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BOUND AND PERIODIC SOLUTIONS OF THE RICCATI EQUATION IN BANACH SPACE

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ABSTRACT

An abstract, nonlinear, differential equation in Banach space is considered. Conditions are presented for the existence of bounded solutions of this equation with a bounded right side, and also for the existence of stationary (periodic) solutions of this equation with a stationary (periodic) process in the right side.

Key words: Abstract Nonlinear Equations, Bounded Solutions, Stationary and Periodic Solution.

AMS (MOS) subject classifications: 60H15, 60H20, 34E10, 34G20, 34K30.

1. Introduction

Stationary processes which are solutions of stochastic differential equations with constant coefficients constitute a class of processes which are studied in detail and which have numerous applications. Comprehensive information about these processes and related bibliographies may be found in the well-known monographs of J.L. Doob and Yu. A. Rozanov. In 1962, the concept of periodic process was introduced by the first-mentioned author of this article. Subsequently, conditions for the existence of stationary or periodic solutions of stochastic equations were obtained by R.Z. Khas'minskiy, B.V. Kolmanovskiy and other mathematicians, first in a finite dimensional setting, and later by other authors, in particular T. Morozan, in Banach space. A detailed bibliography is presented in the cited monograph of the first author of this article. The studied equations were mainly linear or with nonlinearity satisfying Lipschitz condition.

In the theory of differential equations, the special role of Riccati's equation is well known. In modern applications, the matrix and operator equations of the Riccati type are researched intensively. (See [2] and the references therein). This article employs the method used by Dorogovtsev [2] to prove the existence of bounded and periodic solutions of differential, Riccati-type equations in Banach space. It should be mentioned that nonlinearity does not satisfy Lipschitz condition in the whole space and that a stationary or periodic process may not belong to the bounded part of the space with probability 1.

2. Assumptions and Preliminary Facts

Let $(B, \|\cdot\|)$ be a complex Banach space, $\overline{0}$ the zero element in B, and $\mathfrak{L}(B)$ the Banach space of bounded linear operators on B with the operator norm, denoted also by the symbol $\|\cdot\|$. For the function $y:\mathbb{R}\to B$ we define

$$\|y\|_\infty := \sup_{t\in\mathbb{R}} \|y(t)\| \leq +\infty.$$

Function $x:\mathbb{R}\to B$ is differentiable in the point $t_0\in\mathbb{R}$ if an element, $y\in\mathbb{R}$, exists such that

$$\left\|\frac{x(t)-x(t_0)}{t-t_0} - y\right\| \to 0, \quad t\to t_0.$$

The element y is called the derivative of function x at the point t_0 and is denoted by the symbol $x'(t_0)$. With the help of this definition, the class $C^1(\mathbb{R},B)$ is defined by the usual manner. In addition, the continuity of function x at the point t_0 means that

$$\|x(t)-x(t_0)\|\to 0,\ t\to t_0.$$

For the operator $A\in\mathcal{L}(B)$ and function $y\in C(\mathbb{R},B)$ we consider the equation

$$\frac{dx(t)}{dt} = Ax(t)+y(t), \quad t\in\mathbb{R} \tag{1}$$

with respect to function $x\in \mathrm{C}^1(\mathbb{R},B)$. Equation (1) is investigated in detail. We shall formulate the necessary facts about properties of the solutions of equation (1) without proof. Details are presented in [2, 3].

We consider operator A whose spectrum $\sigma(A)$ does note intersect with imaginary axis and which consists of two sets, $\sigma_-(A)$ and $\sigma_+(A)$, where

$$\sigma_-(A) = \sigma(A)\cap\{z \mid \mathrm{Re} z < 0\},$$

$$\sigma_+(A) = \sigma(A)\cap\{z \mid \mathrm{Re} z > 0\}.$$

In addition, $\sigma(A)=\sigma_-(A)\cup\sigma_+(A)$. According to the well-known theorem of M.G. Krein [4], the structure of the spectrum which we have considered is equivalent to a unique solution of equation (1) $x\in C^1(\mathbb{R},B)$ with $\|x\|_\infty < +\infty$ existing for any function $y\in C(\mathbb{R},B)$ with $\|y\|_\infty < +\infty$. Let P_+ and P_- be spectral projectors [2, 3], and P_- and P_+ operators in $\mathcal{L}(B)$ such that

$$P^2_{\mp} = P_{\mp},\ P_+P_- = P_-P_+ = 0 \ \ (0 \text{ is zero operator})$$

$$P_{\mp}A = AP_{\mp},$$

for which $B_- := P_-B$, $B_+ := P_+B$ are invariant subspaces for operator A. The spectrum of operator A in these subspaces coincides with $\sigma_-(A)$ and $\sigma_+(A)$, respectively. Also, we consider operators

$$A_+ := P_+A,\ A_- := P_-A,$$

and, in addition, $A_- + A_+ = A$. With the help of spectral projectors we define the operator valued function

$$G(t) := \begin{cases} e^{tA} - P_-, & t>0 \\ -e^{tA} + P_+, & t>0. \end{cases}$$

It is known [2, 3] that

$$\|G(t)\| \leq \begin{cases} c_- e^{t\alpha_-}, & t<0 \\ c_+ e^{-t\alpha_+}, & t>0 \end{cases} \tag{2}$$

where c_-, c_+ are non-negative numbers and a_-, a_+ are positive numbers. The numbers corresponding to operator A are fixed below. We denote

$$\mathfrak{G}: = \frac{c_-}{a_-} + \frac{c_+}{a_+}.$$

With the help of function G we can write the unique solution $x \in C^1(\mathbb{R}, B)$, $\| x \|_\infty < \infty$ of equation (1) for the function $y \in C(\mathbb{R}, B)$, $\| y \|_\infty < +\infty$ in the form

$$x(t) = \int_{\mathbb{R}} G(t-s)y(s)ds, \quad t \in \mathbb{R}. \tag{3}$$

We understand the integral in (3) as the limit integral sums in B. (For details, see [4].)

3. Existence of Bounded Solutions

Let $b: B \times B \to B$ be a function that is linear in each variable, such that

$$\exists\, C > 0\ \forall \{u, v\} \subset B: \| b(u,v) \| \leq C \| u \| \cdot \| v \|. \tag{4}$$

Theorem 1: *Let operator A satisfy the conditions of p.1; B is a bilinear function such that inequality (4) is true with a number C, and a function $y \in C(\mathbb{R}, B)$, $\| y \|_\infty < +\infty$. Assume that the inequality*

$$4\mathfrak{G}^2 C \| y \|_\infty < 1$$

is true.

Then equation

$$\frac{dx(t)}{dt} = Ax(t) + b(x(t), x(t)) + y(t), \quad t \in \mathbb{R} \tag{5}$$

has a solution x with $\| x \|_\infty < +\infty$ in the class $C^1(\mathbb{R}, B)$.

4. Subsidiary Statements

For the function $y \in C(\mathbb{R}, B)$ with $\| y \|_\infty < +\infty$, equation

$$\frac{dx_0(t)}{dt} = Ax_0(t) + y(t), \quad t \in \mathbb{R} \tag{6}$$

has a unique solution $x_0 \in C^1(\mathbb{R}, B)$ with $\| x_0 \| < +\infty$, moreover,

$$x_0(t) = \int_{\mathbb{R}} G(t-s)y(s)ds, \quad t \in \mathbb{R}. \tag{7}$$

It is easy to verify that

$$\| x_0 \| \leq \mathfrak{G} \| y \|_\infty, \tag{8}$$

and it follows from (4) $\| b(x_0, x_0) \|_\infty < +\infty$. Therefore, equation

$$\frac{dx_1(t)}{dt} = Ax_1(t) + b(x_0(t), x_0(t)), \quad t \in \mathbb{R} \tag{9}$$

has a unique solution $x_1 \in C^1(\mathbb{R}, B)$ with $\| x_1 \|_\infty < +\infty$, moreover,

$$x_1(t) = \int_{\mathbb{R}} G(t-s) b(x_0(s), x_0(s))ds, \quad t \in \mathbb{R} \tag{10}$$

and

$$\| x_1 \|_\infty \leq C\mathfrak{G}^3 \| y \|^2_\infty. \tag{11}$$

Now, we construct the sequence of functions $\{x_n : n \geq 1\}$ as follows: Assume that for $n \geq 2$ functions $x_0, x_1, \ldots, x_{n-1}$ have been already defined as unique solutions of equations in a class $C^1(\mathbb{R}, B) \cap C_\infty(\mathbb{R}, B)$

$$\frac{dx_k(t)}{dt} = Ax_k(t) + y_k(t), \quad t \in \mathbb{R}; \quad 1 \leq k \leq n-1. \tag{12}$$

where

$$y_k: = \sum_{j=0}^{k-1} b(x_j, x_{k-1-j}), \quad k \geq 2.$$

In addition,

$$\| x_k \|_\infty \leq \frac{C_{2k}^k}{k+1} \mathfrak{G}^{2k+1} C^k \| y \|_\infty^{k+1}, \quad 1 \leq k \leq n-1. \tag{13}$$

We also define $y_0: = y$, $y_1: = b(x_0, x_0)$ and note that estimate (13) is true for the solutions x_0 and x_1 defined above, for $k = 0$ and $k = 1$, respectively.

Function y_n constructed by $x_0, x_1, \ldots, x_{n-1}$ with the help of inequalities (13) allows the estimate

$$\| y_n \|_\infty \leq \sum_{j=0}^{n-1} \| b(x_j, x_{n-1-j}) \| \leq C \sum_{j=0}^{n-1} \| x_j \| \cdot \| x_{n-j-1} \|$$

$$\leq C \sum_{j=0}^{n-1} \frac{C_{2j}^j}{j+1} \mathfrak{G}^{2j+1} C^j \| y \|_\infty^{j+1} \cdot \frac{C_{2(n-j-1)}^{n-j-1}}{n-j} \cdot \mathfrak{G}^{2n-2j-1} C^{n-j-1} \| y \|_\infty^{n-j-1} \tag{14}$$

$$= C^n \mathfrak{G}^{2n} \| y \|_\infty^n \sum_{j=0}^{n-1} \frac{C_{2j}^j}{j+1} \cdot \frac{C_{2(n-j-1)}^{n-j-1}}{n-j}$$

$$= \frac{C_{2n}^n}{n+1} \mathfrak{G}^{2n} C^n \| y \|_\infty^n.$$

The last step of the derivation uses identity $11(a)$ from Riordan [6, p. 123].

Now, we define function x_n as a unique solution in the class $C(\mathbb{R}, B) \cap C_\infty(\mathbb{R}, B)$ of equation

$$\frac{dx_n(t)}{dt} = AX_n(t) + y_n(t), t \in \mathbb{R}, \tag{15}$$

moreover,

$$x_n(t) = \int_{\mathbb{R}} G(t-s) y_n(s) ds, \quad t \in \mathbb{R} \tag{16}$$

and it follows from estimates (8) and (14) that

$$\| x_n \|_\infty \leq \frac{C_{2n}^n}{n+1} \mathfrak{G}^{2n+1} C^n \| y \|_\infty^{n+1}. \tag{17}$$

Therefore, estimate (17) is true for x_n, i.e., estimate (13) is true for $k = n$.

5. Proof of Theorem 1

We consider the function

$$x(t): = \sum_{n=0}^{\infty} x_n(t), \quad t \in \mathbb{R}, \tag{18}$$

which has been constructed using the sequence of functions in section 2. It follows from (17) that the series in (18) converges uniformly on $\mathbb{R}$ in the norm, since

$$\| x \|_\infty \leq \sum_{n=0}^{\infty} \frac{C_{2n}^n}{n+1} \mathfrak{G}^{2n+1} C^n \| y \|_\infty^{n+1}$$

$$= \sum_{n=0}^{\infty} \frac{C_{2n}^n}{n+1} (\mathfrak{G}^2 C \| y \|_\infty)^n \mathfrak{G} \| y \|_\infty.$$

Since

$$C_{2n}^{n} \sim \frac{2^{2n}}{\sqrt{\pi n}}, \quad n \to +\infty,$$

the last series converges, if

$$4\mathfrak{G}^2 C \,\| y \|_{\infty} \leq 1.$$

Therefore, $x \in C(\mathbb{R}, B)$. The series of derivatives corresponding to series (18) also converges uniformly on $\mathbb{R}$ in the norm, since

$$\sum_{n=0}^{\infty} x_n'(t) = Ax_0(t) + y(t)$$

$$+ \sum_{n=1}^{\infty} (Ax_n(t) + y_n(t)), \ t \in \mathbb{R},$$

according to the definition of functions x_n, $n \geq 0$, and it follows from estimates (8), (17) and (14) that

$$\sum_{n=0}^{\infty} \| x_n' \|_{\infty} \leq \| A \| \mathfrak{G} \| y \|_{\infty} + \| y \|_{\infty}$$

$$+ \| A \| \sum_{n=1}^{\infty} \frac{C_{2n}^{n}}{n+1} (\mathfrak{G}^2 C \| y \|_{\infty})^n \mathfrak{G} \| y \|_{\infty} + \sum \frac{C_{2n}^{n}}{n+1} (\mathfrak{G}^2 C \| y \|_{\infty})^n < +\infty.$$

Therefore, $x \in C^1(\mathbb{R}, B)$ and

$$x'(t) = \sum_{n=0}^{\infty} x_n'(t) = Ax_0(t) + y(t) + \sum_{n=1}^{\infty} (Ax_n(t) + y_n(t)), \ t \in \mathbb{R}. \tag{19}$$

We note that according to the definition of $\{y_n\}$

$$\sum_{n=1}^{N} y_n(t) = \sum_{n=1}^{N} \sum_{j=0}^{n-1} b(x_j(t), x_{n-1-j}(t)), \quad n \geq 1, \quad t \in \mathbb{R}.$$

Since for each $t \in \mathbb{R}$

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \| b(x_j(t), x_k(t)) \| < +\infty,$$

it follows from condition (4) and estimates (17) that for each $t \in \mathbb{R}$ in the B-norm

$$\lim_{N \to +\infty} \sum_{n=1}^{N} y_n(t) = \lim_{N \to +\infty} \sum_{j,k=0}^{N} b(x_j(t), x_k(t))$$

$$= \lim_{N \to +\infty} b\Big(\sum_{j=0}^{N} x_j(t), \sum_{k=0}^{N} x_k(t)\Big) = b(x(t), x(t)).$$

Therefore, from equality (19) it follows that

$$x'(t) = Ax(t) + y(t) + b(x(t), x(t)), \quad t \in \mathbb{R}.$$

Theorem 1 is proved. □

6. Periodic Solutions

Let $\tau > 0$ be fixed and function $A \in C(\mathbb{R}, \mathcal{L}(B))$ is such that

$$\forall t \in \mathbb{R}: A(t + \tau) = A(\tau).$$

Let $U: \mathbb{R} \to \mathcal{L}(B)$ be a solution of the following problem

$$\begin{cases} U'(t) = A(t)U(t), & t \in \mathbb{R} \\ \quad U(0) = E \end{cases}$$

where E is the identity operator. The properties of the function U below are well-known [5]. The following statement is proved in a similar way for Theorem 1 by results of [1].

Theorem 2: *If $1 \notin \sigma(U(\tau))$, then for every function $y \in C(\mathbb{R}, B)$ which is periodic with period τ, the equation*

$$\frac{dx(t)}{dt} = A(t)x(t) + \epsilon b(x(t), x(t)) + y(t), \quad t \in \mathbb{R}$$

has a τ-periodic solution in the class $C^1(\mathbb{R}, B)$ for every ϵ with small enough $|\epsilon|$.

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