \rightarrow L^2(X^*)$ be the Nemitsky operator corresponding to the map $A(t, x)$; i.e. $(\hat{A}x)(t) = A(t, x(t))$. Then we have:

$$\begin{aligned} ((\ddot{x}, p))_0 &\leq |((\hat{A}(\dot{x}), p))_0| + |((Bx, p))_0| + |((f, p))_0| \\ &\leq [\|\hat{A}(\dot{x})\|_{L^2(X^*)} + \|Bx\|_{L^2(X^*)} + \|f\|_{L^2(X^*)}] \|p\|_{L^2(X)} \\ &\leq [\|a\|_2 + bM_3 + \|B\|_{\mathcal{L}} M_r r^{1/2} + \beta' \|a_1\|_2 + \beta' b_1 M_2 r^{1/2}] \|p\|_{L^2(X)}, \end{aligned}$$

where $\beta' > 0$ is such that $\|\cdot\|_* \leq \beta' \|\cdot\|$. It exists since $H \rightarrow X^*$ continuously. Since $p \in L^2(X)$ was arbitrary, we deduce that there exists $M_5 > 0$ such that for all $x \in S(x_0, x_1)$, we have

$$\|\ddot{x}\|_{L^2(X^*)} \leq M_5. \quad (10)$$

From (8) and (10) above, we deduce that the set

$$S'(x_0, x_0) = \{\dot{x} \in W(T) : x \in S(x_0, x_1)\}$$

is bounded, hence relatively weakly compact in $W(T)$.

Now introduce the following modification of the original orientor field $F(t, x)$:

$$\hat{F}(t, x) = \begin{cases} F(t, x) & \text{if } |x| \leq M_2 \\ F(t, \frac{M_2 x}{|x|}) & \text{if } |x| > M_2. \end{cases}$$

Observe that $\hat{F}(t, x) = F(t, p_{M_2}(x))$, where $p_{M_2}(\cdot)$ is the M_2 -radial retraction in H . Since $p_{M_2}(\cdot)$ is Lipschitz continuous, we have, using hypothesis $H(F)_1$, that $t \rightarrow \hat{F}(t, x)$ is measurable while $x \rightarrow \hat{F}(t, x)$ is u.s.c. from H into H_w . Furthermore, note that $|\hat{F}(t, x)| \leq a(t) + bM_2 = \phi(t)$ a.e., with $\phi(\cdot) \in L^2_+$. Let $K = \{h \in L^2(H) : |h(t)| \leq \phi(t) \text{ a.e.}\}$. This set, endowed with the relative weak $L^2(H)$ -topology, is compactly metrizable. In what follows, this will be the topology considered on K . Let $\gamma: K \rightarrow C(T, X)$ be the map which to each $h \in K$, assigns the unique solution of the initial value problem $\ddot{x}(t) + A(t, \dot{x}(t)) + Bx(t) = h(t)$, $x(0) = x_0$, $\dot{x}(0) = x_1$ (see Zeidler [9], theorem 33.A, p. 224). We claim that $\gamma(\cdot)$ is continuous. To this end, let $h_n \rightarrow h$ in K and let $x_n = \gamma(h_n)$. Recall that $\{\dot{x}_n\}_{n \geq 1} \subseteq W(T)$ is relatively weakly compact. Hence, by passing to a subsequence if necessary, we may assume that $\dot{x}_n \rightharpoonup y$ in $W(T)$. Let $x = \gamma(h)$. We need to show that $y = \dot{x}$. We have:

$$\begin{aligned}
& \langle \ddot{x}_n(t) - \ddot{x}(t), \dot{x}_n(t) - \dot{x}(t) \rangle + \langle A(t, \dot{x}_n(t)) - A(t, \dot{x}(t)), \dot{x}_n(t) - \dot{x}(t) \rangle \\
& + \langle Bx_n(t) - Bx(t), \dot{x}_n(t) - \dot{x}(t) \rangle \\
& = (h_n(t) - h(t), \dot{x}_n(t) - \dot{x}(t)) \text{ a.e.}
\end{aligned}$$

Exploiting the fact that $A(t, \cdot)$ is monotone and using the integration by parts formula for functions in $W(T)$ (see Zeidler [9], proposition 23.23, p. 422), we get

$$\frac{1}{2} \frac{d}{dt} | \dot{x}_n(t) - \dot{x}(t) |^2 + \langle B(x_n(t) - x(t)), \dot{x}_n(t) - \dot{x}(t) \rangle \leq (h_n(t) - h(t), \dot{x}_n(t) - \dot{x}(t)) \text{ a.e.}$$

But, as before, exploiting the symmetry of the operator B , we have

$$\langle B(x_n(t) - x(t)), \dot{x}_n(t) - \dot{x}(t) \rangle = \frac{1}{2} \frac{d}{dt} \langle B(x_n(t) - x(t)), x_n(t) - x(t) \rangle.$$

So we get:

$$\frac{1}{2} \frac{d}{dt} | \dot{x}_n(t) - \dot{x}(t) |^2 + \frac{1}{2} \frac{d}{dt} \langle B(x_n(t) - x(t)), x_n(t) - x(t) \rangle \leq (h_n(t) - h(t), \dot{x}_n(t) - \dot{x}(t)) \text{ a.e.}$$

Integrating and recalling that $x_n(0) = x(0) = x_0$, $\dot{x}_n(0) = \dot{x}(0) = x_1$, we have:

$$\frac{1}{2} | \dot{x}_n(t) - \dot{x}(t) |^2 + \frac{1}{2} \langle B(x_n(t) - x(t)), x_n(t) - x(t) \rangle \leq \int_0^t (h_n(s) - h(s), \dot{x}_n(s) - \dot{x}(s)) ds$$

which yields

$$\frac{c'}{2} \| x_n(t) - x(t) \|^2 \leq \int_0^t (h_n(s) - h(s), \dot{x}_n(s) - \dot{x}(s)) ds$$

which yields

$$\| x_n(t) - x(t) \|^2 \leq \frac{2}{c'} \int_0^t (h_n(s) - h(s), \dot{x}_n(s) - \dot{x}(s)) ds.$$

Note that $h_n \xrightarrow{w} h$ in $L^2(H)$ and $\dot{x}_n \xrightarrow{w} y$ in $W(T)$. Since $W(T) \rightarrow L^2(H)$ compactly, we have that $\dot{x}_n \xrightarrow{s} y$ in $L^2(H)$. Thus we have:

$$\begin{aligned}
& \int_0^t (h_n(s) - h(s), \dot{x}_n(s) - \dot{x}(s)) ds \\
& = \int_0^t (h_n(s) - h(s), \dot{x}_n(s) - y(s)) ds + \int_0^t (h_n(s) - h(s), y(s) - \dot{x}(s)) ds \rightarrow 0 \text{ as } n \rightarrow \infty.
\end{aligned}$$

So $x_n(t) \xrightarrow{s} x(t)$ in X yields $\dot{x} = y \in W(T)$. Now note that

$$\| x_n(t) - x(t) \|^2 \leq \frac{2}{c'} \| h_n - h \|_{L^2(H)} \| \dot{x}_n - \dot{x} \|_{L^2(H)}.$$

Since $h_n \xrightarrow{w} h$ in K , we have $\| h_n - h \|_{L^2(H)} \leq N$ for all $n \geq 1$ and some $N > 0$. Thus

$$\|x_n(t) - x(t)\|^2 \leq \frac{2}{c} N \|\dot{x}_n - \dot{x}\|_{L^2(H)} \rightarrow 0$$

which implies that $\gamma(\cdot)$ is indeed continuous as claimed.

Let $R: K \rightarrow 2^K$ be the multifunction defined by

$$R(h) = S_{\widehat{F}(\cdot, \gamma(h)(\cdot))}^2.$$

First we will show that $R(\cdot)$ has nonempty values. Let $s_n(\cdot)$ be simple functions such that $s_n(t) \xrightarrow{s} \gamma(h)(t)$ a.e. in H . Then because of hypothesis $H(F)_1(1)$, for each $n \geq 1$, $t \mapsto \widehat{F}(t, s_n(t))$ is measurable. Apply Aumann's selection theorem to get $f_n: T \rightarrow H$ measurable such that $f_n(t) \in \widehat{F}(t, s_n(t))$ a.e., $n \geq 1$. Note that $|f_n(t)| \leq \phi(t)$ a.e. with $\phi(\cdot) \in L^2_+$. Hence by passing to a subsequence if necessary, we may assume that $f_n \xrightarrow{w} f$ in $L^2(H)$. Then theorem 3.1 of [6], tells us that

$$\begin{aligned} f(t) &\in \overline{\text{conv}} \, w\text{-}\overline{\text{lim}} \{f_n(t)\}_{n \geq 1} \\ &\subseteq \overline{\text{conv}} \, w\text{-}\overline{\text{lim}} \widehat{F}(t, s_n(t)) \\ &\subseteq \widehat{F}(t, \gamma(t)(t)) \text{ a.e.} \end{aligned}$$

The last inclusion follows from the fact that $\widehat{F}(t, \cdot)$ is u.s.c. from H into H_w and since $s_n(t) \xrightarrow{s} \gamma(h)(t)$ a.e. in H . Therefore $f \in S_{\widehat{F}(\cdot, \gamma(h)(\cdot))}^2$ and so we have established that the values of the multifunction $R(\cdot)$ are nonempty. Also since $F(t, x)$ is $P_{fc}(H)$ -valued, it is clear that for every $h \in K$, $R(h) \in P_{fc}(K)$. Furthermore using theorem 4.2 of [6] and recalling that $\gamma(\cdot)$ is continuous on K into $C(T, X)$, we get that $R(\cdot)$ is u.s.c. Apply the Kakutani-KyFan fixed point theorem to get $h \in R(h)$. Then $x = \gamma(h)$ is a solution of $(*)$, with $F(t, x)$ replaced by $\widehat{F}(t, x)$. But as in the beginning of the proof, with the same *a priori* estimation, we can show that $|x(t)| \leq M_2$ for all $t \in T$ implies that $\widehat{F}(t, x(t)) = F(t, x(t))$ and this yields that $x(\cdot)$ solves $(*)$.

Finally to establish the compactness of $S(x_0, x_1)$ in $C(T, X)$, note that $S(x_0, x_1) \subseteq \gamma(K)$ and the latter is compact in $C(T, X)$ since $\gamma: K \rightarrow C(T, X)$ is continuous. So it suffices to show that $S(x_0, x_1)$ is closed in $C(T, X)$. So let $\{x_n\}_{n \geq 1} \subseteq S(x_0, x_1)$ and assume that $x_n \rightarrow x$ in $C(T, X)$. Then by definition $x_n = \gamma(f_n)$ with $f_n \in S_{F(\cdot, \gamma(f_n)(\cdot))}^2$. Note that because of hypothesis $H(F)_1(3)$ $|f_n(t)| \leq a_1(t) + b_1 \widehat{N}$, where $\widehat{N} = \sup \|x_n\|_{C(T, X)}$. So we may assume that $f_n \xrightarrow{w} f$ in $L^2(H)$ implies that $\gamma(f_n) \rightarrow \gamma(f)$ in $C(T, X)$ which yields $x = \gamma(f)$ and from theorem 3.1 of [6], we have that $f(t) \in \overline{\text{conv}} \, w\text{-}\overline{\text{lim}} \{f_n(t)\}_{n \geq 1} \subseteq \overline{\text{conv}} \, w\text{-}\overline{\text{lim}} F(t, x_n(t)) \subseteq F(t, x(t))$ a.e. which yields $x \in S(x_0, x_1)$. Q.E.D.

Now we consider the case where the multivalued perturbation term $F(t, x)$ is not necessarily convex-valued. We will need the following hypothesis on the orientor field $F(t, x)$.

$H(F_2)$: $F: T \times H \rightarrow P_f(H)$ is a multifunction such that

- (1) $(t, x) \mapsto F(t, x)$ is graph measurable; i.e. $Gr F = \{(t, x, y) \in T \times H \times H : y \in F(t, x)\} \in B(T) \times B(H)$, with $B(T)$ (resp. $B(H)$), being the Borel σ -field of T (resp. of H) (recall that measurability of $F(\cdot, \cdot)$ implies graph measurability).
- (2) $x \mapsto F(t, x)$ is l.s.c.
- (3) $|F(t, x)| = \sup\{|y| : y \in F(t, x)\} \leq a_1(t) + b_1 |x|$ a.e. with $a_1(\cdot) \in L^2_+$, $b_1 > 0$.

Theorem 3.2: *If hypotheses $H(A)$, $H(B)$, $H(F)_2$ hold and $x_0 \in X$, $x_1 \in H$, then $S(x_0, x_1) \neq \emptyset$.*

Proof: As in the proof of theorem 3.1, let $\widehat{F}(t, x) = F(t, p_{M_2}(x))$ (it is clear that the same *a priori* estimation is valid in the present situation). Then given that $p_{M_2}(\cdot)$ is Lipschitz continuous, we have that $(t, x) \rightarrow \widehat{F}(t, x)$ is graph measurable, $x \rightarrow \widehat{F}(t, x)$ is *l.s.c.* and furthermore note that $|\widehat{F}(t, x)| \leq a_1(t) + b_1 M_2 = \phi(t)$ *a.e.* with $\phi(\cdot) \in L^2_+$.

Let $V \subseteq L^1(H)$ be defined by $V = \{h \in L^1(H) : |h(t)| \leq \phi(t) \text{ a.e.}\}$. From proposition 3.1 of [5], we know that V , equipped with the relative weak $L^1(H)$ -topology, is compact metrizable. Consider the multifunction $\Gamma: V \rightarrow P_f(L^1(H))$ defined by $\Gamma(h) = S_{\widehat{F}(\cdot, \gamma(h)(\cdot))}^1$. It is easy to check using the continuity of $\gamma(\cdot)$ and theorem 4.1 of [6], that $\Gamma(\cdot)$ is *l.s.c.* (note that if $h_n \xrightarrow{w} h$ in $V \subseteq L^1(H)$, then $h_n \xrightarrow{w} h$ in $L^2(H)$, since $\phi(\cdot) \in L^2_+$). So, we can apply Fryszkowski's continuous selection theorem [1], to get $k: V \rightarrow V$ continuous such that $k(h) \in \Gamma(h)$. Applying the Schauder-Tichonov fixed point theorem, we get $h \in V$ such that $h = k(h)$. Then $x = p(h)$ solves (*) with $F(t, x)$ replaced by $\widehat{F}(t, x)$. But as before we can check that $|x(t)| \leq M_2$ which implies $\widehat{F}(t, x(t))$ implies that $F(t, x(t))$ which yields $x \in S(x_0, x_1)$. Q.E.D.

4. An Example

In this section we present an example of a nonlinear hyperbolic partial differential inclusion illustrating the applicability of our work.

So let $T = [0, r]$ and Z a bounded domain in $\mathbb{R}^N$, with smooth boundary $\Gamma = \partial Z$. We will consider the following initial-boundary value problem of hyperbolic type with multivalued terms.

$$\left\{ \begin{array}{l} \frac{\partial^2 x}{\partial t^2} - \Delta x - \sum_{i=1}^N D_i(k(t, |Dx_t|^2) D_i x_t) \in [f_1(t, z, x(t, z)), f_2(t, z, x(t, z))] \\ x|_{T \times \Gamma} = 0, x(0, z) = x_0(z), x_t(0, z) = x_1(z). \end{array} \right\} \quad (**)$$

Here $D_i = \frac{\partial}{\partial z_i}$ $i = 1, \dots, N$, $Dx = (D_1 x_1, \dots, D_N x) = \text{grad}(x)$, $Dx Dy = \sum_{i=1}^N D_i x D_i y$ and $|Dx|^2 = \sum_{i=1}^N |D_i x|^2$.

We will need the following hypotheses on the data of (**):

$H(k)$: $k: T \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a function such that

- (1) $t \rightarrow k(t, \mu)$ is measurable,
- (2) $\mu \rightarrow k(t, \mu)$ is continuous,
- (3) $0 \leq k(t, \lambda^2) \leq L$ for all $(t, \lambda) \in T \times \mathbb{R}_+$, with $L > 0$ and $k(t, 0) = 0$,
- (4) $k(t, \lambda^2)\lambda - k(t, \mu^2)\mu \geq d(\lambda - \mu)$ for all $\lambda, \mu \in \mathbb{R}_+$, $\lambda \geq \mu$ and for some $d > 0$.

$H(f)$: $f_1, f_2: T \times Z \times \mathbb{R} \rightarrow \mathbb{R}$ are measurable functions such that $x \rightarrow f_1(t, z, x), -f_2(t, z, x)$ are *l.s.c.* and $|f_i(t, z, x)| \leq a_i(t, z) + b_i(z)|x|$ *a.e.* $i = 1, 2$ with $a_1(\cdot, \cdot) \in L^2(T \times Z)$, $b_1(\cdot) \in L^\infty(Z)$ and $f_1 < f_2$.

A_0 : $x_0(\cdot) \in H_0^1(Z)$, $x_1(\cdot) \in L^2(Z)$.

In this case, $X = H_0^1(Z)$, $H = L^2(Z)$ and $X^* = H_0^1(Z)^* = H^{-1}(Z)$. We know that (X, H, X^*) is an evolution triple with all embeddings being compact (Sobolev embedding theorem). Consider the following Dirichlet forms:

$$a_1(t, x, y) = \int_Z \sum_{i=1}^N k(t, |Dx|^2) D_i x D_i y dz = \int_Z k(t, |Dx|^2) Dx Dy dz$$

and

$$a_2(x, y) = \int_Z \sum_{i=1}^N D_i x D_i y dz = \int_Z Dx Dy dz$$

for all $x, y \in H_0^1(Z)$.

Using hypothesis $H(k)$ (3), we get

$$|a_1(t, x, y)| \leq L \|x\|_{H_0^1(Z)} \|y\|_{H_0^1(Z)}.$$

So there exists a nonlinear operator $A: T \times X \rightarrow X^*$ such that

$$\langle A(t, z), y \rangle = a_1(t, x, y).$$

From Fubini's theorem we have that $t \rightarrow a_1(t, x, y)$ is measurable which implies that $t \rightarrow A(t, x)$ is weakly measurable. But $H^{-1}(Z)$ is a separable Hilbert space. So the Pettis measurability theorem tells us that $t \rightarrow A(t, x)$ is measurable. Also if $x_n \rightarrow x$ in $H_0^1(Z)$, then by passing to a subsequence if necessary, we will have that $|Dx_n(z)|^2 \rightarrow |Dx(z)|^2$ a.e. and since by hypothesis $H(k)(2)$ $k(t, \cdot)$ is continuous we have $k(t, |Dx_n(z)|^2) \rightarrow k(t, |Dx(z)|^2)$ for all $t \in T$ and almost all $z \in Z$. Also $D_i x_n \xrightarrow{s} D_i x$ in $L^2(Z)$. Thus $\int_Z k(t, |Dx_n|^2) Dx_n Dy dz \rightarrow \int_Z k(t, |Dx|^2) Dx Dy dz$ implies that $A(t, x_n) \xrightarrow{w} A(t, x)$ which yields $A(t, \cdot)$ is demicontinuous, this hemicontinuous. Also we have

$$\langle A(t, x) - A(t, y), x - y \rangle = \int_Z (k(t, |Dx|^2) Dx - k(t, |Dy|^2) Dy)(Dx - Dy) dz.$$

Then, because of hypothesis $H(k)(2)$ and lemma 25.26 (b), p. 524 of Zeidler [9], we have

$$\langle A(t, x) - A(t, y), x - y \rangle \geq c \|x - y\|_{H_0^1(Z)}^2, c > 0$$

which yields that $A(t, \cdot)$ is strongly monotone.

Also since $k(t, 0) = 0$ (by hypothesis $H(k)(3)$), we have $A(t, 0) = 0$ yields that $A(t, \cdot)$ is coercive; i.e., $\langle A(t, x), x \rangle \geq c \|x\|_{H_0^1(Z)}^2$. Thus, we satisfied hypothesis $H(A)$.

Next note that by the Cauchy-Schwartz inequality, we have

$$|a_2(x, y)| \leq \|x\|_{H_0^1(Z)} \|y\|_{H_0^1(Z)}.$$

So there exists a continuous linear operator $B: X \rightarrow X^*$ such that

$$\langle Bx, y \rangle = a_2(x, y).$$

Clearly $\langle Bx, y \rangle = \langle x, By \rangle$; i.e. B is symmetric and by Poincaré's inequality, we have $\langle Bx, x \rangle \geq c' \|x\|_{H_0^1(Z)}^2$, $c' > 0$. Therefore, we satisfied hypothesis $H(B)$.

Next let $F: T \times L^2(Z) \rightarrow P_{f_c}(L^2(Z))$ be defined by

$$F(t, x) = \{h \in L^2(Z): f_1(t, z, x(z)) \leq h(z) \leq f_2(t, z, x(z)) \text{ a.e.}\}.$$

Let $\eta: T \times Z \times \mathbb{R} \rightarrow P_{f_c}(\mathbb{R})$ be defined by $\eta(t, z, x) = [f_1(t, z, x), f_2(t, z, x)]$. Because of hypothesis $H(f)$, we deduce that $\eta(\cdot, \cdot, \cdot)$ is measurable while $\eta(t, z, \cdot)$ is u.s.c. (see Klein-Thompson [2], p. 74). Note that $F(t, x) = S_{\eta(t, \cdot, x(\cdot))}^2$. So, from theorem 4.2 of [6], we have that $F(t, \cdot)$ is u.s.c. from H into H_w , while clearly $t \rightarrow F(t, z)$ is measurable. Also, $|F(t, x)| = \sup\{|y|_{L^2(Z)}: y \in F(t, x)\} \leq \hat{a}_1(t) + \hat{b}_1 \|x\|_{L^2(Z)}$, with $\hat{a}_1(t) = \|a(t, \cdot)\|_{L^2(Z)}$, $\hat{b}_1 = \|b\|_{L^\infty(Z)}$. Thus, we satisfied hypothesis $H(F)_1$. Finally, let $\hat{x}_0 = x_0(\cdot) \in H_0^1(Z)$, $\hat{x}_1 = x_1(\cdot) \in L^2(Z)$.

Rewrite (**) in the following equivalent nonlinear evolution inclusion form:

$$\left\{ \begin{array}{l} \dot{x}(t) + A(t, \dot{x}(t)) + Bx(t) \in F(t, x(t)) \\ x(0) = \hat{x}_0, \dot{x}(0) = x_1. \end{array} \right\} \quad (**)'$$

Theorem 4.1: *If hypotheses $H(k)$, $H(f)$ and H_0 hold, then (**) has a solution $x \in C(T, H_0^1(Z))$ such that $\frac{\partial x}{\partial t} \in L^2(T, H_0^1(Z)) \cap C(T, L^2(Z))$ and $\frac{\partial^2 x}{\partial t^2} \in L^2(T, H^{-1}(Z))$. Also, the solution set is compact in $C(T, H_0^1(Z))$.*

Now suppose that (**) corresponds to an optimal control problem; i.e.

$$f_1(t, z, x) = f(t, z, x)u_1(z)$$

and

$$f_2(t, z, x) = f(t, z, x)u_2(z)$$

with a function $f: T \times Z \times \mathbb{R} \rightarrow \mathbb{R}_+$ such that $(t, z) \rightarrow f(t, z, x)$ is measurable, $x \rightarrow f(t, z, x)$ is continuous and $|f(t, z, x)| \leq a_1(t, z) + b_1(z)x$ a.e., with $a_1(\cdot, \cdot) \in L^2(T \times Z)$, $b_1(\cdot) \in L^\infty(Z)$. The control constraint set is defined as

$$U(t, z) = \{v \in \mathbb{R}: u_1(z) \leq v \leq u_2(z)\}$$

with $0 < u_1(z) < u_2(z) \leq M$ a.e.

We are also given a cost functional $J(x) = \int_0^b \int_Z L(t, z, x(t, z)) dz dt$ to be minimized over all admissible trajectories. Assume that $L: T \times Z \times \mathbb{R} \rightarrow \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a measurable integrand such that $L(t, z, \cdot)$ is l.s.c. and $|\phi(t, z) - M(z)|x| \leq L(t, z, x)$ a.e. with $\phi(\cdot, \cdot) \in L^1(T \times Z)$, $M(\cdot) \in L^\infty_+(Z)$. Then, $J(\cdot)$ is l.s.c. on $C(T, H_0^1(Z))$, and so, using theorem 4.1 above, we deduce that this distributed parameter optimal control problem has a solution. Analogous results for parabolic systems can be found in [4].

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