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A BRIEF SURVEY OF SPHERICAL INTERPOLATION AND APPROXIMATION METHODS FOR TEXTURE ANALYSIS

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In texture analysis there are several instances when mathematical methods of spherical interpolation or approximation are required. Ad hoc adaptions of univariate or bivariate methods to the topology of spherical manifolds usually fail in one way or another. Therefore, this contribution will provide a brief survey of genuinely spherical methods.

KEY WORDS: Spherical bilinear in terpolation, quasi-interpolation, spherical singular integrals, spherical multiquadrics, spherical splines.

1 MOTIVATION AND INTRODUCTION

The most prominent functions of texture analysis, the orientation density function describing the probability distribution of proper rotations, and the pole density function describing the probability distribution of crystallographic lattice planes are genuinely spherical functions. The orientation density function is defined on the threedimensional projective space equivalent to the upper (lower) unit sphere $H^3 \equiv S^4_\pm \subset IR^4$, the pole density function is defined on the upper (lower) unit sphere $H^2 \equiv S^3_\pm \subset IR^3$. Thus there are several instances when mathematical methods of spherical interpolation or approximation are required, e.g.

- preprocessing of experimental pole figure data,
- postprocessing of calculated values of a pole density function,
- display of an orientation density function,
- density estimation of individual orientation data;

Ad hoc adaptions of general multivariate methods (cf. Schumaker, 1976a, 1976b; Franke and Schumaker, 1987) to the topology of spherical manifolds usually fail in one way or another. They may be successful provided the domain of the data and of the interpolant/approximant to be constructed is a sufficiently small subset of S^d, or if it suffices that smoothness constraints or boundary conditions are only approximately satisfied for a particular subset of S^d, cf. Dierckx (1984); Gmelig Meyling and Pfluger (1987). However, if the domain of the interpolating or smoothing function to be constructed contains the north (south) pole of the sphere and if smoothness constraints or boundary conditions have to be exactly satisfied genuine spherical methods (cf.

Schumaker and Traas, 1991; Traas *et al.*, 1993) are required, because there is no differentiable mapping of the entire sphere to a bounded hyperplane. Therefore, this contribution will provide a brief survey of genuinely spherical methods. References of the topic include Berens *et al.*, 1968; Pawelke, 1972; Butzer *et al.*, 1979; Freeden, 1981; 1984; Wahba, 1981, 1984; Wehrens, 1981; Lawson, 1984; Renka, 1984; Barnhill *et al.*, 1987; Nielson and Ramaraj, 1987; Foley, 1990; Pottmann and Eck, 1990; Schumaker and Traas, 1991; Hoschek and Seemann, 1992; Taijeron *et al.*, 1994.

2 BILINEAR SPHERICAL INTERPOLATION

From the times of mechanical control of the texture goniometer two "equi-angular" grids to sample a pole density function have survived and are widely used. They are such that

(i) data values $z_p \geq 0$ are sampled at sites $(\phi_i, \theta_j)_{i=1,\dots,P_1,\, j=1,\dots,P_2} \in S^3_+ \subset IR^3$ with $\phi_i = -\pi + i\Delta\phi$, $i = 0, \dots, P_1$; $P_1 = 2\pi/\Delta\phi$ and $\theta_j = j\Delta\theta$, $j = 0, \dots, P_2$; $P_2 = \pi/\Delta\theta$;

(ii) data values $z_p \geq 0$ are sampled at sites $\phi_i = -\pi + (i - \frac{1}{2})\Delta\phi$, $i = 1, \dots, P_1$ and $\theta_j = (j - \frac{1}{2})\Delta\theta$, $j = 1, \dots, P_2 \in S^3_+$.

Thus, the two grids are related to one another by a shift of half the angular step size in each spherical coordinate.

For a conversion of one equi-angular grid to the other bilinear spherical interpolation seems to provide the most simple and actually effective means. For (ϕ, θ) in the patch defined by the four vertices (ϕ_{ip}, θ_{jp}), $p = 1, \dots, 4$, it reads

$$\hat{z}(\phi, \theta) = \sum_{p=1}^{4} w_p z_p = \sum_{p=1}^{4} w_p z(\phi_{ip}, \theta_{jp}) \tag{1}$$

where w_p denotes the areal portion f_p/F with

$$f_p = |\phi - \phi_{ip}| \times |\cos(\theta) - \cos(\theta_{jp})|$$

of the total area $F = \sum_{p=1}^{4} f_p$ "diagonally across" the point (ϕ_{ip}, θ_{jp}). Thus

$$\hat{z}(\phi_i, \theta_j) = z(\phi_i, \theta_j),\ i = 1, \dots, P_1,\ j = 1, \dots, P_2 \tag{2}$$

Particularly, for $\hat{z}(\phi, \pi/2)$, $\phi \in [0, \pi]$,

$$\hat{z}(\phi, \pi/2) = \hat{z}(\phi - \pi, \pi/2) = \frac{z(\phi, \pi/2 - \Delta\theta/2) + z(\phi - \pi, \pi/2 - \Delta\theta/2)}{2}$$

and for $\hat{z}(0, 0)$

$$\hat{z}(0, 0) = \frac{1}{P_1} \sum_{p=1}^{P_1} z(\phi_p, \Delta\theta/2)$$

i.e. no extrapolation is needed.

Obviously, interpolated values are always nonnegative, and the resulting surface $\hat{z}$ is C^0. Repeated conversion of one grid to the other by bilinear interpolation results in an ever smoothed interpolant with gradually diminishing modes. Spherical bilinear interpolation does not seem appropriate for scattered data, i.e. for data sites arbitrarily distributed on the sphere.

3 APPROXIMATION BY SINGULAR CONVOLUTION INTEGRALS, QUASI-INTERPOLATION

The basic idea (cf. Butzer and Nessel, 1971) is to approximate a (given) function f by a function with "better" properties which is to be constructed by some smoothing operation on f itself. The smoothing operation may be provided by convolution, which reads for spherical functions

$$(f * \chi_\rho)(\mathbf{v}) = \text{area}^{-1}(S^d)\int_{S^d} f(\mathbf{u})\chi_\rho(\mathbf{vu})ds(\mathbf{u}),\ \mathbf{u},\ \mathbf{v} \in S^d \tag{3}$$

The convolution integral (3) is called a singlular integral (cf. Aronszajn, 1950) if the sequence $(\chi_\rho)_{\rho\in A}$ is a kernel, or since it is to be applied to probability density functions more specifically a nonnegative approximate identity. For a fixed $\mathbf{v}$ and a variable $\mathbf{u}$ of $S^d \subset IR^d$ the functions $\chi_\rho(\mathbf{vu})$ usually display a bell-shaped graph rotationally symmetric with respect to $\mathbf{v}$ with normalized area under the curve whereby for $\rho \to \rho_0$ the mode at $\mathbf{v}$ becomes larger and narrower in such a way that the area under the curve near $\mathbf{v}$ comes out to be 1. A kernel with some parameter set A, or more precisely a nonnegative approximate identity $(\chi_\rho)_{\rho\in A}$ is defined by its properties

$$0 \leq \chi_\rho(t) \in L^1(-1, 1), \quad \rho \in A \tag{4}$$

$$\frac{1}{2}\int_{(-1,1)} \chi_\rho(t)dt = 1, \quad \rho \in A \tag{5}$$

$$\lim_{n\to\infty} \rho_n = \rho_0 \tag{6}$$

$$\lim_{\rho_n\to\rho_0} \int_{\{-1\leq t\leq 1-\delta\}} \chi_{\rho_n}(t)dt = 0,\ 0 < \delta < 2 \tag{7}$$

$$\lim_{\rho_n\to\rho_0}\ \sup_{-1\leq t\leq 1-\delta} \chi_{\rho_n}(t) = 0,\ 0 < \delta < 2 \tag{8}$$

The last two properties may be used alternatively. The nonnegativity of the kernel implies its uniform boundedness with respect to the L^1 - norm.

The name approximate identity originates in the property that the sequence $\{f * \chi_\rho\}$ tends uniformly to f as $\rho \to \rho_0$ which can be seen from

$$(f * \chi_\rho)(\mathrm{v}) \approx f(\mathrm{v})\ \text{area}^{-1}(S^d) \int_{(1-\delta,\ 1)} \chi_\rho(t)dt$$

One of the most prominent features of the convolution integral $f * \chi_\rho$ is that it preserves the "best" properties of each of its factors, i.e. if χ_ρ is very smooth, then $f * \chi_\rho$ will be so, too; and usually, χ_ρ are chosen to be very smooth.

Now it is presumed that the data values $z_p \geq 0$, $p = 1, ..., P$, have been sampled from a spherical function $f \in \mathcal{L}^q(S^3)$ at arbitrarily scattered data sites $\mathbf{r}_p$, $p = 1, ..., P$, i.e. $(\mathbf{r}_p, z_p) = (\mathbf{r}_p, f(\mathbf{r}_p))$, $p = 1, ..., P$, provides the available data. Then an approximation of f is given by

$$\hat{f}(\mathbf{r};\ \rho_P) = \frac{1}{4\pi}\sum_{p=1}^{P} f(\mathbf{r}_p)\chi_{\rho_P}(\mathbf{rr}_p) = \frac{1}{4\pi}\sum_{p=1}^{P} z_p\chi_{\rho_P}(\cos\ \theta_p) \tag{9}$$

with $\cos\ \theta_p = \mathbf{rr}_p$.

It is emphasized that $\hat{f}(\mathbf{r};\ \rho_P)$ is nonnegative and a density itself of the same class of functions as the approximate identity χ_ρ. The approximant $\hat{f}$ does not interpolate the data. In fact, it is given as superposition of rotationally invariant decreasing functions each of which centered at a data site $\mathbf{r}_p$ and multiplied by $z_p = f(\mathbf{r}_p)$. This approximation method may be referred to as "quasi-interpolation" because the interpolation conditions are thought of as being relaxed by the requirement that the dependence of $\hat{f}(\mathbf{r};\rho_P)$ with respect to each particular $f(\mathbf{r}_p)$ diminishes rapidly with increasing distance of $\mathbf{r}$ from $\mathbf{r}_p$.

Of major interest in practical applications with finite data sets $(\mathbf{r}_p, f(\mathbf{r}_p))$, $p = 1, ..., P$, is not the sequence $f * \chi_\rho$ and its convergence but an individual member $\hat{f} = f * \chi_{\rho_P}$ with a fixed parameter ρ_P depending on the total number P of data. This fixed parameter ρ_P may be interpreted as the "window width" considering the data and controls the degree of smoothing. Therefore the choice of this tuning parameter is crucial. Several mathematical methods exist to optimize the choice of this parameter. Whatever the choice is, the degree of smoothing is usually large because one generally chooses "very smooth" functions as kernel, e.g. spherical analogues of the Gaussian probability density function (Nikolayev and Ullemeyer, these proceedings). Therefore, repeated application of these methods ("iterated singular integrals") tends to make characteristics of the data disappear.

While the parameter ρ_P can be optimized for a given approximate identity, the choice of the approximate identity itself is not obvious. Early constructions of approximate identities which are optimal with respect to the rate of convergence are given by Bartlett (1963), Epachenikov (1969).

As an example for $S^3 \subset IR^3$ without theoretical justification, the exponential function

$$\chi_\rho(t) = c(\rho)\exp(\rho t) \quad t \in (-1, 1),\ \rho \in IR_+ \tag{10}$$

with

$$c(0) = 1 \tag{11}$$

$$c(\rho) = \frac{\rho}{\sin\mathrm{h}(\rho)} \tag{12}$$

$$\lim_{n\to\infty} \rho_n = \rho_0 = \infty \tag{13}$$

borrowed from the zonal Langevin - von Mises - Fisher density is chosen as approximate identity. This choice yields the approximation

$$\hat{f}(\mathbf{r};\ \rho p) = \frac{1}{4\pi}\ \frac{\rho_P}{\sinh\ \rho_P} \sum_{p=1}^{P} f(\mathbf{r}_p)\exp(\rho_P \mathbf{r}\mathbf{r}_p) \tag{14}$$

As another computationally less demanding example one may choose the de la Vallée Poussin approximate identity

$$\chi_n(\cos\ \theta) = \frac{n!^2}{(2n)!}(2\ \cos\frac{\theta}{2})^{2n}\ ,\ n \in IN \tag{15}$$

The reader may find CosiPoWi (Adam, 1989) a rather peculiar reference to the de la Vallée Poussin kernel.

Obviously, approximation methods by singular convolution integral generalize easily to hyperspheres of any dimension.

4 INTERPOLATION BY RADIAL BASIS FUNCTIONS

Employing radial basis functions for the purposes of interpolation/approximation or estimation seems to originate in the earth sciences, cf. Krige, 1951; Matheron, 1967; Crain and Bhattacharyya, 1967; Shepard, 1968. Radial interpolation methods are closely related to approximation by singular integrals (cf. Aronszajn, 1950; Butzer and Nessel, 1971) and splines as delivered by variational calculus (cf. Hogervorst, 1994). A recent survey is given by Kansa (1992).

The basic idea is to start with an analogue of (9), namely

$$\hat{f}(\mathbf{r}) = \sum_{p=1}^{P} \lambda_p \chi_p(\mathbf{r}) = \sum_{p=1}^{P} \lambda_p \chi(\mathbf{r}\mathbf{r}_p) = \sum_{p=1}^{P} \lambda_p \chi(\cos\theta_p) \tag{16}$$

with $\cos\theta_p = \mathbf{r}\mathbf{r}_p$ to construct a smooth function $\hat{f}$ such that (i) it interpolates, i.e.

$$\hat{f}(\mathbf{r}_p) = f(\mathbf{r}_p) = z_p,\ p = 1,\ \ldots,\ P \tag{17}$$

is satisfied, (ii) χ_p depends only on the data site $\mathbf{r}_p$ and not on the surrounding data, and (iii) it is rotationally symmetric, or invariant, with respect to $\mathbf{r}_p$. The unknown coefficients λ_p, $p = 1, \ldots, P$, shall be provided by eqs. (17).

Many schemes arising from this approach have the pleasent properties that (i) interpolation is always possible and unique under very mild conditions on the spatial arrangement of data sites, (ii) the interpolant matches the required smoothness, and (iii) they generalize easily to higher dimensions or non-Euclidean manifolds as hyperspheres.

Examples of radial functions $\chi : IR_+ \mapsto IR$ frequently applied in multi-variate Euclidean interpolation are (cf. Hogervorst, 1994)

surface $\quad \chi(r) = r^{2n-1},\ n \in IN \quad$ (18)

thin plate $\quad \chi(r) = r^{2n}\log r,\ n \in IN \quad$ (19)

multiquadric $\quad \chi(r) = (r^2 + c^2)^{1/2},\ c \in IR \quad$ (20)

inverse multiquadric $\quad \chi(r) = (r^2 + c^2)^{-1/2},\ c \in IR \quad$ (21)

shifted surface $\quad \chi(r) = (r^2 + c^2)^{n-1/2},\ n \in IN,\ c \in IR \quad$ (22)

shifted thin plate $\quad \chi(r) = (r^2 + c^2)^n \log(r^2 + c^2),\ n \in IN,\ c \in IR \quad$ (23)

shifted logarithm $\quad \chi(r) = \log(r^2 + c^2),\ c \in IR \quad$ (24)

Gaussian $\quad \chi(r) = \exp(-cr^2),\ c \in IR_+ \quad$ (25)

where $\chi_p(\mathrm{x}) = \chi(r_p)$ with $r_p = \| \mathrm{x} - \mathrm{x}_p \|$, $\mathrm{x}, \mathrm{x}_p \in IR^d$, $d \in IN$.

The surface and thin plate radial basis function can be derived from variational calculus (cf. Hogervorst, 1994). The multiquadric as well as the inverse multiquadric has been introduced by Hardy (1971, 1990), Hardy and Goepfert (1975), Hardy and Nelson (1986) in geophysics. Shifted relatives have been introduced by Dyn *et al.* (1986), Dyn (1989). The Gaussian reveals most obviously the close relation to approximation by the singular integral of Gauss-Weierstrass. All of them may be read as appropriately smoothed multivariate versions of univariate piecewise linear interpolation provided by $\chi_p(x) = \chi(|x - x_p|)$ which produces discontinuities of the first derivative at the data sites x_p. In case of the multiquadrics smoothing of the distance function is done by using arcs of hyperbolas that possess the r_p values as asymptotes, where c is the tuning parameter which must be supplied by the user. Considering the augmented data sites $(\mathrm{x}_p, 0)$ and (x_p, c) as elements of $I\!R^{d+1}$, the multiquadric $\chi_p(\mathrm{x})$ may be interpreted as the distance between these augmented data sites.

Next, generalizations of the multiquadric and inverse multiquadric approach for scattered spherical data are summarized as they are known for their excellent practical achievements (Foley, 1990; Pottmann and Eck, 1990).

Let $g_p(\mathbf{r}) = \arccos\,(\mathbf{r}\mathbf{r}_p)$ denote the geodesic distance of $\mathbf{r}$ from $\mathbf{r}_p$. Then define

$$\hat{f}(\mathbf{r}) = \sum_{p=1}^{P} \lambda_p \chi(g_p(\mathrm{r})) \tag{26}$$

where $\chi(t)$ is either the multiquadric or the inverse multiquadric. As usually, coefficients λ_p, $p = 1, ..., P$, are determined by solving the linear system of equations

$$\hat{f}(\mathbf{r}_p) = \sum_{p=1}^{P} c_j B(g_j(\mathrm{r}_p)) = z_p, \; p = 1, ..., P \tag{27}$$

Each basis function $\chi(g_p(\mathbf{r}))$ is a C^∞ function on the sphere except at the antipodal point $-\mathbf{r}_p$, where it is not differentiable. Thus the interpolant is $C^0(S^d)$.

Modifications of basis functions such that they are C^2 on S^d can be accomplished by piecewise quintic poynomial blending near the antipodal point.

(i) Modifying the geodesic distance function

$$\hat{f}(\mathbf{r}) = \sum_{p=1}^{P} \lambda_p \chi(h_p(\mathbf{r})) \tag{28}$$

where

$$h_p(\mathbf{r}) = \begin{cases} g_p(\mathbf{r}) \text{ if } g_p(\mathbf{r}) \le 3 \\ H(g_p(\mathbf{r})) \text{ otherwise} \end{cases} \tag{29}$$

and $H(t)$ is the polynomial of degree ≤ 5 that satisfies $H(3) = 3$, $H'(3) = 1$, $H''(3) = 0$ and $H(\pi) = 3.1$, $H'(\pi) = 0$, $H''(\pi) = 0$; $H(t)$ is convex and monotone on $[3, \pi]$. Then $\chi(h_p(\mathbf{r}))$, $\hat{f}(\mathbf{r})$ are C^2 in S^d.

(ii) Modifying the basis function itself

$$\hat{f}(\mathbf{r}) = \sum_{p=1}^{P} \lambda_p \psi(g_p(\mathbf{r})) \tag{30}$$

where

$$\psi(t) = \begin{cases} \chi(t) & \text{if } t \leq 3 \\ \Psi(t) & \text{if } 3 < t \leq \pi \end{cases} \tag{31}$$

and $\Psi(t)$ is the polynomial of degree ≤ 5 that satisfies $\Psi(3) = \chi(3)$, $\Psi(\pi) = \chi(3.1)$, $\Psi'(3) = \chi'(3)$, $\Psi'(\pi) = 0$, $\Psi''(3) = \chi''(3)$, $\Psi''(\pi) = 0$; then $\psi(g_p(\mathbf{r}))$ is C^2 on S^d, hence $\hat{f}(\mathbf{r})$ is so, too.

In practical applications the constant c^2 has to be chosen. Empirically, a value of $c^2 = 2.2$ area$(S^d)/P$ yielded consistently effective results.

Yet another spherical generalization (Hardy and Goepfert, 1975; Pottmann and Eck, 1990) is defined as

$$\hat{f}(\mathbf{r}) = \sum_{p=1}^{P} \lambda_p \chi_p(\mathbf{r}) \tag{32}$$

where

$$\chi_p(\mathbf{r}) = \sqrt{1 + c^2 - 2c(\mathbf{r}\mathbf{r}_p)} \tag{33}$$

usually with $0 < c < 1$, but $c > 1$ is feasible, too. $c = 3/8$ yields $\chi_p(\mathbf{r}_p) = 5/8$; for large P a smaller value of c is to be preferred; an optimal choice of c is unknown.

The linear system of equations to determine the coefficients has always a unique solution (Micchelli, 1986). However, the system may be ill conditioned, in particular for large P, and require preconditioning (cf. Hogervorst, 1994). The interpolant is not necessarily nonnegative. If the nonnegativity constraint is actually violated for one tuning parameter c_1 another value c_2 may apply.

5 APPROXIMATION BY SPHERICAL SPLINES

Polynomial splines may roughly be thought of as piecewise polynomials of a given polynomial order which are joined together to form a unique function at their defining knots. With polynomial interpolation/approximation in mind one may think of the decoupling of the total number of interpolation conditions (data) and the polynomial order of the interpolant as one major motivation of the development of polynomial splines. In accordance with the best known natural cubic splies the term spline is nowadays used for a function that is defined piecewise or satisfies a specific optimality constraint or both. Key references for splines in probability and statistics are provided by Boneva *et al.*, 1971; Wegman and Wright, 1983; Silverman, 1989.

This communication is confined to an approximation scheme provided by a tensor product of a polynomial and a periodic trigonometric spline of order $m = n = 3$ (Schumaker and Traas, 1991; Traas *et al.*, 1993).

It is again presumed that the data values $z_p \geq 0$, $p = 1, ..., P$, have been sampled from a spherical function $f \in \mathcal{L}^q(S^3)$ at arbitrarily scattered data sites $\mathbf{r}_p$, $p = 1, ..., P$, i.e. $(\mathbf{r}_p, z_p) = (\mathbf{r}_p, f(\mathbf{r}_p))$, $p = 1, ..., P$, provides the available data. Let the function $f(\mathbf{r}) = f(\vartheta, \varphi)$ be given in terms of spherical coordinates lattitude $\vartheta \in [0, \pi]$ and longitude $\varphi \in [0, 2\pi]$ of $\mathbf{r} \in S^3$.

Assume the spline approximation $\hat{f}(\mathbf{r})$ of $f(\mathbf{r})$ is given as

$$\hat{f}(\mathbf{r}) = \hat{f}(\vartheta, \varphi) = \sum_{i=1}^{M} \sum_{j=1}^{N+2} c_{ij} N_i^m(\vartheta) T_j^n(\varphi) \tag{34}$$

where N_i^m, $m = 1, ..., M$, are the usual normalized B-splines of order m associated with the extended partition

$$x_1 = ... = x_m = -\frac{\pi}{2} < x_{m+1} < ... < x_M < \frac{\pi}{2} = x_{M+1} = ... = x_{M+m} \quad (35)$$

and where T_j^n, $n = 1, ..., N$, are the normalized periodic trigonometric B-splines of order n associated with the partition

$$0 = y_n < y_{n+1} < ... < y_{N+n-1} < y_{N+n} = 2\pi \quad (36)$$

$$y_i = y_{i+N} - 2\pi \text{ and } y_{N+n+i} = y_{n+i} + 2\pi, \; i = 1, ..., n-1 \quad (37)$$

Desirable properties of a well defined $\hat{f}(\vartheta, \varphi)$ are

- data fit in the sense of

$$\hat{f}(\mathbf{r}_p) \approx f(\mathbf{r}_p), \; p = 1, ..., P \quad (38)$$

- continuity

$$f(\vartheta, 0) = f(\vartheta, 2\pi), \; -\pi/2 \leq \vartheta \leq \pi/2 \quad (39)$$

$$f(-\pi/2, \varphi) = f_S \text{ and } f(\pi/2, \varphi) = f_N, \; 0 \leq \varphi \leq 2\pi \quad (40)$$

- continuously varying tangent plane

$$\frac{\partial}{\partial\varphi} f(\vartheta, 0) = \frac{\partial}{\partial\varphi} f(\vartheta, 2\pi), \; -\pi/2 \leq \vartheta \leq \pi/2 \quad (41)$$

$$\frac{\partial}{\partial\vartheta} f(-\pi/2, \varphi) = A_S \cos \varphi + B_S \sin \varphi, \; 0 \leq \varphi \leq 2\pi \quad (42)$$

$$\frac{\partial}{\partial\vartheta} f(\pi/2, \varphi) = A_N \cos \varphi + B_N \sin \varphi, \; 0 \leq \varphi \leq 2\pi \quad (43)$$

where A_S, B_S, A_N, B_N are constants.
These side conditions can be written in the form

$$B\mathbf{c} = 0 \quad (44)$$

Data is fit by minimizing the mean square error

$$E(\mathbf{c}) = \sum_{p=1}^{P} [\hat{f}(\mathbf{r}_p) - f(\mathbf{r}_p)]^2 \rightarrow \min \quad (45)$$

Summarily, that is to find the best l^2 solution of the system of equations

$$\sum_{i=1}^{M} \sum_{j=1}^{N+2} c_{ij} N_i^m(\vartheta_p) T_j^n(\varphi_p) = f(\mathbf{r}_p), \; p = 1, ..., P \quad (46)$$

or equivalently

$$Ac = (f(\mathbf{r}_1), \ldots, f(\mathbf{r}_p))^t \tag{47}$$

subject to

$$Bc = 0 \tag{48}$$

A solution can be obtained be the Householder transformation method. The approximate is C^1 on S^3.

This scheme of spline approximation generalizes to a better order of smoothness as well as to hyperspheres in a straightforward manner. Its major advantage is that repeated application does not change the initial spline approximation, because the spline approximation of a given spline is this spline itself. While B-splines are nonnegative functions any linear combination need not to be nonnegative. If the required nonnegativity of $\hat{f}$ is violated it may be corrected by appropriate subdivision (cf. Dyn *et al.*, 1990) and final correction of corresponding control points or some other numerical heuristics. Promising results have been reported by Traas *et al.* (1993).

6 CONCLUSIONS

This communication is merely a brief review of methods of genuinely spherical interpolation and approximation. It is in no way complete, but rather a subjective selection of suggested readings. Any major omission is owed to the author's limited knowledge.

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