Journal of Applied Mathematics and Stochastic Analysis, **10**:2 (1997), 179-186.

REACTION DIFFUSION EQUATIONS AND QUADRATIC CONVERGENCE

A.S. VATSALA and MOHAMED A. MAHROUS
University of Southwestern Louisiana
Department of Mathematics
Lafayette, LA 70504-1010 USA

HADI YAHYA ALKAHBY
Dillard University
Department of Mathematics
New Orleans, LA 70122-3097 USA

(Received January, 1996; Revised August, 1996)

In this paper, the method of generalized quasilinearization has been extended to reaction diffusion equations. The extension includes earlier known results as special cases. The earlier results developed are when (i) the right-hand side function is the sum of a convex and concave function, and (ii) the right-hand function can be made convex by adding a convex function. In our present result, if the monotone iterates are mildly nonlinear, we establish the quadratic convergence as in the quasilinearization method. If the iterates are totally linear then the iterates converge semi-quadratically.

Key words: Generalized Quasilinearization, Upper and Lower Solutions.

AMS subject classifications: 35K57, 35A35.

1. Introduction

The method of quasilinearization [1, 2, 3] is known to be a constructive approach to prove the existence of a solution of initial and boundary value problems. However, this method is applicable only if the right-hand side function is convex or concave. Also, the method yields either an increasing or decreasing sequence of approximate solutions which converge quadratically to the exact solution. The main advantage of the method is that the iterates are solutions of linear differential equations. Recently, the method has been extended, generalized, and revitalized so that it applies to a larger class of functions. See [6-13, 15-19] for details. In addition, two-sided bounds for the solution are obtained as in the monotone method. This method is now referred to as *generalized quasilinearization*. Recently, the method of generalized quasilinearization was extended to a dynamic system on time scales [13] so that it applies to many situations. This paper deals with an extension of the method of generalized quasilinearization to reaction diffusion equations. The present result yields the earlier know results [19, 21] as special cases.

2. Preliminaries

In this section we list the assumptions and recall some known existence and comparison theorems which are needed to establish our main result. See [4, 5, 15, 21] for more details.

Consider the reaction diffusion system with initial and boundary value problem (IBVP for short) of the form

$$\begin{aligned} \mathfrak{L}u &= f(t,x,u) \text{ in } Q_T \\ Bu &= \phi \text{ on } \Gamma_T \\ u(0,x) &= u_0(x) \text{ in } \bar{\Omega}, \end{aligned} \tag{2.1}$$

where Ω is a bounded domain in R^m with boundary $\partial\Omega \in C^{2+\alpha}$ and closure $\bar{\Omega}$, $Q_T = (0,T] \times \Omega$, $\Gamma_T = (0,T) \times \partial\Omega$, $\bar{Q}_T = [0,T] \times \bar{\Omega}$, $\bar{\Gamma}_T = [0,T] \times \partial\Omega$, $T > 0$. Here $\mathfrak{L}$ is a second order differential operator defined by

$$\mathfrak{L} = \frac{\partial}{\partial t} - L \tag{2.2}$$

$$L = \sum_{i,j=1}^{m} a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i(t,x)\frac{\partial}{\partial x_i} \tag{2.3}$$

and B is the boundary operator given by

$$Bu = p(t,x)u + q(t,x)\frac{du}{d\gamma}, \tag{2.4}$$

where $\frac{du}{d\gamma}$ denotes the normal derivative of u, and $\gamma(t,x)$ is the unit outward normal vector field on $\partial\Omega$ for $t \in [0,T]$.

We list the following assumptions for convenience.

(A_0) (i) For each $i,j = 1,\ldots,m, a_{ij}, b_j \in C^{\alpha/2,\alpha}[\bar{Q}_T, R]$ and $\mathfrak{L}$ is strictly uniformly parabolic in $\bar{Q}_T$;

(ii) $p,q \in C^{1+\alpha/2,1+\alpha}[\bar{\Gamma}_T, R], p(t,x) > 0,\ q(t,x) \geq 0$ on Γ_T;

(iii) $\partial\Omega$ belongs to $C^{2+\alpha}$;

(iv) $f \in C^{\alpha/2,\alpha}[[0,T] \times \bar{\Omega} \times R, R]$, that is $f(t,x,u)$ is Hölder continuous in t and (x,u) with exponent $\frac{\alpha}{2}$ and α respectively;

(v) $\phi \in C^{1+\alpha/2,1+\alpha}[\bar{\Gamma}_T, R]$, and $u_0(x) \in C^{2+\alpha}[\bar{\Omega}, R]$;

(vi) The initial boundary value problem (2.1) satisfies the compatibility condition of order $[\frac{(1+\alpha)}{2}]$. See [4] for definition.

Definition 2.1: We say a function $v_0 \in C^{1,2}[\bar{Q}_T, R]$ is called a *lower solution* of (2.1), if

$$\mathfrak{L}v_0 \leq f(t,x,v_0),$$
$$v_0(0,x) \leq u_0(x), Bv_0(t,x) \leq \phi(x),$$

and *upper solution* of (2.1) if reversed inequality holds.

We denote the closed set

$$\Lambda = [u: v_0(t,x) \leq u \leq w_0(t,x), (t,x) \in \bar{Q}_T].$$

We recall a known existence result which proves the existence of a solution of (2.1) in the closed set defined by means of the upper and lower solution of (2.1).

Theorem 2.1: *Assume* (A_0) *holds, and that there exists* v_0 *and* $w_0 \in C^{1,2}[\bar{Q}_T, R]$ *which are lower and upper solutions of* (2.1) *such that* $v_0(t,x) \leq w_0(t,x)$ *on* $\bar{Q}_T$. *Then the initial boundary value problem* (2.1) *has a solution belonging to* $C^{1+\alpha/2, 2+\alpha}[\bar{Q}_T, R]$ *such that* $v_0(t,x) \leq u(t,x) \leq w_0(t,x)$ *on* $\bar{Q}_T$.

See [4, 14, 19] for details. Next we give two comparison theorems which we need in the main result to prove the monotonicity of the iterates and quadratic convergence part respectively.

Theorem 2.2: *Assume that*

(*i*) $v, w \in C^{1,2}[\bar{Q}_t, R], f \in C[\bar{Q}_T \times R, R]$ *and*

$$\mathfrak{L}v \leq f(t,x,v),$$

$$\mathfrak{L}w \geq f(t,x,w) \text{ on } \bar{Q}_T,$$

(*ii*) (*a*) $v(0,x) \leq w(0,x), x \in \bar{\Omega}$,

(*b*) $Bv(t,x) \leq Bw(t,x)$ *on* Γ_T, *where the boundary operator* B *is as in* (2.4) *such that* $p(t,x) > 0$, $q(t,x) \geq 0$ *and* $p(t,x) + q(t,x) > 0$ *on* Γ_T.

Then if $f(t,x,u)$ *is Lipschitzian in* u *for a constant* $L > 0$, *then* $v(t,x) \leq w(t,x)$.

See [4] for the details for the proof.

The next result is a special case of Theorem 10.2.1 of [5].

Theorem 2.3: *Suppose that*

(*i*) $m \in C^{1,2}[\bar{Q}_T, R_+]$ *such that* $\mathfrak{L}m \leq f(t,x,m)$ *where* $f(t,x,u) \in C[Q_T \times R, R]$ *where the operator* $\mathfrak{L}$ *is parabolic,*

(*ii*) $g \in C[[0,T] \times R_+, R]$ *and let* $r(t,0,y_0) \geq 0$ *be the maximal solution of the differential equations*

$$y' = g(t,y), y(0) = y_0 \geq 0,$$

existing for $t \geq 0$ *and*

$$f(t,x,z) \leq g(t,z),\ z \geq 0;$$

(*ii*) $m(0,x) \leq r(0,0,y_0)$ *for* $x \in \bar{\Omega}$.

Then $m(t,x) \leq r(t,0,y_0)$ *on* $\bar{Q}_T$.

3. Generalized Quasilinearization

Theorem 3.1: *Suppose that there exist functions* $v_0, w_0, S_j, j = 1,2$ *under the following assumptions:*

(A_1) $v_0, w_0 \in C^{1,2}[\bar{Q}_T, R]$, $\mathfrak{L}v_0 \leq S_j(t,x,v_0,v_0,w_0)$ *and* $\mathfrak{L}w_0 \geq S_j(t,x,w_0,v_0,w_0)$ *for* $j = 1,2$ *such that* $v_0(0,x) \leq u_0(x) \leq w_0(0,x)$ *in* $\bar{\Omega}$, $Bv_0(x) \leq \varphi(x) \leq Bw_0(x)$ *on* Γ_T, $v_0 \leq w_0$ *on* $\bar{Q}_T$;

(A_2) $S_j \in C^{\alpha/2,\alpha}[[0,T] \times \bar{\Omega} \times \Lambda^3, R]$, *that is* S_j *is Hölder continuous in* t *and* x,

u with exponent $\alpha/2$ and α respectively, where $S_j(t,x,u,v,w)$ is such that

$$S_1(t,x,u,u,w) = f(t,u), S_2(t,x,u,v,u) = f(t,u),$$
$$\text{and } S_j(t,x,u,u,u) = f(t,u);$$

(A_3) *$S_1(t,x,u,v,w) \le S_1(t,x,u,u,w)$ if $v \le u$ for each w on Λ and $S_2(t,x,u,v,w) \ge S_2(t,x,u,v,u)$ if $w \le u$ for each v on Λ;*

(A_4) *Further, S_j's are such that*

$$|\, S_j(t,x,u,u,u) - S_j(T,x,u_1,v,w)\,|$$
$$\le M\,|\, u - u_1 \,| + N[\,|\, u - v\,|^{1+\eta} + |\, u - w \,|^{1+\eta}]$$

for $0 < \eta \le 1$, where M, N are nonnegative constants.

Then, there exist monotone sequences $\{v_n(t,x)\}$, and $\{w_n(t,x)\}$ which converge uniformly to the unique solution of (2.1) on $\bar{Q}_T$ and the convergence is superlinear.

Proof: Consider the initial boundary value problems

$$\left.\begin{aligned} \pounds v_1 &= S_1(t,x,v_1,v_0,w_0) \text{ in } Q_T, \\ v_1(0,x) &= u_0(x) \text{ on } \bar{\Omega},\ Bv_1(t,x) = \phi \text{ on } \Gamma_T, \end{aligned}\right] \tag{3.1}$$

and

$$\left.\begin{aligned} \pounds w_1 &= S_2(t,x,w_1,v_0,w_0) \text{ in } Q_T, \\ w_1(0,x) &= u_0(x) \text{ on } \bar{\Omega},\ Bw_1(t,x) = \phi \text{ on } \Gamma_T, \end{aligned}\right] \tag{3.2}$$

where $v_0(0,x) \le u_0(x) \le w_0(0,x)$ and $Bv_0(t,x) \le \phi \le Bw_0(t,x)$ on $\bar{\Omega}$ and Γ_T, respectively. With assumptions (A_1) and (A_2) we have

$$\pounds v_0 \le f(t,x,v_0) = S_1(t,x,v_0,v_0,w_0)$$

and

$$\pounds w_0 \ge f(t,x,w_0) = S_1(t,x,w_0,v_0,w_0).$$

Consequently, Theorem 2.1 yields the existence of a unique solution $v_1(t,x)$ of (3.1) satisfying $v_0(t,x) \le v_1(t,x) \le w_0(t,x)$ on $\bar{Q}_T$.

Similarly, in view of (A_1) and (A_2), we also have

$$\pounds v_0 \le f(t,x,v_0) \le S_2(t,x,v_0,v_0,w_0),$$
$$\pounds w_0 \ge f(t,x,w_0) \ge S_2(t,x,w_0,v_0,w_0);$$

which, by Theorem 2.1, yields the existence of a unique solution $w_1(t,x)$ of (3.2) with $v_0(t,x) \le w_1(t,x) \le w_0(t,x)$ on $\bar{Q}_T$.

Now, since $v_0 \le v_1$ and $w_1 \le w_0$ on $\bar{Q}_T$, using (A_3) we have,

$$\pounds v_1 \le S_1(t,x,v_1,v_0,w_0) \le S_1(t,x,v_1,v_1,w_0) = f(t,v_1)$$
$$\pounds w_1 \ge S_2(t,x,w_1,v_0,w_0) \ge S_2(t,x,w_1,v_0,w_1) = f(t,w_1).$$

Hence, by Theorem 2.2, we get $v_1(t,x) \le w_1(t,x)$ on $\bar{Q}_T$ and this proves that

$$v_0 \le v_1 \le w_1 \le w_0 \text{ on } \bar{Q}_T. \tag{3.3}$$

Furthermore, it proves that v_1 and w_1 are lower and upper solutions of (2.1).

Assume now that for some $k > 1$ and for $(t,x) \in \bar{Q}_T$,

$$\begin{aligned} \mathcal{L}v_k &\leq f(t,x,v_k) \text{ in } Q_T, \\ v_k(0,x) &= u_0(x) \text{ on } \bar{\Omega}, \\ Bv_k(t,x) &= \phi \text{ on } \Gamma_T, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \mathcal{L}w_k &\geq f(t,x,w_k) \text{ in } Q_T, \\ w_k(0,x) &= u_0(x) \text{ on } \bar{\Omega}, \\ Bw_k(t,x) &= \phi \text{ on } \Gamma_T, \end{aligned} \tag{3.5}$$

and $v_0 \leq v_k \leq w_k \leq w_0$ on $\bar{Q}_T$. Certainly it holds true for $k = 1$. Then consider the initial boundary value problems

$$\begin{aligned} \mathcal{L}v_{k+1} &= S_1(t,x,v_{k+1},v_k,w_k) \text{ on } Q_T, \\ v_{k+1}(0,x) &= u_0(x) \text{ on } \bar{\Omega}, \\ Bv_{k+1}(t,x) &= \phi \text{ on } \Gamma_T, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \mathcal{L}w_{k+1} &= S_2(t,x,w_{k+1},v_k,w_k) \text{ on } Q_T, \\ w_{k+1}(0,x) &= u_0(x) \text{ on } \bar{\Omega}, \\ Bv_{k+1}(t,x) &= \phi \text{ on } \Gamma_T. \end{aligned} \tag{3.7}$$

It is easy to see from assumptions (A_2), that

$$\begin{aligned} \mathcal{L}v_k \leq f(t,x,v_k) &= S_1(t,x,v_k,v_k,w_k) \text{ in } Q_T, \\ v_k(0,x) &= u_0(x) \text{ on } \bar{\Omega}, \\ Bv_k(0,x) &= \phi \text{ on } \Gamma_T, \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}w_k \geq f(t,x,w_k) &= S_2(t,x,w_k,v_k,w_k) \text{ in } Q_T, \\ w_k(0,x) &= u_0(x) \text{ on } \bar{\Omega}, \\ Bw_k(0,x) &= \phi \text{ on } \Gamma_T. \end{aligned}$$

By Theorem 2.1, there exists a unique solution $v_{k+1}(t,x)$ of (3.6) satisfying

$$v_k(t,x) \leq v_{k+1}(t,x) \leq w_k(t,x) \text{ on } \bar{Q}_T.$$

Similarly, one can show the existence of a unique solution $w_k(t,x)$ of (3.7) satisfying $v_k(t,x) \leq w_{k+1}(t,x) \leq w_k(t,x)$ on Q_T. Using (A_3) and the facts that $v_k \leq v_{k+1}$ and $w_{k+1} \leq w_k$, we can see that

$$\mathcal{L}v_{k+1} = S_1(t,x,v_{k+1},v_k,w_k) \leq S_1(t,x,v_{k+1},v_{k+1},w_k) = f(t,x,v_{k+1})$$

and

$$\mathcal{L}w_{k+1} = S_2(t,x,w_{k+1},v_k,w_k) \geq S_2(t,x,w_{k+1},v_k,w_{k+1}) = f(t,x,w_{k+1}).$$

By Theorem 2.2, it follows that $v_{k+1} \leq w_{k+1}$ on $\bar{Q}_T$. Thus we have

$$v_k \leq v_{k+1} \leq w_{k+1} \leq w_k \text{ on } \bar{Q}_T.$$

By induction, we then we have for all n,

$$v_0 \leq v_1 \leq v_2 \leq \ldots \leq v_n \leq w_n \leq \ldots \leq w_1 \leq w_0 \text{ on } \bar{Q}_T,$$

with

$$\mathcal{L}v_{n+1} = S_1(t,x,v_{n+1},v_n,w_n) \text{ in } Q_T,$$
$$v_{n+1}(0,x) = u_0(x) \text{ on } \bar{\Omega},$$
$$Bv_{n+1}(t,x) = \phi \text{ on } \Gamma_T,$$

and

$$\mathcal{L}w_{n+1} = S_2 t,x,w_{n+1},v_n,w_n) \text{ in } Q_T,$$
$$w_{n+1}(0,x) = u_0(x) \text{ on } \bar{\Omega},$$
$$Bw_{n+1}(t,x) = \phi \text{ on } \Gamma_T.$$

Employing standard arguments and using Theorem 2.2, we can conclude that the sequences $\{v_n(t,x)\}$, and $\{w_n(t,x)\}$ converge uniformly and monotonically to the unique solution $u(t,x)$ on (2.1) on $\bar{Q}_T$.

In order to prove superlinear convergence of $v_n(t,x)$ and $w_n(t,x)$ to $u(t,x)$, we set $p_{n+1}(t,x) = u(t,x) - v_n(t,x)$ and $q_{n+1}(t,x) = w_n(t,x) - u(t,x)$ so that $p_{n+1}(t,x) \geq 0$ and $q_{n+1}(t,x) \geq 0$ on $\bar{Q}_T$. Also, we have $p_{n+1}(0,x) = 0 = q_{n+1}(0,x)$ on $\bar{\Omega}$ and $Bp_{n+1}(t,x) = 0 = Bq_{n+1}(t,x)$ on Γ_T. Using (A_4) we obtain

$$\mathcal{L}p_{n+1}(t) \leq Mp_{n+1}(t,x) + N[\,|\,p_n(t,x)\,|^{1+\eta} + \,|\,q_n(t,x)\,|^{1+\eta}], \text{ on } \bar{Q}_T.$$

Now using Theorem 2.3 and computing the solution of the corresponding ordinary linear differential equation we get

$$0 \leq p_{n+1}(t,x) \leq \int_0^t e^{M(t-s)} N \max_{Q_T}[\,|\,p_n(s)\,|^{1+\eta} + \,|\,q_n(s)\,|^{1+\eta}]ds.$$

This in turn proves

$$\max_{Q_T}|\,u(t,x) - v_{n+1}(t,x)\,| \leq \frac{(e^{MT}-1)N}{M}[\max_{Q_T}|\,u(t,x) = v_n(t,x)\,|^{1+\eta}$$
$$+ \max_{Q_T}|\,w_n(t,x) - u(t,x)\,|^{1+\eta}].$$

Similarly, we can get the estimate

$$\max_{Q_T}|\,w_{n+1}(t,x) - u(t,x)\,| \leq \frac{(e^{MT}-1)N}{M}[\max_{Q_T}|\,u(t,x) - v_n(t,x)\,|^{1+\eta}$$
$$+ \max_{Q_T}|\,w_n(t,x) - u(t,x)\,|^{1+\eta}].$$

This completes the proof.

The following result can be proved as an application of Theorem 3.1.

Theorem 3.2: *Assume that all of* (A_0) *holds except (iv). Furthermore, assume that*

(A_1) v_0 *and* $w_0 \in C^{1,2}[\bar{Q}_T, R]$ *which are lower and upper solutions of* (2.1) *such* *that* $v_0(t,x) \leq w_0(t,x)$ *on* $\bar{Q}_T$.

(A_2) *Let* $f(t,x,u) = f_1(t,x) + f_2(t,x,u) + f_3(t,x,u)$ *are such that* $f_1(t,x,u) + \Phi(t,x,u)$ *and* $\Phi(t,x,u)$ *are uniformly convex in* u *on* Λ *(i.e.,* $f_{1uu} + \Phi_{uu} \geq 0$ *and* $\Phi(tx,u) \geq 0$*). Also let* $f_2(t,x,u) + \Psi(t,x,u)$ *and* $\Psi(t,x,u)$ *be uniformly concave in* u *(i.e.,* $f_{2uu} + \Psi_{uu} \leq 0$ *and* $\Psi(t,x,u) \leq 0$*) on* Λ, *and* $f_3(t,x,u)$ *be Lipschitzian in* u *on* Λ, *i.e.,*

$$| f_3(t,x,u_1) - f_3(t,x,u_2) | \leq \ell \, | u_1 - u_2 |;$$

In addition, let $F(t,x,u) = f_{1u}(t,x,u) + \Phi_u(t,x,u), G(t,x,u) = f_{2u}(t,x,u) + \Psi(t,x,u)$ *and* $f_3(t,x,u) \in C^{\alpha/2,\alpha}[[0,T] \times \bar{\Omega} \times R, R]$. *That is,* $F(t,x,u)$, $G(t,x,u)$, $f_3(t,x,u)$ *are Hölder continuous in* t, x *and* u *of order* $\alpha/2$, α *respectively. Then there exists monotone sequences* $\{v_n(t,x)\}$ *and* $\{w_n(t,x)\}$ *which converge uniformly and monotonically to the unique solution of* (2.1) *and the convergence is quadratic.*

Proof: Choose S_j as follows:

$$S_1(t,x,u,v,w) = f_1(t,x,v) + f_2(t,x,v) + f_3(t,x,u)$$
$$+ [F_u(t,x,v) + G_u(t,x,w) - \Phi_u(t,x,w) - \Psi_u(t,x,v)](u - v)$$

and

$$S_2(t,x,u,v,w) = f_1(t,x,w) + f_2(t,x,w) + f_3(t,x,u)$$
$$+ [F_u(t,x,v) + G_u(t,x,w) - \Phi_u(t,x,w) - \Psi_u(t,x,v)](u - w).$$

One can easily verify that S_j, $j = 1,2$, defined above, satisfy all the hypotheses of Theorem 3.1. Hence the conclusion follows.

We note that Theorem 3.2 includes results of [21] as a special case if we choose $f_2 = f_3 = 0$ in Theorem 3.2. Also the iterates generated from (3.6) and (3.7) from the S_j defined above are nonlinear due to $f_3(t,x,u)$ term. If we make it linear as in the monotone method we get semi-quadratic convergence as in [18] for initial value problems.

References

[1] Bellman, R., *Methods of Nonlinear Analysis*, Vol. II, Academic Press, New York 1973.

[2] Bellman, R., and Kalaba, R., *Quasilinearization and Nonlinear Boundary Value Problems*, American Elsevier, New York 1965.

[3] Chan, C.Y., Positive solutions for nonlinear parabolic second initial boundary value problem, *Quart. Appl. Math.* **31** (1974), 443-454.

[4] Ladde, G.S., Lakshmikantham, V. and Vatsala, A.S., *Monotone Iterative Techniques for Nonlinear Differential Equations*, Pitman, Boston 1985.

[5] Lakshmikantham, V., Leela, S., *Differential and Integral Inequalities*, Vol. II, Academic Press, New York 1968.

[6] Lakshmikantham, V., and Malek, S., Generalized quasilinearization, *Nonlinear World* **1** (1994), 59-64.

[7] Lakshmikantham, V., Leela, S., and McRae, F.A., Improved generalized quasilinearization method, *Nonlinear Anal.*, to appear.

[8] Lakshmikantham, V., An extension of the method of quasilinearization, *JOTA*,

to appear.

[9] Lakshmikantham, V., Leela, S. and Sivasundaram, S., Extensions of the method of quasilinearization, *JOTA*, to appear.

[10] Lakshmikantham, V. and Köksal, S., Another extension of the method of quasilinearization, *Proc. of Dynamic Systems and Applications* **1** (1994), 205-211.

[11] Lakshmikantham, V., Further improvement of generalized quasilinearization method, *Nonlinear Anal.*, to appear.

[12] Lakshmikantham, V. and Shahzad, N., Further generalization of a generalized quasilinearization method, *J. Appl. Math. Stoch. Anal.* **7**:4 (1994).

[13] Leela, S. and Vatsala, A.S., Dynamic systems on time scale and generalized quasilinearization, *Nonlinear Studies*, to appear.

[14] Malek, S. and Vatsala, A.S., Method of generalized quasilinearization of for second order boundary value problem, *Inequalities and Applications* **3**, Volume Dedicated to W. Walter, to appear.

[15] Pao, C.V., *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, New York 1992.

[16] Shahzad, N. and Vatsala, A.S., Improved generalized quasilinearization method for second order boundary value problem, *Dyn. Syst. and Appl.* **4** (1995), 79-86.

[17] Shahzad, N. and Vatsala, A.S., Extension of the method of generalized quasilinearization for second order boundary value problems, *Appl. Anal.* **58** (1995), 77-83.

[18] Stutson, D. and Vatsala, A.S., Quadratic and semi-quadratic convergence of IVP, *Neural Parallel Sci. Comp.* **3** (1995), 235-248.

[19] Vatsala, A.S., An extension of the method of quasilinearization for reaction diffusion equations, *Comparison Methods and Stability Theory* **162** (1994), 331-337.

[20] Vatsala, A.S., Recent advances in the method of quasilinearization, *Proc. on 4th Colloquium on Differential Equations*, VSP, The Netherlands (1994), 277-285.

[21] Vatsala, A.S., Generalized quasilinearization and reaction diffusion equations, *Nonlinear Times and Digest* **1**:1 (1994), 211-220.

[22] Walter, W., *Differential and Integral Inequalities*, Springer-Verlag, New York 1970.