

QUASI-RETRACTIVE REPRESENTATION OF SOLUTION SETS TO STOCHASTIC INCLUSIONS

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Continuous dependence and retraction properties of solution sets to stochastic inclusions $x_t - x_s \in \int_s^t F_\tau(x_\tau) d\tau + \int_s^t G_\tau(x_\tau) d\omega_\tau + \int_s^t \int_{\mathbb{R}^n} H_{\tau,r}(x_\tau) \tilde{\nu}(d\tau, dr)$ are considered.

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1. Introduction

Properties of solution sets to stochastic inclusions play a crucial role in stochastic optimal control theory. The first results dealing with this topic are given in the author's paper [4], in which, by rather strong assumptions the weak compactness of the set of all solutions to stochastic inclusions

$$x_t - x_s \in \int_s^t F_\tau(x_\tau) d\tau + \int_s^t G_\tau(x_\tau) d\omega_\tau + \int_s^t \int_{\mathbb{R}^n} H_{\tau,r}(x_\tau) \tilde{\nu}(d\tau, dr)$$

has been obtained. In the present paper, we show that for a given random variable λ , the solution set \mathcal{C}_λ to an initial value problem

$$x_t - x_s \in \int_s^t F_\tau(x_\tau) d\tau + \int_s^t G_\tau(x_\tau) d\omega_\tau + \int_s^t \int_{\mathbb{R}^n} H_{\tau,r}(x_\tau) \tilde{\nu}(d\tau, dr), x_0 = \lambda,$$

has quasi-retractive representation. As a result, we obtain lower semicontinuous dependence of solution set \mathcal{C}_λ on an initial date.

We begin with basic notations dealing with set-valued stochastic integrals. Some properties of fixed point sets to subtrajectory integral mappings are investigated. Hence, the main results of this paper readily follow.

2. Basic Definitions and Notations

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ be a complete, filtered probability space. Given $T > 0$, let $I = [0, T]$ and let $\mathfrak{B}(I)$ denote the Borel σ -algebra on I . We consider set-valued stochastic processes $(F_t)_{t \in I}$, $(\mathcal{G}_t)_{t \in I}$, and $(\mathfrak{R}_{t,r})_{t \in I, r \in \mathbb{R}^n}$, taking on values from the space $\text{Conv}(\mathbb{R}^n)$ of all nonempty, compact convex subsets of the n -dimensional Euclidean space \mathbb{R}^n . These processes are assumed to be nonanticipative such that $\int_0^T \|F_t\|^2 dt < \infty$; $\int_0^T \|\mathcal{G}_t\|^s dt < \infty$; and $\int_0^T \int_{\mathbb{R}^n} \|\mathfrak{R}_{t,z}\|^2 dt q(dz) < \infty$, a.s., where q is a measure on a Borel σ -algebra \mathfrak{B}^n of \mathbb{R}^n , $A \in \text{Conv}(\mathbb{R}^n)$, and $\|A\| := \sup\{|a| : a \in A\}$. The space $\text{Conv}(\mathbb{R}^n)$ is endowed with the Hausdorff metric h defined in the usual way (i.e., $h(A, B) = \max\{\bar{h}(A, B), \bar{h}(B, A)\}$, for $A, B \in \text{Conv}(\mathbb{R}^n)$, where $\bar{h}(A, B) = \{\text{dist}(a, B) : a \in A\}$ and $\bar{h}(B, A) = \{\text{dist}(b, A) : b \in B\}$). $\text{Cl}(X)$ denotes the family of all nonempty closed subsets of a metric space (X, ρ) .

Filtered, complete probability spaces $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ are assumed to satisfy the usual hypotheses: (i) \mathcal{F}_0 contains all the P -null sets of \mathcal{F} ; and (ii) $\mathcal{F}_t = \bigcap_{u > t} \mathcal{F}_u$, all $t, 0 \leq t < \infty$. As usual, we shall consider a set $I \times \Omega$ as a measurable space with the product σ -algebra $\mathfrak{B}(I) \otimes \mathcal{F}$.

$(X_t)_{t \in I}$ denotes an n -dimensional stochastic process x , understood as a function $x: I \times \Omega \rightarrow \mathbb{R}^n$ with \mathcal{F} -measurable sections x_t , each $t \in I$. This process is measurable if x is $\mathfrak{B}(I) \otimes \mathcal{F}$ -measurable. The process $(x_t)_{t \in I}$ is \mathcal{F}_t -adapted or adapted if x_t is \mathcal{F}_t -measurable for $t \in I$. Every measurable and adapted process is called *nonanticipative*.

The Banach spaces $L^2(\Omega, \mathcal{F}_t, P, \mathbb{R}^n)$ and $L^2(\Omega, \mathcal{F}, P, \mathbb{R}^n)$, with the usual norm $\|\cdot\|_{L_n^2}$, are denoted by $L_n^2(\mathcal{F}_t)$ and $L_n^2(\mathcal{F})$, respectively. $\mathcal{M}^2(\mathcal{F}_t)$ denotes the family (i.e., equivalence classes) of all n -dimensional nonanticipative processes $(f_t)_{t \in I}$ such that $\int_0^T |f_t|^2 dt < \infty$, a.s. We shall also consider a subspace $\underline{\mathcal{L}}^2$ of $\mathcal{M}^2(\mathcal{F}_t)$ defined by $\underline{\mathcal{L}}^2 = \{(f_t)_{t \in I} \in \mathcal{M}^2(\mathcal{F}_t) : |f|_{\underline{\mathcal{L}}^2} < \infty\}$, with $|f|_{\underline{\mathcal{L}}^2}^2 = E \int_0^T |f_t|^2 dt$. Finally, $M_n(\mathcal{F}_t)$ we denote the space (i.e., equivalence classes) of all n -dimensional \mathcal{F}_t -measurable mappings.

$(w_t)_{t \in I}$ defines a one-dimensional \mathcal{F}_t -Brownian motion starting at 0. $\nu(t, A)$ denotes a \mathcal{F}_t -Poisson measure on $I \times \mathfrak{B}^n$. We define a \mathcal{F}_t -centered Poisson measure $\tilde{\nu}(t, A)$, $t \in I$, $A \in \mathfrak{B}^n$ by taking $\tilde{\nu}(t, A) = \nu(t, A) - tq(A)$, $t \in I$, $A \in \mathfrak{B}^n$, where q is a measure on \mathfrak{B}^n such that $E\nu(t, B) = tq(B)$ and $q(B) < \infty$ for $B \in \mathfrak{B}_0^n := \{A \in \mathfrak{B}^n : 0 \notin \bar{A}\}$.

$\mathcal{M}^2(\mathcal{F}_t, q)$ denotes the family (i.e., equivalence classes) of all $\mathfrak{B}(I) \otimes \mathcal{F} \otimes \mathfrak{B}^n$ -measurable and \mathcal{F}_t -adapted functions $h: I \times \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $\int_0^T \int_{\mathbb{R}^n} |h_{t,r}|^2 dt q(dr) < \infty$, a.s.

Recall, a function $h: I \times \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be \mathcal{F}_t -adapted or adapted if $h(t, \cdot, r)$ is \mathcal{F}_t -measurable for every $r \in \mathbb{R}^n$ and $t \in I$. Elements of $\mathcal{M}^2(\mathcal{F}_t, q)$ will be denoted by $h = (h_{t,r})_{t \in I, r \in \mathbb{R}^n}$. Finally, we let $\mathcal{W}_n^2 = \{h \in \mathcal{M}^2(\mathcal{F}_t, q) : |h|_{\mathcal{W}_n^2} < \infty\}$, where $|h|_{\mathcal{W}_n^2}^2 = E \int_0^T \int_{\mathbb{R}^n} |h_{t,r}|^2 dt q(dr)$.

Given $f, g \in \mathcal{M}^2(\mathcal{F}_t)$ and $h \in \mathcal{M}^2(\mathcal{F}_t, q)$, $(\int_0^t f_\tau d\tau)_{t \in I}$, $(\int_0^t g_\tau dw_\tau)_{t \in I}$, and

$(\int_0^t \int_{\mathbb{R}^n} h_{\tau,r} \tilde{\nu}(d\tau, dr))_{t \in I}$ denote their stochastic integrals with respect to Lebesgue measure on \mathbb{R}^+ , the \mathcal{F}_t -Brownian motion $(w_t)_{t \in I}$, and the \mathcal{F}_t -centered Poisson measure $\tilde{\nu}(t, A), t \in I, A \in \mathcal{B}^n$, respectively. For fixed $t \in I$ and $(f, g, h) \in \mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2$, we equate $\mathfrak{J}_t(f) = \int_0^t f_{\tau} d\tau$, $\mathfrak{J}_t(g) = \int_0^t g_{\tau} dw_{\tau}$, and $\mathcal{T}_t(h) = \int_0^t \int_{\mathbb{R}^n} h_{\tau,z} \tilde{\nu}(d\tau, dz)$.

$\mathfrak{J}, \mathfrak{J}$, and \mathcal{T} denote linear mappings defined by $\mathcal{L}^2 \ni f \rightarrow (\mathfrak{J}_t(f))_{t \in I} \in D$, $\mathcal{L}^2 \ni g \rightarrow (\mathfrak{J}_t(g))_{t \in I} \in D$, and $\mathcal{W}^2 \ni h \rightarrow (\mathcal{T}_t(h))_{t \in I} \in D$, respectively. Here, D denotes the family of all n -dimensional \mathcal{F}_t -adapted càdlàg (see [7]) processes $(x_t)_{t \in I}$ such that $E \sup_{t \in I} |x_t|^2 < \infty$. The space D is considered a normed space with norm $\|\xi\|_{\varrho} = \|\sup_{t \in I} |\xi_t|\|_{L^2}$ for $\xi = (\xi_t)_{t \in I} \in D$. It can be verified that $(D, \|\cdot\|_{\varrho})$ is a Banach space.

Given a measure space (X, \mathcal{B}, m) , a set-valued function $\mathfrak{R}: X \rightarrow \text{Cl}(\mathbb{R}^n)$ is said to be \mathcal{B} -measurable if $\{x \in X: \mathfrak{R}(x) \cap C \neq \emptyset\} \in \mathcal{B}$ for every closed set $C \subset \mathbb{R}^n$. For such a multifunction, we define subtrajectory integrals as a set $\mathcal{Y}(\mathfrak{R}) = \{g \in L^2(X, \mathcal{B}, m, \mathbb{R}^n): g(x) \in \mathfrak{R}(x) \text{ a.e.}\}$. We shall assume that the \mathcal{B} -measurable, set-valued function $\mathfrak{R}: X \rightarrow \text{Cl}(\mathbb{R}^n)$ is square integrable bounded (i.e., a real-valued mapping $X \ni x \rightarrow \|\mathfrak{R}(x)\| \in \mathbb{R}_+$ belongs to $L^2(X, \mathcal{B}, m, \mathbb{R})$).

Let $\mathfrak{G} = (\mathfrak{G}_t)_{t \in I}$ be a set-valued stochastic process with values in $\text{Cl}(\mathbb{R}^n)$, (i.e., a family of \mathcal{F} -measurable set-valued mappings $\mathfrak{G}_t: \Omega \rightarrow \text{Cl}(\mathbb{R}^n), t \in I$). We call \mathfrak{G} measurable if it is $\mathcal{B}(I) \otimes \mathcal{F}$ -measurable. Similarly, \mathfrak{G} is said to be \mathcal{F}_t -adapted or adapted if \mathfrak{G}_t is \mathcal{F}_t -measurable for each $t \in I$. A measurable and adapted set-valued stochastic process is called nonanticipative.

We shall also consider $\mathcal{B}(I) \otimes \mathcal{F} \otimes \mathcal{B}^n$ -measurable set-valued mappings $\mathfrak{R}: I \times \Omega \times \mathbb{R}^n \rightarrow \text{Cl}(\mathbb{R}^n)$. These mappings will be denoted by $(\mathfrak{R}_{t,r})_{t \in I, r \in \mathbb{R}^n}$, and called measurable set-valued stochastic processes depending on a parameter $r \in \mathbb{R}^n$. The process $\mathfrak{R} = (\mathfrak{R}_{t,zr})_{t \in I, r \in \mathbb{R}^n}$ is said to be \mathcal{F}_t -adapted or adapted if $\mathfrak{R}_{t,r}$ is \mathcal{F}_t -measurable for each $t \in I$ and $z \in \mathbb{R}^n$. We call this process nonanticipative if it is measurable and adapted.

$\mathcal{M}_{s-v}^2(\mathcal{F}_t)$ and $\mathcal{M}_{s-v}^2(\mathcal{F}_t, q)$ denote families of all nonanticipative set-valued processes $\mathfrak{G} = (\mathfrak{G}_t)_{t \in I}$ and $\mathfrak{R} = (\mathfrak{R}_{t,r})_{t \in I, r \in \mathbb{R}^n}$, respectively, such that

$$\int_0^T \|\mathfrak{G}_t\|^2 dt < \infty \text{ and } \int_0^T \int_{\mathbb{R}^n} \|\mathfrak{R}_{t,r}\|^2 dt dq(dr) < \infty, \text{ a.s.}$$

From Kuratowski and Ryll-

Nardzewski measurable selection theorem (see [3]) it immediately follows that for every $F, \mathfrak{G} \in \mathcal{M}_{s-v}^2(\mathcal{F}_t)$ and $\mathfrak{R} \in \mathcal{M}_{s-v}^2(\mathcal{F}_t, q)$, their subtrajectory integrals $\mathcal{Y}(F) := \{f \in \mathcal{M}^2(\mathcal{F}_t): f_t(\omega) \in F_t(\omega), dt \times P\text{-a.e.}\}$, $\mathcal{Y}(\mathfrak{G}) := \{g \in \mathcal{M}^2(\mathcal{F}_t): g_t(\omega) \in \mathfrak{G}_t(\omega), dt \times P\text{-a.e.}\}$, and $\mathcal{Y}_q(\mathfrak{R}) := \{h \in \mathcal{M}^2(\mathcal{F}_t, q): h_{t,r}(\omega) \in \mathfrak{R}_{t,r}(\omega), dt \times P \times q\text{-a.e.}\}$ are nonempty. Indeed, we let $\Sigma = \{Z \in \mathcal{B}(I) \otimes \mathcal{F}: Z_t \in \mathcal{F}_t, \text{ each } t \in I\}$, where Z_t denotes a section of Z determined by $t \in I$. Σ is a σ -algebra on $I \times \Omega$, and a function $f: I \times \Omega \rightarrow \mathbb{R}^n$ (a multifunction $F: I \times \Omega \rightarrow \text{Cl}(\mathbb{R}^n)$) is nonanticipative if and only if it is Σ -measurable. Therefore, by Kuratowski and Ryll-Nardzewski measurable selection theorem, every nonanticipative set-valued function admits a nonanticipative selector. It is clear that for $F \in \mathcal{M}_{s-v}^2(\mathcal{F}_t)$, such selectors belong to $\mathcal{M}^2(\mathcal{F}_t)$. Similarly, we define on $I \times \Omega \times \mathbb{R}^n$ a σ -algebra $\Sigma = \{Z \in \mathcal{B}(I) \otimes \mathcal{F} \otimes \mathcal{B}^n: Z_t^u \in \mathcal{F}_t, \text{ each } t \in I \text{ and } u \in \mathbb{R}^n\}$, where $Z_t^u = (Z^u)_t$, and Z^u denotes a section of Z determined by $u \in \mathbb{R}^n$. The foregoing arguments can be repeated to obtain the above result for nonanticipative, set-valued

processes depending on a parameter $r \in \mathbb{R}^n$.

It can be verified (see [2, 3]) that for given $F = (F_t)_{t \in I} \in \mathcal{M}_{s-v}^2(\mathcal{F}_t)$, $\mathcal{G} = (\mathcal{G}_t)_{t \in I} \in \mathcal{M}_{s-v}^2(\mathcal{F}_t)$, and $\mathcal{R} = (\mathcal{R}_{t,r})_{t \in I, r \in \mathbb{R}^n} \in \mathcal{M}_{s-v}^2(\mathcal{F}_t, q)$, their stochastic

integrals are defined as families $(\int_0^t F_\tau d\tau)_{t \in I}, (\int_0^t \mathcal{G}_\tau dw_\tau)_{t \in I}$, and

$(\int_0^t \int_{\mathbb{R}^n} \mathcal{R}_{\tau,z} \tilde{\nu}(d\tau, dz))_{t \in I}$ of the subsets of $M(\mathcal{F}_t)$, of the form $\int_0^t \mathcal{F}_\tau d\tau = \{\int_0^t f_\tau d\tau:$

$f \in \mathcal{Y}(F)\}$, $\int_0^t \mathcal{G}_\tau dw_\tau = \{\int_0^t g_\tau dw_\tau: g \in \mathcal{Y}^2(\mathcal{G})\}$ and $\int_0^t \int_{\mathbb{R}^n} h_{\tau,z} \tilde{\nu}(d\tau, dz): h \in \mathcal{Y}_q(\mathcal{R})\}$.

Given $0 \leq \alpha < \beta < \infty$, we also define $\int_\alpha^\beta F_s ds = \{\int_\alpha^\beta f_s ds: f \in \mathcal{Y}^p(F)\}$, $\int_\alpha^\beta \mathcal{G}_s dw_s =$

$\{\int_\alpha^\beta g_s dw_s: g \in \mathcal{Y}^2(\mathcal{G})\}$, and $\int_\alpha^\beta \int_{\mathbb{R}^n} \mathcal{R}_{s,r} \tilde{\nu}(ds, dz) = \{\int_\alpha^\beta \int_{\mathbb{R}^n} h_{s,r} \tilde{\nu}(ds, dz): h \in \mathcal{Y}_q(\mathcal{R})\}$.

The following selection property of set-valued stochastic integrals has been obtained in [5]:

Proposition 1. *Let $F, \mathcal{G} \in \mathcal{M}_{s-v}^2(\mathcal{F}_t), \mathcal{R} \in \mathcal{M}_{s-v}^2(\mathcal{F}_t, q)$, and $(x_t)_{t \in I} \in D$. Then:*

$$x_t - x_s \in \int_s^t F_\tau d\tau + \int_s^t G_\tau dw_\tau + \int_s^t \int_{\mathbb{R}^n} \mathcal{R}_{\tau,r} \tilde{\nu}(d\tau, dr)$$

for $0 \leq s \leq t \leq T$ if and only if there exists $(f, g, h) \in \mathcal{Y}(\mathcal{F}) \times \mathcal{Y}(\mathcal{G}) \times \mathcal{Y}_q(\mathcal{R})$ such that

$$x_t = \int_0^t f_\tau d\tau + \int_0^t g_\tau dw_\tau + \int_0^t \int_{\mathbb{R}^n} h_{\tau,r} \tilde{\nu}(d\tau, dr)$$

for $t \in I$.

3. Stochastic Inclusions and Subtrajectory Integrals Depending on Parameters

Let

$$F = \{(F_t(x))_{t \in I}: x \in \mathbb{R}^n\}, G = \{(G_t(x))_{t \in I}: x \in \mathbb{R}^n\},$$

$$\text{and } H = \{(H_{t,r}(x))_{t \in I, r \in \mathbb{R}^n}: x \in \mathbb{R}^n\}.$$

Assume F, G , and H are such that $(F_t(x))_{t \in I} \in \mathcal{M}_{s-v}^p(\mathcal{F}_t)$, $(G_t(x))_{t \in I} \in \mathcal{M}_{s-v}^2(\mathcal{F}_t)$, and $(H_{t,r}(x))_{t \in I, r \in \mathbb{R}^n} \in \mathcal{M}_{s-v}^2(\mathcal{F}_t^n, q)$, $x \in \mathbb{R}^n$. A stochastic inclusion denoted by $SI(F, G, H)$, corresponding to the aforementioned F, G , and H is the relation:

$$x_t - x_s \in \int_s^t F_\tau(x_\tau) d\tau + \int_s^t G_\tau(x_\tau) dw_\tau + \int_s^t \int_{\mathbb{R}^n} H_{\tau,r}(x_\tau) \tilde{\nu}(d\tau, dz),$$

which is satisfied for every $0 \leq s < t < T$ by a stochastic process $x = (x_t)_{t \in I} \in D$ such that $F \circ x \in \mathcal{M}_{s-v}^2(\mathcal{F}_t)$, $G \circ x \in \mathcal{M}_{s-v}^2(\mathcal{F}_t)$, and $H \circ x \in \mathcal{M}_{s-v}^2(\mathcal{F}_t, q)$, where $F \circ x = (F_t(x_t))_{t \in I}$, $G \circ x = (G_t(x_t))_{t \in I}$, and $H \circ x = (H_{t,r}(x_t))_{t \in I, r \in \mathbb{R}^n}$. Every stochastic process $(x_t)_{t \in I} \in D$, satisfying conditions mentioned above, is said to be a global solution to $SI(F, G, H)$. Given $\lambda \in L_n^2(\mathcal{F}_0)$ we shall consider $SI(F, G, H)$ together with an initial value condition $x_0 = \lambda$. This type of initial value problem will be denoted by $SI_\lambda(F, G, H)$.

We shall assume that F, G , and H satisfy the following condition:

- (\mathcal{A}_1): (i) $F = \{(F_t(x))_{t \in I} : x \in \mathbb{R}^n\}$, $G = \{(G_t(x))_{t \in I} : x \in \mathbb{R}^n\}$, and $H = \{(H_{t,r}(x))_{t \in I, r \in \mathbb{R}^n} : x \in \mathbb{R}^n\}$, such that mappings $\mathbb{R}^+ \times \Omega \times \mathbb{R}^n \ni (t, \omega, x) \rightarrow F_t(x)(\omega) \in \text{Conv}(\mathbb{R}^n)$, $I \times \Omega \times \mathbb{R}^n \ni (t, \omega, x) \rightarrow G_t(x)(\omega) \in \text{conv}(\mathbb{R}^n)$, and $I \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n \ni (t, \omega, r, x) \rightarrow H_{t,r}(x)(\omega) \in \text{Conv}(\mathbb{R}^n)$ are $\Sigma \otimes \mathcal{B}^n$ and $\tilde{\Sigma} \otimes \mathcal{B}^n$ -measurable, respectively;
- (ii) $(F_t(x))_{t \in I}$, $(G_t(x))_{t \in I}$, and $(H_{x,r}(x))_{t \in I, r \in \mathbb{R}^n}$ are uniformly square integrable bounded (i.e., functions $(t, \omega) \rightarrow \sup_{x \in \mathbb{R}^n} \|F_t(x)(\omega)\| \in \mathbb{R}^+$, $(t, \omega) \rightarrow \sup_{x \in \mathbb{R}^n} \|G_t(x)(\omega)\| \in \mathbb{R}^+$, and $(t, \omega, r) \rightarrow \sup_{x \in \mathbb{R}^n} \|H_{t,r}(x)(\omega)\| \in \mathbb{R}^+$) are square integrable on $\mathbb{R}^+ \times \Omega$ and $\mathbb{R}^+ \times \Omega \times \mathbb{R}^n$, respectively.

We denote $B_F = \{u \in \mathcal{L}^2 : |u_t| \leq \sup_{x \in \mathbb{R}^n} \|F_t(x)\| \text{ a.e. on } I \times \Omega\}$, $B_G = \{u \in \mathcal{L}^2 : |v_t| \leq \sup_{x \in \mathbb{R}^n} \|G_t(x)\| \text{ a.e. on } I \times \Omega\}$, and $B_H = \{z \in \mathcal{W}^2 : |z_{t,r}| \leq \sup_{x \in \mathbb{R}^n} \|H_{t,r}(x)\| \text{ a.e. on } I \times \Omega \times \mathbb{R}^n\}$. Then we define $B = B_F \times B_G \times B_H$.

Corollary 1. *If F, G, H satisfy (\mathcal{A}_1) , then B is a nonempty convex and weakly compact subset of $\mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2$. Moreover, for every $(x_t)_{t \in I} \in D$, one has $F \circ x, G \circ x \in \mathcal{M}_{s-v}^2(\mathcal{F}_t)$ and $H \circ x \in \mathcal{M}_{s-v}^2(\mathcal{F}_t^q)$.*

Let Φ be a linear mapping on $\mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2$ defined by $\Phi = \mathfrak{J} + \mathfrak{J} + \mathfrak{T}$, (i.e., $\Phi(f, g, h) = (\mathfrak{J}_t f + \mathfrak{J}_t g + \mathfrak{T}_t h)_{t \in I}$ for $(f, g, h) \in \mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2$). For fixed $\lambda \in L_n^2(\mathcal{F}_0)$, Φ^λ denotes an affine mapping on $\mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2$ defined by $\Phi^\lambda(u, v, r) = \lambda + \Phi(f, g, h)$ for $(f, g, h) \in \mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2$. Given F, G , and H and $\lambda \in L_n^2(\mathcal{F}_0)$, we set

$$\mathfrak{H}_\lambda(x) = \Phi^\lambda(\mathcal{F}(F \circ x) \times \mathcal{F}(G \circ x) \times \mathcal{F}_q(H \circ x))$$

for $x = (x_t)_{t \in I} \in D$. It can be verified (see [4, 5]) that for every $x \in D$, $\mathfrak{H}_\lambda(x)$ is a convex, weakly compact subset of D . \mathfrak{H}_λ denotes a set-valued mapping $D \ni x \rightarrow \mathfrak{H}_\lambda(x) \subset D$. From Proposition 1, it immediately follows that for every $\lambda \in L_n^2(\mathcal{F}_0)$, and F, G , and H satisfying condition (\mathcal{A}_1) , $x \in D$ is a solution to $SI_\lambda(F, G, h)$ if and only if x is a fixed point to \mathfrak{H}_λ .

Suppose $F = \{(F_t(x))_{t \in I} : x \in \mathbb{R}^n\}$, $G = \{(G_t(x))_{t \in I} : x \in \mathbb{R}^n\}$, and $H\{(H_{t,r}(x))_{t \in I, z \in \mathbb{R}^n} : x \in \mathbb{R}^n\}$ satisfy condition (\mathcal{A}_1) and the following condition

- (\mathcal{A}_2) There are $k_F, k_G \in L_1^2(\mathcal{B}(I))$ and $m \in L_1^2(\mathcal{B}(I) \times \mathcal{B}^n)$ such that $h(F_t(x_2), F_t(x_1)) \leq k_F(t) |x_2 - x_1|$, $h(G_t(x_2), G_t(x_1)) \leq k_G(t) |x_2 - x_1|$, and $h(H_{t,r}(x_2), H_{t,r}(x_1)) \leq m(t, r) |x_2 - x_1|$ a.s., each $t \in I$ and $x_1, x_2 \in \mathbb{R}^n$.

Consider for fixed $\lambda \in L_n^2(\mathcal{F}_0)$ a subtrajectory integrals mapping S_λ defined by:

$$S_\lambda(u, v, r) = \mathcal{F}(F \circ \Phi^\lambda(u, v, r)) \times \mathcal{F}(G \circ \Phi^\lambda(u, v, r)) \times \mathcal{F}_q(H \circ \Phi^\lambda u, v, r))$$

for $(u, v, r) \in \mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2$. It is clear that for $\lambda \in L_n^2(\mathcal{F}_0)$, $(u, v, r) \in \mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2$; and F, G , and H satisfying condition (\mathcal{A}_1) , one has $S_\lambda(u, v, r) \in \text{Cl}(\mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2)$. We shall show that if condition (\mathcal{A}_2) is satisfied, then it is possible to renorm a space $\mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2$ by an equivalent norm $\|\cdot\|$ such that $S_\lambda(\cdot)$ is a contraction from $(\mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2, \|\cdot\|)$ into $(\text{Cl}(\mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2), \ell)$, where ℓ is the Hausdorff metric induced by $\|\cdot\|$. A similar result is also true for $S_\cdot(u, v, r): L_n^2(\mathcal{F}_0) \ni \lambda \rightarrow S_\lambda(u, v, z) \in \text{Cl}(\mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2)$. Observe that a norm $\|\cdot\|$ is defined by $\|(u, v, r)\| = \max(\|u\|_{\mathcal{L}^2}, \|v\|_{\mathcal{L}^2}, \|z\|_{\mathcal{W}^2})$, where $\|\cdot\|_{\mathcal{L}^2}$ and $\|\cdot\|_{\mathcal{W}^2}$ are appropriate norms on \mathcal{L}^2 and \mathcal{W}^2 equivalent to $|\cdot|_{\mathcal{L}^2}$ and $|\cdot|_{\mathcal{W}^2}$ defined above.

Finally, observe that for every $A, \tilde{A}, B, \tilde{B} \in \text{Cl}(\mathcal{L}^2)$ and $C, \tilde{C} \in \text{Cl}(\mathcal{W}^2)$ one has:

$$\ell(A \times B \times C, \tilde{A} \times \tilde{B} \times \tilde{C}) \leq \max\{\ell_{\mathcal{L}^2}(A, \tilde{A}), \ell_{\mathcal{L}^2}(B, \tilde{B}), \ell_{\mathcal{W}^2}(C, \tilde{C})\}$$

where $\ell_{\mathcal{L}^2}$ and $\ell_{\mathcal{W}^2}$ are Hausdorff metrics on $\text{Cl}(\mathcal{L}^2)$ and $\text{Cl}(\mathcal{W}^2)$ induced by the norms $\|\cdot\|_{\mathcal{L}^2}$ and $\|\cdot\|_{\mathcal{W}^2}$, respectively.

Proposition 2. *Suppose F, G , and H satisfy (\mathcal{A}_1) and (\mathcal{A}_2) . For every $L > 0$, there are norms $\|\cdot\|_{\mathcal{L}^2}$ and $\|\cdot\|_{\mathcal{W}^2}$ on \mathcal{L}^2 and \mathcal{W}^2 equivalent to $|\cdot|_{\mathcal{L}^2}$ and $|\cdot|_{\mathcal{W}^2}$, respectively, such that:*

$$\begin{aligned} & \ell_{\mathcal{L}^2}(S(F \circ \Phi^\lambda(u, v, r)), S(F \circ \Phi^\lambda(\tilde{u}, \tilde{v}, \tilde{z}))) \\ & \leq L \max(\|u - \tilde{u}\|_{\mathcal{L}^2}, \|v - \tilde{v}\|_{\mathcal{L}^2}, \|z - \tilde{z}\|_{\mathcal{W}^2}), \end{aligned}$$

$$\ell_{\mathcal{L}^2}(S(G \circ \Phi^\lambda(u, v, r)), S(G \circ \Phi^\lambda(\tilde{u}, \tilde{v}, \tilde{z})))$$

$$\leq L \max(\|u - \tilde{u}\|_{\mathcal{L}^2}, \|v - \tilde{v}\|_{\mathcal{L}^2}, \|z - \tilde{z}\|_{\mathcal{W}^2}),$$

and

$$\ell_{\mathcal{L}^2}(S_q(H \circ \Phi^\lambda(u, v, r)), S_q(H \circ \Phi^\lambda(\tilde{u}, \tilde{v}, \tilde{z})))$$

$$\leq L \max(\|u - \tilde{u}\|_{\mathcal{L}^2}, \|v - \tilde{v}\|_{\mathcal{L}^2}, \|z - \tilde{z}\|_{\mathcal{W}^2}),$$

for $(u, v, r), (\tilde{u}, \tilde{v}, \tilde{z}) \in \mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2$.

Proof. Let $L > 0$ be given and fix $(u, v, z), (\tilde{u}, \tilde{v}, \tilde{z}) \in \mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2$. For every $f \in S(F \circ \Phi_t^\lambda(u, v, z))$, there is $\tilde{f} \in S(F \circ \Phi_t^\lambda(\tilde{u}, \tilde{v}, \tilde{z}))$ such that:

$$\begin{aligned} |f_t - \tilde{f}_t| & \leq h(F_t(\Phi_t^\lambda(u, v, z)), F_t(\Phi_t^\lambda(\tilde{u}, \tilde{v}, \tilde{z}))) \leq k_F(t) |\Phi_t^\lambda(u, v, z) - \Phi_t^\lambda(\tilde{u}, \tilde{v}, \tilde{z})| \\ & \leq k_F(t) \left\{ \int_0^t |u_r - \tilde{u}_r| d\tau + \left| \int_0^t [v_r - \tilde{v}_r] d\omega_\tau \right| + \left| \int_0^t \int_{\mathbb{R}^m} [z_{\tau,r} - \tilde{z}_{\tau,r}] \tilde{v} (d\tau, dr) \right| \right\} \end{aligned}$$

a.s., each $t \in I$. Similarly, for every $g \in S(G \circ \Phi^\lambda(u, v, z))$ and $h \in S_q(H \circ \Phi^\lambda(u, v, z))$, there are $\tilde{g} \in S(G \circ \Phi^\lambda(\tilde{u}, \tilde{v}, \tilde{z}))$ and $\tilde{h} \in S_q(H \circ \Phi^\lambda(\tilde{u}, \tilde{v}, \tilde{z}))$ such that:

$$\begin{aligned}
 & |g_t - \tilde{g}_t| \\
 \leq & k_G(t) \left\{ \int_0^t |u_\tau - \tilde{u}_\tau| d\tau + \left| \int_0^t [v_\tau - \tilde{v}_\tau] d\omega_\tau \right| + \left| \int_0^t \int_{\mathbb{R}^m} [z_{\tau,r} - \tilde{z}_{\tau,r}] \tilde{v}(d\tau, dr) \right| \right\} \\
 \text{and} & \\
 & |h_{t,r} - \tilde{h}_{t,r}| \\
 \leq & m(t,r) \left\{ \int_0^t |u_\tau - \tilde{u}_\tau| d\tau + \left| \int_0^t [v_\tau - \tilde{v}_\tau] d\omega_\tau \right| + \left| \int_0^t \int_{\mathbb{R}^m} [z_{\tau,r} - \tilde{z}_{\tau,r}] \tilde{v}(d\tau, dr) \right| \right\}
 \end{aligned}$$

a.s., each $t \in I$. Let:

$$\gamma = \max\{(3/L)^2 T, (3/L)^2\}, k^2(t) = \max\{k_F^2(t), k_G^2(t), \int_{\mathbb{R}^m} m^2(t,r) q(dr)\},$$

and $\mathfrak{K}(t) = \int_0^t k^2(\tau) d\tau$ for $t \in I$. Let us renorm \mathcal{L}^2 and \mathcal{W}^2 with equivalent norms

$\|\cdot\|_{\mathcal{L}^2}$ and $\|\cdot\|_{\mathcal{W}^2}$ defined by:

$$\|u\|_{\mathcal{L}^2} = \left(E \int_0^T e^{-\gamma \mathfrak{K}(t)} |u_t|^2 dt \right)^{1/2}$$

and

$$\|z\|_{\mathcal{W}^2} = \left(E \int_0^T \int_{\mathbb{R}^m} e^{-\gamma \mathfrak{K}(t)} |z_{t,r}|^2 q(dr) dt \right)^{1/2}$$

for $u \in \mathcal{L}^2$ and $z \in \mathcal{W}^2$. We obtain:

$$\begin{aligned}
 \|f - \tilde{f}\|_{\mathcal{L}^2} & \leq \left(E \int_0^T k^2(t) e^{-\gamma \mathfrak{K}(t)} \left[\int_0^t |u_\tau - \tilde{u}_\tau| d\tau \right]^2 dt \right)^{1/2} \\
 & + \left(E \int_0^T k^2(t) e^{-\gamma \mathfrak{K}(t)} \left| \int_0^t [v_\tau - \tilde{v}_\tau] d\omega_\tau \right|^2 dt \right)^{1/2} \\
 & + \left(E \int_0^T k^2(t) e^{-\gamma \mathfrak{K}(t)} \left| \int_0^t \int_{\mathbb{R}^m} [z_{\tau,r} - \tilde{z}_{\tau,r}] \tilde{v}(d\tau, dr) \right|^2 dt \right)^{1/2}.
 \end{aligned}$$

We have:

$$E \int_0^T k^2(t) e^{-\gamma \mathfrak{K}(t)} \left[\int_0^t |u_\tau - \tilde{u}_\tau| d\tau \right]^2 dt$$

$$\begin{aligned} & |g_t - \tilde{g}_t| \\ & \leq k_G(t) \left\{ \int_0^t |u_\tau - \tilde{u}_\tau| d\tau + \left| \int_0^t [v_\tau - \tilde{v}_\tau] d\omega_\tau \right| + \left| \int_0^t \int_{\mathbb{R}^m} [z_{\tau,r} - \tilde{z}_{\tau,r}] \tilde{v}(d\tau, dr) \right| \right\} \\ \text{and} & \\ & |h_{t,r} - \tilde{h}_{t,r}| \\ & \leq m(t,r) \left\{ \int_0^t |u_\tau - \tilde{u}_\tau| d\tau + \left| \int_0^t [v_\tau - \tilde{v}_\tau] d\omega_\tau \right| + \left| \int_0^t \int_{\mathbb{R}^m} [z_{\tau,r} - \tilde{z}_{\tau,r}] \tilde{v}(d\tau, dr) \right| \right\} \end{aligned}$$

a.s., each $t \in I$. Let:

$$\gamma = \max\{(3/L)^2 T, (3/L)^2\}, k^2(t) = \max\{k_F^2(t), k_G^2(t), \int_{\mathbb{R}^m} m^2(t,r) q(dr)\},$$

and $\mathfrak{K}(t) = \int_0^t k^2(\tau) d\tau$ for $t \in I$. Let us renorm \mathcal{L}^2 and \mathcal{W}^2 with equivalent norms

$\|\cdot\|_{\mathcal{L}^2}$ and $\|\cdot\|_{\mathcal{W}^2}$ defined by:

$$\|u\|_{\mathcal{L}^2} = \left(E \int_0^T e^{-\gamma \mathfrak{K}(t)} |u_t|^2 dt \right)^{1/2}$$

and

$$\|z\|_{\mathcal{W}^2} = \left(E \int_0^T \int_{\mathbb{R}^m} e^{-\gamma \mathfrak{K}(t)} |z_{t,r}|^2 q(dr) dt \right)^{1/2}$$

for $u \in \mathcal{L}^2$ and $z \in \mathcal{W}^2$. We obtain:

$$\begin{aligned} \|f - \tilde{f}\|_{\mathcal{L}^2} & \leq \left(E \int_0^T k^2(t) e^{-\gamma \mathfrak{K}(t)} \left[\int_0^t |u_\tau - \tilde{u}_\tau| d\tau \right]^2 dt \right)^{1/2} \\ & \quad + \left(E \int_0^T k^2(t) e^{-\gamma \mathfrak{K}(t)} \left| \int_0^t [v_\tau - \tilde{v}_\tau] d\omega_\tau \right|^2 dt \right)^{1/2} \\ & \quad + \left(E \int_0^T k^2(t) e^{-\gamma \mathfrak{K}(t)} \left| \int_0^t \int_{\mathbb{R}^m} [z_{\tau,r} - \tilde{z}_{\tau,r}] \tilde{v}(d\tau, dr) \right|^2 dt \right)^{1/2}. \end{aligned}$$

We have:

$$E \int_0^T k^2(t) e^{-\gamma \mathfrak{K}(t)} \left[\int_0^t |u_\tau - \tilde{u}_\tau| d\tau \right]^2 dt$$

Therefore,

$$\begin{aligned} & \ell_{\underline{L}2}(S(F \circ \Phi^\lambda(u, v, z)), S(F \circ \Phi^\lambda(\tilde{u}, \tilde{v}, \tilde{z}))), \\ & \leq L \max(\|u - \tilde{u}\|_{\underline{L}2}, \|v - \tilde{v}\|_{\underline{L}2}, \|z - \tilde{z}\|_{\mathcal{W}2}). \end{aligned}$$

Similarly,

$$\begin{aligned} & \ell_{\underline{L}2}(S(G \circ \Phi^\lambda(u, v, z)), S(G \circ \Phi^\lambda(\tilde{u}, \tilde{v}, \tilde{z}))), \\ & \leq L \max(\|u - \tilde{u}\|_{\underline{L}2}, \|v - \tilde{v}\|_{\underline{L}2}, \|z - \tilde{z}\|_{\mathcal{W}2}). \end{aligned}$$

Finally,

$$\begin{aligned} & \|h - \tilde{h}\|_{\mathcal{W}2} \\ & \leq \left(E \int_0^T e^{-\gamma \mathcal{K}(t)} \int_{\mathbb{R}^m} m^2(t, r) \left| \int_0^t [u_\tau - \tilde{u}_\tau] d\tau \right|^2 q(d\tau) dt \right)^{1/2} \\ & + \left(E \int_0^T e^{-\gamma \mathcal{K}(t)} \int_{\mathbb{R}^m} m^2(t, r) \left| \int_0^t [v_\tau - \tilde{v}_\tau] d\omega_\tau \right|^2 q(d\tau) dt \right)^{1/2} \\ & + \left(E \int_0^T e^{-\gamma \mathcal{K}(t)} \int_{\mathbb{R}^m} m^2(t, r) \left| \int_0^t \int_{\mathbb{R}^m} [z_{\tau, r} - \tilde{z}_{\tau, r}] \nu(d\tau, dr) \right|^2 q(d\tau) dt \right)^{1/2}. \end{aligned}$$

Similarly, as above,

$$\begin{aligned} & E \int_0^T e^{-\gamma \mathcal{K}(t)} \int_{\mathbb{R}^m} m^2(t, r) \left| \int_0^t [u_\tau - \tilde{u}_\tau] d\tau \right|^2 q(d\tau) dt \\ & \leq TE \int_0^T e^{-\gamma \mathcal{K}(t)} \int_0^t |u_\tau - \tilde{u}_\tau|^2 d\tau dt \leq (L/3)^2 \|u - \tilde{u}\|_{\underline{L}2}^2, \\ & E \int_0^T e^{-\gamma \mathcal{K}(t)} \int_{\mathbb{R}^m} m^2(t, r) \left| \int_0^t [v_\tau - \tilde{v}_\tau] d\omega \right|^2 q(d\tau) dt \\ & \leq E \int_0^T e^{-\gamma \mathcal{K}(t)} \int_0^t |v_\tau - \tilde{v}_\tau|^2 d\tau dt \leq (L/3)^2 \|v - \tilde{v}\|_{\underline{L}2}^2, \end{aligned}$$

and

$$E \int_0^T e^{-\gamma \mathcal{K}(t)} \int_{\mathbb{R}^m} m^2(t, r) \left| \int_0^t \int_{\mathbb{R}^m} [z_{\tau, r} - \tilde{z}_{\tau, r}] \tilde{\nu}(d\tau, dr) \right|^2 q(d\tau) dt$$

$$\leq E \int_0^T e^{-\gamma \mathfrak{K}(t)} \int_0^t \int_{\mathbb{R}^m} |z_{\tau,r} - \tilde{z}_{\tau,r}|^2 q(dr) d\tau dt \leq (L/3)^2 \|z - \tilde{z}\|_{\mathfrak{W}^2}^2.$$

Therefore,

$$\begin{aligned} & \ell_{\mathfrak{W}^2}(S_q(H \circ \Phi^\lambda(u, v, r)), S_q(H \circ \Phi^\lambda(\tilde{u}, \tilde{v}, \tilde{z}))) \\ & \leq L \max(\|u - \tilde{u}\|_{\mathfrak{L}^2}, \|v - \tilde{v}\|_{\mathfrak{L}^2}, \|z - \tilde{z}\|_{\mathfrak{W}^2}). \end{aligned} \quad \square$$

Now we can prove the following basic lemma.

Lemma 1. *Suppose F, G , and H satisfy (\mathcal{A}_1) and (\mathcal{A}_2) . There is a norm $\|\cdot\|$ on $\mathfrak{L}^2 \times \mathfrak{L}^2 \times \mathfrak{W}^2$ equivalent to the norm defined on $\mathfrak{L}^2 \times \mathfrak{L}^2 \times \mathfrak{W}^2$ by $|\cdot|_{\mathfrak{L}^2}$ and $|\cdot|_{\mathfrak{W}^2}$ such that $S_\lambda(\cdot)$ and $S_\cdot(u, v, z)$ are contractions from $(\mathfrak{L}^2 \times \mathfrak{L}^2 \times \mathfrak{W}^2, \|\cdot\|)$ and $(L_n^2(\mathfrak{F}_0), \|\cdot\|_{L_n^2})$, respectively, into $(\text{Cl}(\mathfrak{L}^2 \times \mathfrak{L}^2 \times \mathfrak{W}^2), \ell)$, where ℓ is the Hausdorff metric induced by the norm $\|\cdot\|$.*

Proof. Let $L \in (0, 1)$ and $\|\cdot\|_{\mathfrak{L}^2}$ and $\|\cdot\|_{\mathfrak{W}^2}$ be such as in Proposition 2, corresponding to the given L . Set $\|(u, v, z)\| = \max(\|u\|_{\mathfrak{L}^2}, \|v\|_{\mathfrak{L}^2}, \|z\|_{\mathfrak{W}^2})$ and let ℓ be the Hausdorff metric on $\text{Cl}(\mathfrak{L}^2 \times \mathfrak{L}^2 \times \mathfrak{W}^2)$ induced by the norm $\|\cdot\|$. By Proposition 2, we obtain

$$\ell(S_\lambda(u, v, z), S_\lambda(\tilde{u}, \tilde{v}, \tilde{z})) \leq L \|(u, v, z) - (\tilde{u}, \tilde{v}, \tilde{z})\|$$

for $\lambda \in L_n^2(\mathfrak{F}_0)$ and $(u, v, z), (\tilde{u}, \tilde{v}, \tilde{z}) \in \mathfrak{L}^2 \times \mathfrak{L}^2 \times \mathfrak{W}^2$. Quite similarly,

$$\ell(S_\lambda(u, v, z), S_{\tilde{\lambda}}(u, v, z)) \leq L \|\lambda - \tilde{\lambda}\|_{L_n^2}$$

for $\lambda, \tilde{\lambda} \in L_n^2(\mathfrak{F}_0)$ and $(u, v, z) \in \mathfrak{L}^2 \times \mathfrak{L}^2 \times \mathfrak{W}^2$. □

4. Quasi-Retractive Representation of Solution Set

We shall show that if conditions (\mathcal{A}_1) and (\mathcal{A}_2) are satisfied, then the solution set mapping $\lambda \rightarrow \mathfrak{C}_\lambda$, where \mathfrak{C}_λ denotes a set of all solutions to an initial value problem $SI_\lambda(F, G, H)$, has quasi-retractive representation. In particular, it will follow that this mapping is lower semicontinuous. Moreover, it will follow that in some special cases the solution set \mathfrak{C}_λ is weakly compact in $(D, \|\cdot\|_\ell)$. These results are consequences of Lemma 1 and a general retractive representation theorem presented in [1].

Let Λ be a topological space and $(X, |\cdot|)$ be a Banach space. Denote $\mathcal{N}(X) = \{A \subset X : A \neq \emptyset\}$. Given $S: \Lambda \rightarrow \mathcal{N}(X)$ and $C \subset X$, let $S^-(C) = \{\lambda \in \Lambda : S(\lambda) \cap C \neq \emptyset\}$. We say that $S: \Lambda \rightarrow \mathcal{N}(X)$ is lower semicontinuous (l.s.c.) [upper semicontinuous (u.s.c.)] if $S^-(C)$ is open [closed] for every open [closed] set $C \subset X$. A set-valued mapping $S: \Lambda \rightarrow \mathcal{N}$ is said to be W -upper semicontinuous (W -u.s.c.) if for every $x \in X$ the function $\lambda \rightarrow \text{dist}(x, S(\lambda))$ is lower semicontinuous in the usual sense. Finally, S is said to be W -continuous if it is l.s.c. and W -u.s.c.

We say that $S: \Lambda \rightarrow \mathcal{N}(X)$ has a retractive representation if there exists a set $B \in \mathcal{N}(X)$ and a continuous mapping $p: \Lambda \times B \rightarrow B$ such that $p(\lambda, x) \in S(\lambda)$ for every $(\lambda, x) \in \Lambda \times B$ and $p(\lambda, x) = x$ if and only if $x \in S(\lambda)$.

We say that the solution set mapping $L_n^2(\mathcal{F}_0) \ni \lambda \rightarrow \mathcal{C}_\lambda \subset D$ has quasi-retractive representation if there is a set-valued mapping $S: L_n^2(\mathcal{F}_0) \rightarrow \mathcal{N}(\mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2)$ having a retractive representation $p: \Lambda \times B \rightarrow B$ such that $\mathcal{C}_\lambda = \Phi^\lambda(p(\lambda, B))$, each $\lambda \in L_n^2(\mathcal{F}_0)$.

We present the following general results (see [1, 8]) dealing with retractive representation of set-valued mappings.

Theorem 2. ([8], Th. 1) *Let Λ be a paracompact and perfectly normal topological space, $(X, |\cdot|)$ be a Banach space, and $B \in \text{Cl}(X)$. Suppose $\mathfrak{P}: \Lambda \times B \rightarrow \text{Cl}(X)$ takes on convex values and is such that:*

- (i) *for every $x \in B$ the set-valued mapping $\mathfrak{P}(\cdot, x)$ is W -continuous,*
- (ii) *there is $L \in [0, 1)$ such that $h(\mathfrak{P}(\lambda, x), \mathfrak{P}(\lambda, \tilde{x})) \leq L|x - \tilde{x}|$ for fixed $\lambda \in \Lambda$ and $x, \tilde{x} \in B$, where h is Hausdorff metric on $\text{Cl}(X)$ induced by the norm $|\cdot|$.*

Let $S_{\mathfrak{P}}(\lambda) := \{x \in B: x \in \mathfrak{P}(\lambda, x)\}$, each $\lambda \in \Lambda$. A set-valued mapping $S_{\mathfrak{P}}: \Lambda \ni \lambda \rightarrow S_{\mathfrak{P}}(\lambda) \in \mathcal{N}(B)$ has a retractive representation $p: \Lambda \times B \rightarrow B$.

We now apply Theorem 2 and Lemma 1 to the subtrajectory integrals mapping S defined above. Recall that for given F, G , and H satisfying condition (\mathcal{A}_1) , we can define a convex, weakly compact set B (see Corollary 1), where B is a subset of a Banach space $(\mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{W}^2, \|\cdot\|)$ with a norm $\|\cdot\|$ defined in Lemma 1 corresponding to any $L \in [0, 1)$, containing the set $C(\lambda)$ of all fixed points to subtrajectory integrals mapping $S_\lambda(\cdot)$. From Theorem 2 and Lemma 1 we immediately obtain the following result.

Lemma 3. *Suppose F, G , and H satisfy conditions (\mathcal{A}_1) and (\mathcal{A}_2) . A set-valued mapping $C: L_n^2(\mathcal{F}_0) \ni \lambda \rightarrow C(\lambda) \in \mathcal{N}(B)$ has a retractive representation $p: L_n^2(\mathcal{F}_0) \times B \rightarrow B$.*

Corollary 2. *Let F, G , and H satisfy conditions (\mathcal{A}_1) and (\mathcal{A}_2) , and $p: L_n^2(\mathcal{F}_0) \times B \rightarrow B$ be a retractive representation for S . Then $C(\lambda) = p(\lambda, B)$, each $\lambda \in L_n^2(\mathcal{F}_0)$.*

Corollary 3. *The set-valued mapping $\lambda \rightarrow C(\lambda)$ is continuous as a mapping from $L^2(\mathcal{F}_0)$ into a metric space $(\text{Cl}(\mathcal{L}^2 \times \mathcal{L}^2 \times \mathcal{L}^2, \ell)$.*

Given F, G , and H satisfying conditions (\mathcal{A}_1) and (\mathcal{A}_2) , \mathcal{C}_λ denotes a set of all solutions to the initial value problem $SI_\lambda(F, G, H)$. As an immediate consequence of Proposition 1, we obtain $\mathcal{C}_\lambda = \Phi^\lambda(C(\lambda))$, where $C(\lambda)$ is defined as above. \mathcal{C} denotes a set-valued mapping $L_n^2(\mathcal{F}_0) \ni \lambda \rightarrow \mathcal{C}_\lambda \subset D$. From the above definitions, Lemma 3, and properties of Φ^λ , we immediately obtain the following main result of this paper.

Theorem 4. *If F, G , and H satisfy conditions (\mathcal{A}_1) and (\mathcal{A}_2) , then \mathcal{C} has a quasi-retractive representation and is l.s.c. on $L^2(\mathcal{F}_0)$.*

Proof. Let $p: L_n^2(\mathcal{F}_0) \times B \rightarrow B$ be a retractive representation for the set-valued mapping C defined in Lemma 3. We have $\mathcal{C}_\lambda = \Phi^\lambda(C(\lambda))$ and $C(\lambda) = p(\lambda, B)$, each $\lambda \in L_n^2(\mathcal{F}_0)$. Therefore, \mathcal{C} has a quasi-retractive representation. Moreover, a function $L_n^2(\mathcal{F}_0) \ni \lambda \rightarrow \Phi^\lambda(p(\lambda, x)) \in D$ is continuous for fixed $x \in B$. Therefore, a set-valued mapping \mathcal{C} (see [3], Proposition II 2.5) is l.s.c. on $L^2(\mathcal{F}_0)$. \square

Corollary 4. *If F, G , and H satisfy conditions (\mathcal{A}_1) and (\mathcal{A}_2) and are such that a set-valued mapping C has a retractive representation $p: L_n^2(\mathcal{F}_0) \times B \rightarrow B$ that is weakly-weakly continuous, then \mathcal{C}_λ is a weakly compact subset of D for every $\lambda \in L_n^2(\mathcal{F}_0)$ and a set-valued mapping \mathcal{C} is weak-weak continuous on $L_n^2(\mathcal{F}_0)$.*

Proof. Indeed, if p has properties mentioned above, then (see [3], Th. II 2.6)

$p(\lambda, B)$ is a weakly compact subset of B for each $\lambda \in L_n^2(\mathcal{F}_0)$. \mathcal{C}_λ is also a weakly compact subset of B for each $\lambda \in L_n^2(\mathcal{F}_0)$ because $\mathcal{C}_\lambda = \Phi^\lambda(p(\lambda, B))$. Finally, by weak-weak continuity of the linear mapping $L_n^2(\mathcal{F}_0) \times B \ni (\lambda, x) \rightarrow \Phi^\lambda(x)$, weak compactness of B , and an equality $\mathcal{C}_\lambda = \Phi^\lambda(p(\lambda, B))$, each $\lambda \in L_n^2(\mathcal{F}_0)$, it follows (see again [3], Proposition II 2.5), that \mathcal{C} is weak-weak continuous on $L_n^2(\mathcal{F}_0)$. \square

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