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A NEW CRITERION FOR CLOSE-TO-CONVEXITY OF PARTIAL SUMS OF CERTAIN HYPERGEOMETRIC FUNCTIONS[1]

MASSOUD JAHANGIRI
Kent State University, Department of Mathematics
Burton, Ohio 44021 USA

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We consider the partial sums of certain hypergeometric functions and establish conditions imposed on the locations of zeros of those polynomials in order to be close-to-convex in the open unit disk.

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1. Introduction

Let α_i $(i = 1, 2, \ldots p)$ and β_j $(j = 1, 2, \ldots, q)$ be complex numbers with $\beta_j \neq 0, -1, -2, \ldots$; $j = 1, 2, \ldots, q$. The generalized hypergeometric function ${}_pF_q(z)$ is defined by

$$ {}_pF_q(z) = {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \cdot \frac{z^n}{n!}, \tag{1} $$

where $p \leq q + 1$, $(\lambda)_0 = 1$, and $(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \lambda(\lambda + 1)\ldots(\lambda + n - 1)$ if $n = 1, 2, \ldots$. The series given by (1) converges absolutely for $|z| < \infty$ if $p < q + 1$, and for z in the open unit disk $U = \{z: |z| < 1\}$ if $p = q + 1$. For suitable values of α_i and β_j, ${}_pF_q(z)$ is closely related to classes of analytic and univalent functions. A quote from Miller and Mocanu [12] reads: "The surprising use of hypergeometric functions in the recent proof of the Bieberbach conjecture by L. de Branges [4] in 1985 has prompted renewed interest in these classes of functions. Prior to this proof, there had been only a few articles in the literature dealing with the relationship between these special functions and univalent function theory." It is well-known that hypergeometric and univalent functions play important roles in a large variety of problems encountered in applied mathematics, probability and statistics, operations research, signal theory, moment problems, and other areas. For further references and applications see Exton [6, 7] and Rönning [16]. In this paper we introduce a new approach for studying the relationships between classes of hypergeometric and analytic univalent functions. We

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hope this approach will motivate further work in this direction. In Section 2 we discuss the convolution properties of classes of hypergeometric functions ${}_2F_1$. In Section 3 we determine the conditions on the location of the zeros of the partial sums of ${}_2F_1$ that are close-to-convex of order α; $\alpha > 0$.

2. Convolution Properties of ${}_2F_1$

For $p = q+1 = 2$, the series defined by (1) gives rise to the Gaussian hypergeometric series ${}_2F_1(a, b: c: z)$. This reduces to the elementary Gaussian geometric series $1 + z + z^2 + \ldots$ if (i) $a = c$ and $b = 1$ or (ii) $a = 1$ and $b = c$. When $Re(c) > Re(b) > 0$, we obtain

$$ {}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a} dt. $$

As a special case, we see that

$$ {}_2F_1(1, 1; a; z) = (a-1) \int_0^1 \frac{(1-t)^{a-2}}{1-tz} dt $$

and

$$ {}_2F_1(a, 1; 1; z) = \frac{1}{(1-z)^a}, $$

so that

$$ {}_2F_1(1, 1; a; z) * {}_2F_1(a, 1; 1; z) = \frac{1}{1-z} = {}_2F_1(1, 1; 1; z). $$

The operator $*$ stands for the Hadamard product or convolution of two power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, that is, $(f*g)(z) = f(z)*g(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. If f and g are analytic in U then the convolution $f*g$ is also analytic in U. An alternative representation for the Hadamard product is the convolution integral

$$ (f*g)(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \zeta^{-1} f(\tfrac{z}{\zeta}) g(\zeta) d\zeta, \ |z| < 1. $$

Two power series f and g are said to be the *convolution inverses* of each other whenever the convolution $f*g$ gives the identity power series $1/(1-z)$. In this case we write $f = g^{(-1)}$ or $g = f^{(-1)}$ So, ${}_2F_1(1, 1; a; z)$ and ${}_2F_1(a, 1; 1; z)$ are convolution inverses of each other. The function $z\,{}_2F_1(1, 1; 1; z) = z/(1-z)$ is a typical example of a convex univalent function. A function $f(z)$ which is analytic in U is said to be *convex univalent in* U [5] if $f(z)$ is univalent and conformally maps the disk $\{z: |z| < r < 1\}$ onto a convex region so that the boundary of the region is a simple closed convex curve. Alexander [3] showed that $f(z)$ is convex in U if and only if $zf'(z)$ is starlike in U. A function $f(z)$ is said to be *starlike in* U [5] if every point of the image of $\{z: |z| = r < 1\}$ under the conformal mapping f is "visible" from the origin. A necessary and sufficient condition for $f(z)$ to be analytic and starlike of order α; $\alpha \le 1$ in U is that $Re\{\frac{zf'(z)}{f(z)}\} \ge \alpha$, $z \in U$. Let $S^*(\alpha)$ denote the class of functions f that are analytic and starlike of order α; $\alpha \leq 1$ in U. With a simple calculation [17] we see that if $f \in S^*(\alpha)$, $\alpha \leq 1$, then there exists a probability measure μ on ∂U such that

$$f(z) = \int_{\partial U} z_2F_1(2-2\alpha, 1; 1; z\zeta) d\mu(\zeta).$$

In particular,

$$f(z) \ll z_2F_1(2-2\alpha, 1;; z) = \frac{z}{(1-z)^{2-2\alpha}}.$$

For the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $F(z) = \sum_{n=0}^{\infty} A_n z^n$ convergent in U we say $f(z)$ is *dominated* or *marjorized by* $F(z)$, in notation, $f(z) \ll F(z)$, if $|a_n| \leq A_n$.

For $t > -1/2$, $k > -1/2$, and $|x| \leq 1$ we define $F(t,k,x)$ by

$$F(t,k,x) = \sum_{n=0}^{\infty} \frac{P_n^{(t,k)}(x)}{P_n^{(t,k)}(1)} z^{n+1},$$

where $z \in U$, and $P_n^{(t,k)}(x)$ are (see Lewis [11]) the Jacobi polynomials

$$P_n^{(t,k)}(x) = \frac{(1+t)n}{n!} {}_2F_1(-n, t+k+n+1; t+1; \frac{1-x}{2}).$$

From the definitions of $F(t,k,x)$, convex, and starlike functions, it follows that $F(\frac{1}{2}, \frac{1}{2}, x)$ is convex in U and $F(0,0,x)$ is starlike of order $1/2$ in U. Next, we use starlike functions to construct a class of analytic functions which is the subject to our investigation in the following section. We say a function $f(z)$ is *analytic and close-to-convex* (or *linearly accessible*) *in* U [5] if the "complement" of the image of $\{z: |z| \leq r < 1\}$ under conformal mapping f is the union of a family of non-intersecting half-lines. A function $f(z)$ is said to be *analytic and close-to-convex of order* α; $\alpha > 0$ *in* U, in notation $C(\alpha)$, if and only if there exists a function g in $S^*(0)$ such that

$$|\arg \frac{zf'(z)}{g(z)}| < \frac{\alpha\pi}{2}, \quad z \in U.$$

Many authors including ([1, 2 and 10-18]) studied the properties of classes of convex, starlike, and close-to-convex hypergeometric functions. In this paper we investigate the locations of the zeros of partial sums of close-to-convex hypergeometric series. We are not aware of any previous work that has adopted this approach.

3. Partial Sums of ${}_2F_1$

Let ${}_2F_1(a,b;c;z)$ be so that its n-th partial sums ${}_2F_1(a,b;c;z)_n$ can be written as

$$ {}_2F_1(a,b;c;z)_n = \prod_{k=1}^{n} (1 + ze^{i\phi_k}), \tag{2}$$

where $0 \leq \phi_1 \leq \phi_2 \leq \ldots \leq \phi_n \leq \phi_1 + 2\pi$ and $z \in U$. For such polynomials we have

$$z_3F_2(a,b,1;c,2;z)_n = \int_0^z {}_2F_1(a,b;c;\zeta)_n d\zeta.$$

Actually, this is true even without truncating the Gaussian hypergeometric series and holds also true under certain convergence conditions for the infinite generalized hypergeometric series defined by (1).

Using the change of argument properties of close-to-convex functions in conjunction with a result due to the author (let $\beta = 2+\alpha$ in [8]) we have the following theorem.

Theorem 1: *Let $1 \leq \alpha \leq n$. Then $z_3F_2(a,b,1;c,2;z)_n$ is close-to-convex of order α: $\alpha > 0$, if and only if*

$$max\{0,\frac{2(m+1-\alpha)\pi}{n+2}\} \leq \phi_{l+m} - \phi_l \leq min\{\frac{2(m+1+\alpha)\pi}{n+2}, 2\pi\}, \tag{3}$$

where $2 \leq l+m \leq n$ and $1 \leq l,\ m \leq n-1$.

To see the relations between the parameters a, b, c, and ϕ_k we examine a special case when $\alpha = 1$ and $n = 2$. For $\alpha = 1$, inequality (3) reduces to

$$\frac{2m\pi}{n+2} \leq \phi_{l+m} - \phi_l \leq \frac{2(m+2)\pi}{n+2}. \tag{4}$$

Consequently, for $n = 2$ we deduce that if $z_3F_2(a,b,1;c,2;z)_2$ is in $C(1)$, then

$${}_2F_1(a,b;c;z)_2 \ll 1 + \sqrt{2}z + z^2. \tag{5}$$

An extremal case, which satisfies condition (5), is

$${}_2F_1\left(a, \frac{\sqrt{2}-2-2a}{a+2}; \frac{a(\sqrt{2}-2-2a)}{(a+2)\sqrt{2}}; z\right)_2.$$

For a non-extremal case, let $a = \frac{1}{2}$, $b = 2$, and $c = \frac{-1+\sqrt{10}}{2}$. Therefore,

$${}_2F_1\left(\frac{1}{2}, 2; \frac{1}{2}(-1+\sqrt{10}); z\right)_2 = \prod_{k=1}^{2} (1 + ze^{i\phi_k}).$$

In this case, $\phi_1 = \cos^{-1}(\frac{1+\sqrt{10}}{2})$ and $\phi_2 = 2\pi - \phi_1$, which satisfies condition (4) when $n = 2$. As a more general case, let $2 \leq \alpha \leq n = 4$. A necessary condition for $z_3F_2(a,b,1;c,2;z)_4$ to be close-to-convex of order α is that

$${}_2F_1(a,b;c;z)_4 \ll P_4(z;\alpha) = (1+z)^{\alpha-1}\prod_{j=1}^{5-\alpha}(1 + ze^{\frac{(2j+\alpha-6)\pi i}{6}}).$$

The polynomial $P_4(z;\alpha)$ plays an important role in the convolution of close-to-convex hypergeometric functions. The following theorem is a consequence of the above argument and an application of a result due to the author [9].

Theorem 2: *If $z_3F_2(a_1,b_1,1;c_1,2;z)_4$ and $z_3F_2(a_2,b_2,1;c_2,2;z)_4$ are close-to-convex of order α; $\alpha = 2, 3$ and 4, then the convolution $z\{{}_5F_3(a_1,a_2,b_1,b_2,1;c_1,c_2,2;z)_4 * P_4(z;\alpha)\}$ has the same property.*

The above theorem for the case $\alpha = 1$ was proved by Suffridge ([19], Theorem 5). We do not know if a similar convolution invariance property holds for the general case $\alpha > 0$. This remains open. A potential candidate for further investigation is the extremal polynomial

$$P_n(z;\alpha) = (1+z)^{[\alpha]-1} \prod_{j=1}^{n+1-[\alpha]} \left(1 + ze^{\frac{(2j+[\alpha]-n-2)\pi i}{n+2}}\right),$$

where $[\alpha]$ stands for the integer part of α. (See also the Conjecture in [9].)

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