MEAN NUMBER OF REAL ZEROS OF A RANDOM TRIGONOMETRIC POLYNOMIAL IV

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If a_j ($j = 1, 2, \dots, n$) are independent, normally distributed random variables with mean 0 and variance 1, if p is one half of any odd positive integer except one, and if ν_{np} is the mean number of zeros on $(0, 2\pi)$ of the trigonometric polynomial $a_1 \cos x + 2^p a_2 \cos 2x + \dots + n^p a_n \cos nx$, then $\nu_{np} = \mu_p \{(2n+1) + D_{1p} + (2n+1)^{-1} D_{2p} + (2n+1)^{-2} D_{3p}\} + O\{(2n+1)^{-3}\}$, in which $\mu_p = \{(2p+1)/(2p+3)\}^{1/2}$, and D_{1p} , D_{2p} and D_{3p} are explicitly stated constants.

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1. Introduction

Suppose that a_j ($j = 1, 2, \dots, n$) are independent, normally distributed random variables with mean 0 and variance 1, and that ν_{np} is the mean value of the number of zeros on the interval $(0, 2\pi)$ of the random trigonometric polynomial

$$\sum_{j=1}^n j^p a_j \cos jx, \quad (1.1)$$

in which p is a nonnegative real number. Das [1] has shown that, for large n ,

$$\nu_{np} = 2\mu_p n + O(n^{1/2}), \mu_p = \{(2p+1)/(2p+3)\}^{1/2}. \quad (1.2)$$

When $p = 0$, the author [2] has shown that the error term $O(n^{1/2})$ is actually $O(1)$. Moreover, the error term is also $O(1)$ when p is a positive integer [3]. In fact, if p is a nonnegative integer, there exist constants $D_{0p} = 1$, D_{1p} , D_{2p} and D_{3p} such that

$$\nu_{np} = (2n+1)\mu_p \sum_{r=0}^3 (2n+1)^{-r} D_{rp} + O\{(2n+1)^{-3}\}. \quad (1.3)$$

The author and Souter [4] have also derived a relation of the form (1.3) when $p = 1/2$.

In this paper we show that (1.3) remains valid when $p = (2s+1)/2$, in which s is a positive integer. In combination with the earlier results, this implies that a relation of the form (1.3) is valid when $2p$ is any nonnegative integer, although we have not been able to construct a unified derivation that covers the various cases in [2], [3], [4]

and this paper. We have also not been able to extend our techniques to the case in which $2p$ is not a nonnegative integer.

The logical organization of our analysis is identical to that in [3], although the algebraic details are occasionally different. For this reason we rely heavily on the contents of [3], recording in Section 2 only those portions of the analysis that are essentially new.

2. Derivation of (1.3)

Exactly as in [3], we find that

$$\nu_{np} = 4\pi^{-1} \int_0^{\pi/2} F_{np}(x) dx. \quad (2.1)$$

when $n \geq 2$, in which

$$F_{np} = A_{np}^{-1} (A_{np} C_{np} - B_{np}^2)^{1/2}, \quad (2.2)$$

$$A_{np} = \sum_{j=1}^n j^{2p} \cos^2 jx, \quad (2.3)$$

$$B_{np} = \sum_{j=1}^n j^{2p+1} \sin jx \cos jx, \quad (2.4)$$

$$C_{np} = \sum_{j=1}^n j^{2p+2} \sin^2 jx. \quad (2.5)$$

If we define the constant S_{np} so that

$$2S_{np} = \sum_{j=1}^n j^{2p}, \quad (2.6)$$

and assume that $2p = 2s + 1$, in which s is a positive integer, it is then clear that

$$A_{np} = S_{np} + (-4)^{-s} d^{2s} A_{n,1/2} / dx^{2s}. \quad (2.7)$$

It is known [4, Eqs. (2.4), (2.7), (2.8) and (2.9)] that

$$8A_{n,1/2} = (2n+1)^2 g_0(z) + (2n+1)g_1 + g_2, \quad (2.8)$$

in which

$$z = (2n+1)x, f(x) = \csc x - x^{-1}, \varphi(x) = f^2(x) + 2x^{-1}f(x) = \csc^2 x - x^{-2}, \quad (2.9)$$

$$g_0(z) = (1/2) + z^{-1} \sin z - z^{-2}(1 - \cos z), \quad g_1 = f(x) \sin z, \quad (2.10)$$

$$g_2 = -\{(1/2) + \varphi(x) + f'(x) \cos z\}. \quad (2.11)$$

Lemma 1: *If the constants γ_{rp} ($p - 1/2 = 0, 1, \dots; r = 0, 1, \dots, p + 1/2$) are defined so that*

$$\gamma_{0p} = (2p+1)^{-1}, \gamma_{rp} = {}_{2p}C_{2r-1} f^{(2r-1)}(0) \quad (r = 1, 2, \dots, s), \quad (2.12)$$

$$\gamma_{s+1,p} = \varphi^{(2s)}(0) + f^{(2s+1)}(0),$$

in which ${}_hC_k$ is the binomial coefficient $h!/\{k!(h-k)!\}$, then

$$4^{p+1}S_{np} = \sum_{r=0}^{s+1} (-1)^r \gamma_{rp} (2n+1)^{2p+1-2r}. \quad (2.13)$$

We start the proof of Lemma 1 with the inference from (2.7), (2.8), (2.9), (2.10) and (2.11) that

$$\begin{aligned} 2^{2s+3}A_{np} &= 2^{2s+3}S_{np} + (-1)^s(2n+1)^{2s+2}g_0^{(2s)}(z) - (-1)^s\varphi^{(2s)}(x) \\ &+ (-1)^s(2n+1)\sum_{r=0}^{2s} 2_s C_r f^{(r)}(x)(2n+1)^{2s-r}(\sin z)^{(2s-r)} \\ &- (-1)^s\sum_{r=0}^{2s} 2_s C_r f^{(r+1)}(x)(2n+1)^{2s-r}(\cos z)^{(2s-r)}. \end{aligned} \quad (2.14)$$

If we replace x by 0 in (2.14) and note that $g_0^{(2s)}(0) = (-1)^s(2p+1)^{-1}$, that $A_{np}(0) = 2S_{np}$, and that $f^{(2r)}(0) = 0$ if r is a nonnegative integer, some simple manipulations suffice to prove the lemma.

We will need the explicit representations of A_{np} , B_{np} and C_{np} stated in the following lemma, whose proof is essentially the same as that of Lemma 2 in [3].

Lemma 2: *It is true that*

$$2^{2p+2}A_{np} = (2n+1)^{2p+1} \sum_{r=0}^{2p+1} g_{rp} (2n+1)^{-r}, \quad (2.15)$$

$$2^{2p+3}B_{np} = (2n+1)^{2p+2} \sum_{r=0}^{2p+2} h_{rp} (2n+1)^{-r}, \quad (2.16)$$

$$2^{2p+4}C_{np} = (2n+1)^{2p+3} \sum_{r=0}^{2p+3} k_{rp} (2n+1)^{-r}, \quad (2.17)$$

if the coefficients g_{rp} , h_{rp} and k_{rp} are defined as they were in Lemma 2 in [3], with the following exceptions:

- When in [3, Eqs. (2.19a)-(2.21d)] the letter p occurs as a superscript, or in a range of values of r , it should be replaced by the letter s .
- The coefficients not defined in [3] are defined as follows:

$$g_{2p+1,p} = (-1)^{s+1} \{ \gamma_{s+1,p} + f^{(2s+1)}(x) \cos z + \varphi^{(2s)}(s) \}, \quad (2.18)$$

$$h_{2p+2,p} = (-1)^s \{ f^{(2s+2)}(x) \cos z + \varphi^{(2s+1)}(x) \}, \quad (2.19)$$

$$\begin{aligned} k_{2p+3,p} &= (-1)^{s+1} \{ \varphi^{(2s+2)}(x) - \varphi^{(2s+2)}(0) + f^{(2s+3)}(x) \cos z \\ &- f^{(2s+3)}(0) \}. \end{aligned} \quad (2.20)$$

If we start from (2.15), (2.16) and (2.17), we can reproduce the statements and proofs of Lemmas 3 through 7 of [3] almost verbatim. (The quantity $O(1)$ in line 8, p. 587 of [3] should have been $o(1)$. The quantity g in line 9, p. 587 of [3] could have been, and in this paper should be, g_0 .) The last Lemma 7 exhibits quantities v_{rp} such that

$$\nu_{np} = (2n+1) \sum_{r=0}^{\infty} (2n+1)^{-r} v_{rp} \quad (2.21)$$

when n is sufficiently large. The definition v_{rp} is the same as in [3], although the underlying functions g_0 , g_1 and g_2 are different, and the quantities γ_{mp} , g_{mp} , h_{mp} and k_{mp} have been modified as described in Lemmas 1 and 2 above.

In a similar manner, the constants S_{rmp} ($0 \leq r+m \leq 3$) and S_{rp} ($r=0,1,2,3$) exhibited in Lemmas 8 through 11 of [3] can be shown by arguments essentially the same as those in [3] to be such that

$$\mu_p^{-1}v_{rp} = \sum_{m=0}^{3-r} (2n+1)^{-m}S_{rmp} + (-1)^n(2n+1)^{r-3}S_{rp} + O\{(2n+1)^{r-4}\} \quad (2.22)$$

when $r=0,1,2,3$. (The coefficient $8p^2+6p+3$ of $\cos 2z$ in Eq. (3.6) of [3] should have been $8p^2+12p+3$.) The desired result (1.3) now follows from (2.21) and (2.22) if the coefficients D_{rp} are defined so that

$$D_{rp} = \sum_{m=0}^r S_{r-m,mp} \quad (r=0,1,2,3). \quad (2.23)$$

Just as in [3] we then find the following explicit formulas for D_{rp}

$$\begin{aligned} D_{0p} &= 1, D_{1p} = 2\pi^{-1} \int_0^\infty \{\mu_p^{-1}G_p(z) - 1\}dz, D_{2p} = -(4p+3)/6, \\ D_{3p} &= (3\pi)^{-1} \int_0^\infty \{\mu_p^{-1}J_p(z) + 4p+3 - (4p^2+6p+3)\cos z\}dz - (3\pi)^{-1} \\ &\cdot \int_0^\infty [\mu_p^{-1}H_p(z) - 2(p+1)\sin z - (2z)^{-1}\{4p+3 + (8p^2+12p+3)\cos 2z\}]zdz. \end{aligned} \quad (2.24)$$

These formulas for the coefficients D_{rp} are identical with Equations (3.28) and (3.29) in [3], apart from three inexplicable typographical errors in (3.29). They are, therefore, valid when $2p$ is any nonnegative integer except 0 and 1. On the other hand, the results of [2] and [4] show that $D_{20} = -0.25973$ and $D_{2,1/2} = -1/2$ instead of the values $-1/2$ and $-5/6$ predicted by (2.24).

References

- [1] Das, M., The average number of real zeros of a random trigonometric polynomial, *Proc. Cambridge Philos. Soc.* **64** (1968), 721-729.
- [2] Wilkins, Jr., J.E., Mean number of real zeros of a random trigonometric polynomial, *Proc. Amer. Math. Soc.* **111** (1991), 851-863.
- [3] Wilkins, Jr., J.E., Mean number of real zeros of a random trigonometric polynomial. II. In: *Topics in Polynomials of One and Several Variables and Their Applications* (ed. by Th. M. Rassias, H.M. Srivastava and A. Yanushauskas), World Scientific Co., Singapore (1993), 581-594.
- [4] Wilkins, Jr., J.E. and Souter, S.A., Mean number of real zeros of a random trigonometric polynomial. III, *J. Appl. Math Stoch. Anal.* **8** (1995), 299-317.