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SOJOURN TIMES FOR THE BROWNIAN MOTION

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In this paper explicit formulas are given for the distribution function, the density function and the moments of the sojourn time for the reflecting Brownian motion process.

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1. Introduction

Let $\{\xi(t), t \geq 0\}$ be a standard Brownian motion process. We have $\mathbf{P}\{\xi(t) \leq x\} = \Phi(x/\sqrt{t})$ for $t > 0$ where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du \tag{1}$$

is the normal distribution function. We also use the notation

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{2}$$

for the normal density function.

Let us define

$$\tau(\alpha) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \text{ measure } \{t: \alpha \leq \xi(t) < \alpha + \varepsilon, 0 \leq t \leq 1\} \tag{3}$$

for any real α. The limit (3) exists with probability one and $\tau(\alpha)$ is a nonnegative random variable which is called the local time at level α. We define also

$$\omega(\alpha) = \int_0^1 \delta(\xi(t) > \alpha) dt \tag{4}$$

for $\alpha \geq 0$ where $\delta(S)$ denotes the indicator variable of any event S, that is, $\delta(S) = 1$ if S occurs and $\delta(S) = 0$ if S does not occur. The integral (4) exists with probability one and $\omega(\alpha)$ is a nonnegative random variable which is called the sojourn time of the process $\{\xi(t), t \geq 0\}$ spent in the set (α, ∞) in the time interval $(0,1)$. We also consider the reflecting Brownian motion process $\{\,|\,\xi(t)\,|\,, t \geq 0\}$ and define

$$\omega^*(\alpha) = \int_0^1 \delta(\,|\,\xi(t)\,|\, > \alpha)dt \tag{5}$$

for $\alpha \geq 0$ as the sojourn time of the process $\{\,|\,\xi(t)\,|\,, t \geq 0\}$ spent in the set (α, ∞) in the time interval $(0,1)$.

Our main object is to determine the distribution and the moments of $\omega^*(\alpha)$ for $\alpha > 0$. In principle, we can apply the method of M. Kac [6] to find the distribution of $\omega^*(\alpha)$. His method requires the inversion of a double Laplace transform which can be obtained by solving a certain Sturm-Liouville differential equation. Our approach is combinatorial and we shall find explicit formulas for the distribution function and the moments of $\omega^*(\alpha)$.

Let us define

$$\mathbf{E}\{[\tau(\alpha)]^r\} = m_r(\alpha), \tag{6}$$

$$\mathbf{E}\{[\omega(\alpha)]^r\} = M_r(\alpha) \tag{7}$$

and

$$\mathbf{E}\{[\omega^*(\alpha)]^r\} = M_r^*(\alpha) \tag{8}$$

for $r = 1, 2, \ldots$ and $\alpha \geq 0$. We shall prove the following surprisingly simple formulas for the moments (7) and (8):

$$M_r(\alpha) = m_{2r}(\alpha)/(2^r r!) \tag{9}$$

and

$$M_r^*(\alpha) = \frac{(r-1)!}{2^{r-1}} \sum_{k=1}^{r} \frac{m_{2r}((2k-1)\alpha)}{(r-k)!(r+k-1)!} \tag{10}$$

if $r = 1, 2, \ldots$ and $\alpha > 0$. Equations (9) and (10) make it possible to determine the distribution function $\mathbf{P}\{\omega^*(\alpha) \leq x\} = G_\alpha(x)$ explicitly. We shall prove that

$$G_\alpha(x) = 2F_\alpha(x) - 1 + 2\sum_{k=2}^{\infty} \sum_{j=2}^{k} \frac{(-1)^j j!}{(k+j-1)!} \binom{k-2}{j-2} \frac{d^{k-1} x^{k-1}[1 - F_{(2j-1)\alpha}(x)]}{dx^{k-1}} \tag{11}$$

if $0 \leq x < 1$ and $\alpha > 0$, and $G_\alpha(1) = 1$. In (11), $F_\alpha(x) = \mathbf{P}\{\omega(\alpha) \leq x\}$. We have

$$F_\alpha(x) = 1 - \frac{1}{\pi} \int_0^{1-x} \frac{e^{-\alpha^2/(2u)}}{\sqrt{u(1-u)}} du \tag{12}$$

for $0 < x \leq 1$, and $\alpha \geq 0$, and

$$F_\alpha(0) = 2\Phi(\alpha) - 1 \tag{13}$$

for $\alpha \geq 0$. The distribution function $F_\alpha(x)$ was found by P. Lévy [9] p. 326 in 1939. If, in particular, $x = 0$ in (11), we obtain that

$$G_\alpha(0) = 1 + 4\sum_{k=1}^{\infty}(-1)^k[1 - \Phi((2k-1)\alpha)] = \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{2j-1}e^{-(2j-1)^2\pi^2/(8\alpha^2)} \tag{14}$$

for $\alpha > 0$.

We note that

$$\mathbf{P}\{\tau(\alpha) \le x\} = 2\Phi(\alpha + x) - 1 \tag{15}$$

if $x \ge 0$ and $\alpha \ge 0$, and

$$m_r(\alpha) = 2r\int_0^\infty x^{r-1}[1 - \Phi(\alpha + x)]dx \tag{16}$$

if $\alpha \ge 0$ and $r \ge 1$ where $\Phi(x)$ is defined by (1). Explicitly,

$$m_r(\alpha) = 2(-1)^r\{a_r(\alpha)[1 - \Phi(\alpha)] - b_r(\alpha)\varphi(\alpha)\} \tag{17}$$

for $r = 1, 2, \ldots$ where

$$a_r(\alpha) = r!\sum_{j=0}^{[r/2]}\frac{\alpha^{r-2j}}{2^j j!(r-2j)!} \tag{18}$$

and

$$b_r(\alpha) = \sum_{j=0}^{[(r-1)/2]}\binom{r-1-j}{j}\frac{j!\alpha^{r-1-2j}}{2^j}\sum_{\nu=0}^{j}\binom{r}{\nu} \tag{19}$$

for $r \ge 1$. See L. Takács [13].

Our approach is based on a symmetric random walk $\{\zeta_r, r \ge 0\}$ where $\zeta_r = \xi_1 + \xi_2 + \ldots + \xi_r$ for $r \ge 1$, $\zeta_0 = 0$, and $\{\xi_r, r \ge 1\}$ is a sequence of independent and identically distributed random variables for which

$$\mathbf{P}\{\xi_r = 1\} = \mathbf{P}\{\xi_r = -1\} = 1/2. \tag{20}$$

Let us define $\tau_n(a)$ as the number of subscripts $r = 0, 1, \ldots, n$ for which $\zeta_r = a$ where $a = 0, 1, 2, \ldots$. Furthermore, define $\omega_n(a)$ as the number of subscripts $r = 0, 1, \ldots, n$ for which $\zeta_r \ge a$ where $a = 0, 1, 2, \ldots$, and $\omega_n^*(a)$ as the number of subscripts $r = 1, 2, \ldots, n$ for which $|\zeta_r| \ge a$ where $a = 1, 2, \ldots$.

By the results of M.D. Donsker [2], if $n \to \infty$, the process $\{\zeta_{[nt]}/\sqrt{n}, 0 \le t \le 1\}$ converges weakly to the Brownian motion $\{\xi(t), 0 \le t \le 1\}$. See also I.I. Gikhman and A.V. Skorokhod [4] pp. 490-495.

In 1965, F.B. Knight [7] proved that

$$\lim_{n\to\infty}\mathbf{P}\left\{\frac{\tau_n([\alpha\sqrt{n}])}{\sqrt{n}} \le x\right\} = \mathbf{P}\{\tau(\alpha) \le x\} \tag{21}$$

for $\alpha \ge 0$ and $x > 0$. Since the integrals (4) and (5) are continuous functionals of the process $\{\xi(t), 0 \le t \le 1\}$, we can conclude that

$$\lim_{n\to\infty}\mathbf{P}\{\omega_n([\alpha\sqrt{n}]) \le nx\} = \mathbf{P}\{\omega(\alpha) \le x\} \tag{22}$$

and

$$\lim_{n\to\infty} \mathbf{P}\{\omega_n^*([\alpha\sqrt{n}]) \le nx\} = \mathbf{P}\{\omega^*(\alpha) \le x\} \tag{23}$$

for $\alpha > 0$ and $x \ge 0$.

We shall determine the distributions and the moments of the random variables $\tau_n(a)$, $\omega_n(a)$ and $\omega_n^*(a)$, and their asymptotic behavior in the case where $a = [\alpha\sqrt{n}]$, $\alpha > 0$, and $n \to \infty$. We shall prove that

$$\lim_{n\to\infty} \mathbf{E}\left\{\left(\frac{\tau_n([\alpha\sqrt{n}])}{\sqrt{n}}\right)^r\right\} = m_r(\alpha) \tag{24}$$

for $r \ge 1$ and $\alpha \ge 0$ where $m_r(\alpha)$ is given by (16). Furthermore, we shall determine (7) and (8) by calculating the following limits

$$\lim_{n\to\infty} \mathbf{E}\left\{\left(\frac{\omega_n([\alpha\sqrt{n}])}{n}\right)^r\right\} = M_r(\alpha) \tag{25}$$

and

$$\lim_{n\to\infty} \mathbf{E}\left\{\left(\frac{\omega_n^*([\alpha\sqrt{n}])}{n}\right)^r\right\} = M_r^*(\alpha) \tag{26}$$

for $r \ge 1$ and $\alpha > 0$. The moments $M_r(\alpha)$, $(r \ge 1)$, and $M_r^*(\alpha)$, $(r \ge 1)$, uniquely determine the distribution functions $\mathbf{P}\{\omega(\alpha) \le x\}$ and $\mathbf{P}\{\omega^*(\alpha) \le x\}$.

2. The Random Walk $\{\zeta_r, r \ge 0\}$

Let us recall some results for $\{\zeta_r, r \ge 0\}$ which we need in this paper. See L. Takács [12]. We have

$$\mathbf{P}\{\zeta_n = 2j - n\} = \binom{n}{j}\frac{1}{2^n} \tag{27}$$

for $j = 0, 1, \ldots, n$, and by the central limit theorem

$$\lim_{n\to\infty} \mathbf{P}\left\{\frac{\zeta_n}{\sqrt{n}} \le x\right\} = \Phi(x) \tag{28}$$

where $\Phi(x)$ is defined by (1).

Let us define $\rho(a)$ as the first passage time through a $(a = 0, \pm 1, \pm 2, \ldots)$, that is,

$$\rho(a) = \inf\{r : \zeta_r = a \text{ and } r \ge 0\}. \tag{29}$$

We have

$$\mathbf{P}\{\rho(a) = a + 2j\} = \frac{a}{a+2j}\binom{a+2j}{j}\frac{1}{2^{a+2j}} \tag{30}$$

for $a \ge 1$ and $j \ge 0$. If $1 \le a \le n$, then

$$\mathbf{P}\{\rho(a) \le n\} = \mathbf{P}\{\zeta_n \ge a\} + \mathbf{P}\{\zeta_n > a\}. \tag{31}$$

By (30),

$$\sum_{n=0}^{\infty} \mathbf{P}\{\rho(a) = n\} w^n = [\gamma(w)]^a \tag{32}$$

for $a \ge 1$ and $|w| \le 1$ where $\gamma(0) = 0$ and

$$\gamma(w) = (1 - \sqrt{1 - w^2})/w \tag{33}$$

for $0 < |w| \leq 1$. The identity

$$\sum_{j=0}^{n} \mathbf{P}\{\rho(a) = j\}\mathbf{P}\{\rho(b) = n-j\} = \mathbf{P}\{\rho(a+b) = n\} \tag{34}$$

is valid for any $a \geq 1$, $b \geq 1$ and $n \geq 1$.

We note that

$$\mathbf{P}\{\rho(1) = 2n+1\} = C_n / 2^{2n+1} \tag{35}$$

for $n = 0, 1, 2, \ldots$ where

$$C_n = \binom{2n}{n} \frac{1}{n+1} \tag{36}$$

is the nth Catalan number.

Let us define

$$P(n,\nu) = \sum_{\substack{\alpha_1 + \alpha_2 + \ldots + \alpha_n = \nu \\ \alpha_1 + 2\alpha_2 + \ldots + n\alpha_n = n}} \frac{\nu!}{\alpha_1!\alpha_2!\ldots\alpha_n!} C_0^{\alpha_1} C_1^{\alpha_2} \ldots C_{n-1}^{\alpha_n} \tag{37}$$

for $1 \leq \nu \leq n$. To evaluate (37) let us express each Catalan number in (37) by (35). By the repeated applications of (34) we obtain that

$$P(n,\nu) = 2^{2n-\nu}\mathbf{P}\{\rho(\nu) = 2n-\nu\} = \binom{2n-\nu}{n} \frac{\nu}{2n-\nu}. \tag{38}$$

By (31) we obtain that

$$\sum_{s=j}^{r} \mathbf{P}\{\rho(2j) = 2s\} = \mathbf{P}\{\rho(2j) \leq 2r+1\}$$

$$= 2\mathbf{P}\{\zeta_{2r+1} \geq 2j+1\} = \sum_{s=j}^{r} \binom{2r+1}{r-s} \frac{1}{2^{2r}} \tag{39}$$

for $j = 0, 1, \ldots, r$.

If $a \geq 1$ and $b \geq 1$, let us define $\Theta(a,b)$ as the smallest $r = 0, 1, \ldots$ for which either $\zeta_r = a$ or $\zeta_r = -b$. We can interpret $\Theta(a,b)$ as the duration of games in the classical ruin problem. See L. Takács [11]. By the results of P.S. Laplace [8], p. 228 we have

$$\mathbf{E}\{w^{\Theta(a,b)}\} = \frac{[\gamma(w)]^a + [\gamma(w)]^b}{1 + [\gamma(w)]^{a+b}} \tag{40}$$

if $|w| \leq 1$ where $\gamma(w)$ is defined by (33). See also I. Todhunter [15], p. 169.

3. Sojourn Times

Let us consider now a stochastic process $\{(t), t \geq 0\}$ with state space $A \cup B$ where A and B are disjoint Borel sets. Let $\mathbf{P}\{\zeta(0) \in A\} = 1$ and denote by $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots$ the lengths of the successive intervals spent in states A and B respectively in the interval $(0, \infty)$. We suppose that $\{\alpha_i\}$ and $\{\beta_i\}$ are discrete random variables which take on positive integers only. Define $\gamma_n = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ for $n \geq 1$ and $\gamma_0 = 0$. Furthermore, let $\delta_n = \beta_1 + \beta_2 + \ldots + \beta_n$ for $n \geq 1$ and $\delta_0 = 0$.

Denote by $\beta(n+1)$ the total time spent in state B in the interval $(0, n+1)$. If the two sequences $\{\alpha_i\}$ and $\{\beta_i\}$ are independent, then we have

$$\mathbf{P}\{\beta(n+1) \leq k\} = \sum_{r \geq 0} \mathbf{P}\{\delta_r \leq k\}[\mathbf{P}\{\gamma_r \leq n-k\} - \mathbf{P}\{\gamma_{r+1} \leq n-k\}] \tag{41}$$

for $0 \leq k \leq n$.

Proof of (41): Denote by $\alpha(t)$ the total time spent in state A in the time interval $(0,t)$ and by $\beta(t)$ the total time spent in state B in the time interval $(0,t)$. If $0 \leq k \leq n$, denote by $\tau = \tau(n-k)$ the smallest $u \in [0,\infty)$ for which $\alpha(u) = n-k+1$. Then we have $\{\beta(n+1) \leq k\} \equiv \{\beta(\tau) \leq k\}$. This follows from the following identities

$$\begin{aligned}\{\beta(n+1) \leq k\} &\equiv \{\alpha(\tau) \leq \alpha(n+1)\} \equiv \{\tau \leq n+1\} \\ &\equiv \{\alpha(\tau) + \beta(\tau) \leq n+1\} \equiv \{\beta(\tau) \leq k\}.\end{aligned} \tag{42}$$

Since $\beta(\tau) = \delta_r$ $(r = 0,1,2,\ldots)$ if $\gamma_r < n+1-k \leq \gamma_{r+1}$, it follows from (42) that

$$\mathbf{P}\{\beta(n+1) \leq k\} = \sum_{r \geq 0} \mathbf{P}\{\delta_r \leq k \text{ and } \gamma_r \leq n-k < \gamma_{r+1}\} \tag{43}$$

for $0 \leq k \leq n$. This proves (41).

By forming generating functions, we obtain from (41) that

$$\begin{aligned}&(1-w)(1-zw)\sum_{n=0}^{\infty} \mathbf{E}\{z^{\beta(n+1)}\}w^n \\ &= (1-w)z + (1-z)\sum_{r=0}^{\infty} \mathbf{E}\{(zw)^{\delta_r}\}[\mathbf{E}\{w^{\gamma_r}\} - \mathbf{E}\{w^{\gamma_r+1}\}]\end{aligned} \tag{44}$$

if $|w| < 1$ and $|zw| < 1$.

Now we consider the case where $\{\alpha_i\}$ and $\{\beta_i\}$ are independent sequences of independent random variables such that $\alpha_2, \alpha_3, \ldots$ are identically distributed, but α_1 may have a different distribution, and $\beta_1, \beta_2, \ldots$ are identically distributed. Let us write $\mathbf{E}\{z^{\alpha_1}\} = a_0(z)$, $\mathbf{E}\{z^{\alpha_i}\} = a(z)$ for $i = 2,3,\ldots$ and $\mathbf{E}\{z^{\beta_i}\} = b(z)$ for $i = 1,2,\ldots$. In this case by (44) we have

$$(1-w)(1-zw)\sum_{n=0}^{\infty} \mathbf{E}\{z^{\beta(n+1)}\}w^n = 1 - zw - (1-z)\frac{[1-b(zw)]a_0(w)}{1-a(w)b(zw)} \tag{45}$$

if $|w| < 1$ and $|zw| < 1$.

4. On a Formula of Faà di Bruno

The nth derivative of the compound function $f = f(y)$ where $y = y(z)$ is given by Faà di Bruno's formula

$$\frac{d^n f}{dz^n} = \sum_{\nu=1}^{n} \frac{d^\nu f}{dy^\nu} Y_{n,\nu}(y) \tag{46}$$

where

$Y_{n,\nu}(y)$

$$= \sum_{\substack{\alpha_1 + \alpha_2 + \ldots + \alpha_n = \nu \\ \alpha_1 + 2\alpha_2 + \ldots + n\alpha_n = n}} \frac{n!}{\alpha_1!\alpha_2!\ldots\alpha_n!} \left(\frac{y^{(1)}(z)}{1!}\right)^{\alpha_1} \left(\frac{y^{(2)}(z)}{2!}\right)^{\alpha_2} \ldots \left(\frac{y^{(n)}(z)}{n!}\right)^{\alpha_n}. \quad (47)$$

See Faà di Bruno [3] and Ch. Jordan [5], p. 34.

In this paper we need to calculate the rth derivative of a function of the form $f(\gamma(zw))$ where $\gamma(w)$ is given by (33) for $|w| \le 1$. In what follows we use the abbreviation $\gamma = \gamma(w)$ for a fixed w. Since

$$w\gamma^2 - 2\gamma + w = 0 \quad (48)$$

for $|w| \le 1$, we can easily see that

$$\gamma^{(i)}(w) = \frac{i!(1+\gamma^2)^{i+1}}{(1-\gamma^2)^{2i-1}} g_i(\gamma) \quad (49)$$

for $i = 1, 2, \ldots$ and $|w| < 1$ where $g_i(x)$ is a polynomial of degree $3(i-1)$ in x. In particular, $2g_1(x) = 1$, $4g_2(x) = 3x - x^3$ and $8g_3(x) = 1 + 11x^2 - 5x^4 + x^6$. For the determination of $g_i(x)$, $i \ge 1$, we have the recurrence formula

$$(i+1)g_{i+1}(x) = [3ix + (i-2)x^3]g_i(x) - \tfrac{1}{2}(x^4 - 1)g_i'(x). \quad (50)$$

If we apply (46) to the function $f(\gamma(zw))$, we obtain that

$$\left(\frac{d^n f(\gamma(zw))}{dz^n}\right)_{z=1} = \frac{2^n\gamma^n}{(1-\gamma^2)^{2n}} \sum_{\nu=1}^{n} f^{(\nu)}(\gamma)(1-\gamma^4)^\nu Q_{n,\nu}(\gamma) \quad (51)$$

where

$$Q_{n,\nu}(\gamma) = \sum_{\substack{\alpha_1 + \alpha_2 + \ldots + \alpha_n = \nu \\ \alpha_1 + 2\alpha_2 + \ldots + n\alpha_n = n}} \frac{n!}{\alpha_1!\alpha_2!\ldots\alpha_n!} [g_1(\gamma)]^{\alpha_1}[g_2(\gamma)]^{\alpha_2}\ldots[g_n(\gamma)]^{\alpha_n} \quad (52)$$

for $1 \le \nu \le n$. Clearly, $Q_{n,\nu}(\gamma)$ is a polynomial of degree $3(n-\nu)$ in γ.

By (50) we obtain that $g_i(1) = C_{i-1}/2$ for $i = 1, 2, \ldots$ where C_{i-1} is a Catalan number defined by (36). By (38) we have

$$Q_{n,\nu}(1) = \frac{n!P(n,\nu)}{2^\nu \nu!} = \frac{\nu(2n-1-\nu)!}{2^\nu \nu!(n-\nu)!} \quad (53)$$

if $1 \le \nu \le n$. We have also

$$\sum_{\nu=i}^{n} \binom{\nu-1}{i-1} P(n,\nu) = \frac{i}{n}\binom{2n}{n-i} \quad (54)$$

if $1 \le i \le n$.

We shall use the definition and some properties of the Kummer hypergeometric function

$$M(a,b,z) = {}_1F_1(a;b;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n}\frac{z^n}{n!} \quad (55)$$

where $(a)_0 = 1$ and $(a)_n = a(a+1)\ldots(a+n-1)$ for $n \ge 1$. We have

$$M(a,b,-z) = e^{-z}M(b-a,b,z) \quad (56)$$

and
$$\int_0^\infty t^{s-1}M(a,b,-t)dt = \frac{\Gamma(b)\Gamma(s)\Gamma(a-s)}{\Gamma(a)\Gamma(b-s)} \tag{57}$$

if $0 < Re(s) < Re(a)$. See L.J. Slater [10] and M. Abramowitz and I.A. Stegun [1].

5. The Local Time $\tau_n(a)$

We defined $\tau_n(a)$ as the number of subscripts $r = 0,1,2,\ldots,n$ for which $\zeta_r = a$ where $a \geq 0$. If $a \geq 1$, then

$$\mathbf{P}\{\tau_n(a) = 0\} = \mathbf{P}\{\rho(a) > n\}, \tag{58}$$

and if $a \geq 1$ and $k \geq 1$, then

$$\mathbf{P}\{\tau_n(a) \geq k\} = \mathbf{P}\{\rho(a+k-1) \leq n+1-k\}. \tag{59}$$

The distribution of $\rho(a)$ is given by (30).

Equation (58) is trivially true. To prove (59) let us denote by $\theta_1, \theta_1+\theta_2, \ldots, \theta_1 + \ldots + \theta_i, \ldots$ the successive values of $r = 1,2,\ldots$ for which $\zeta_r = a$. The random variables θ_i, $(i \geq 1)$, are independent, θ_1 has the same distribution as $\rho(a)$ and θ_i, $(i \geq 2)$, has the same distribution as $\rho(1)+1$. Since

$$\mathbf{P}\{\tau_n(a) \geq k\} = \mathbf{P}\{\theta_1 + \theta_2 + \ldots + \theta_k \leq n\} \tag{60}$$

we obtain (59) by (34).

We note that
$$\mathbf{P}\{\tau_n(0) > k\} = \mathbf{P}\{\rho(k) \leq n-k\} \tag{61}$$
if $1 \leq k \leq n$.

If in (59) we put $a = [\alpha\sqrt{n}]$ where $\alpha > 0$ and $k = [x\sqrt{n}]$ where $x \geq 0$ and let $n \to \infty$, then by (28) and (31) we obtain that
$$\lim_{n\to\infty} \mathbf{P}\left\{\frac{\tau_n([\alpha\sqrt{n}])}{\sqrt{n}} \leq x\right\} = 2\Phi(\alpha+x) - 1 \tag{62}$$
for $x \geq 0$ and $\alpha > 0$. This proves (21).

By (32) and (59) we can prove that

$$\Psi_r(a) = (1-w)\sum_{n=0}^{\infty} \mathbf{E}\left\{\binom{\tau_n(a)}{r}\right\} w^n = \frac{2^{r-1}\gamma^{a+2r-2}(1+\gamma^2)}{(1-\gamma^2)^r} \tag{63}$$

if $|w| < 1$, $r \geq 1$ and $a \geq 1$. In (63) we used the abbreviation $\gamma = \gamma(w)$, where $\gamma(w)$ is defined by (33), and we took into consideration that $w = 2\gamma/(1+\gamma^2)$.

We observe that if $n+a$ is odd, then $\tau_n(a)$ has the same distribution as $\tau_{n-1}(a)$. If $n+a$ is even, then by expanding (63) into Taylor series at $w = 0$, we obtain that

$$\mathbf{E}\left\{\binom{\tau_n(a)}{r}\right\} = \sum_{j=0}^{a+r-1} (-1)^j \binom{a+r-1}{j}\binom{(n+a+r-j)/2}{(n+a)/2} \tag{64}$$

for $a \geq 1$, $n \geq 1$ and $r \geq 1$.

We can prove that

$$\mathbf{E}\left\{\binom{\tau_n(a)}{r}\right\} = 2^{r+1}\mathbf{E}\left\{\binom{[(\zeta_{n+1} - a)/2]^+}{r}\right\} \tag{65}$$

if $n + a$ is odd, and

$$\mathbf{E}\left\{\binom{\tau_n(a)}{r}\right\} = 2^{r+1}\mathbf{E}\left\{\binom{[(\zeta_{n+2} - a)/2]^+}{r}\right\} \tag{66}$$

if $n + a$ is even.

Theorem 1: *If $a = [\alpha\sqrt{n}]$ where $\alpha > 0$, then*

$$\lim_{n\to\infty} \mathbf{E}\left\{\left(\frac{\tau_n(a)}{\sqrt{n}}\right)^r\right\} = m_r(\alpha) \tag{67}$$

for $r \geq 1$ where $m_r(\alpha)$ is given by (16).

Proof: If $a = [\alpha\sqrt{n}]$ where $\alpha > 0$ and $n \to \infty$, then by (65) and (66) we obtain that

$$\mathbf{E}\{[\tau_n(a)]^r\} \sim 2\mathbf{E}\{([\zeta_n - a]^+)^r\} \tag{68}$$

for $r = 1, 2, \ldots$. Accordingly,

$$\lim_{n\to\infty} \mathbf{E}\left\{\left(\frac{\tau_n([\alpha\sqrt{n}])}{\sqrt{n}}\right)^r\right\} = 2\lim_{n\to\infty} \mathbf{E}\left\{\left(\left[\frac{\zeta_n}{\sqrt{n}} - \alpha\right]^+\right)^r\right\} = 2\mathbf{E}\{([\xi - a]^+)^r\} \tag{69}$$

for $r = 1, 2, \ldots$ where $\mathbf{P}\{\xi \leq x\} = \Phi(x)$ and $\Phi(x)$ is defined by (1). Since

$$2\mathbf{E}\{([\xi - a]^+)^r\} = m_r(\alpha) \tag{70}$$

for $\alpha > 0$ and $r \geq 1$, where $m_r(\alpha)$ is given by (16), (69) implies (67). □

The limit theorem (67) proves (24). We note that if in (67) $a = a_n$ where $\lim_{n\to\infty} a_n/\sqrt{n} = \alpha > 0$, then the right-hand side of (67) remains unchanged.

Finally, we note that if $n + a$ is odd then by (65) we can write that

$$\mathbf{E}\{\tau_n(a)\} = (n + a + 1)\mathbf{P}\{\zeta_n = a + 1\} - 2a\mathbf{P}\{\zeta_n \geq a + 1\} \tag{71}$$

and

$$\mathbf{E}\{[\tau_n(a)]^2\}$$
$$= 2(n + a^2 + a + 1)\mathbf{P}\{\zeta_n \geq a + 1\} - (n + a + 1)(a + 2)\mathbf{P}\{\zeta_n = a + 1\}. \tag{72}$$

Similar expressions can be derived for $\mathbf{E}\{[\tau_n(a)]^r\}$ if $r \geq 2$.

6. The Sojourn Time $\omega_n(a)$

We defined $\omega_n(a)$ as the number of subscripts $r = 0, 1, 2, \ldots, n$ for which $\zeta_r \geq a$ where

$a = 0, 1, 2, \ldots$. If $1 \le a \le n$, then evidently

$$\mathbf{P}\{\omega_n(a) = 0\} = \mathbf{P}\{\rho(a) > n\} \tag{73}$$

and the distribution of $\rho(a)$ is given by (30). If $1 \le j \le n+1-a$, then we can write that

$$\mathbf{P}\{\omega_n(a) = j\} = \tfrac{1}{2}\mathbf{P}\{\rho(1) \ge j\}[\mathbf{P}\{\rho(a) > n-j\} - \mathbf{P}\{\rho(a-1) > n-j\}]. \tag{74}$$

See Theorem 2 in L. Takács [14]. By (74) we can prove that

$$\mathbf{P}\{\omega_n(a) = j\} = \binom{j-1}{[(j-1)/2]}\binom{n-j}{[(n+1-a-j)/2]}\frac{1}{2^n} \tag{75}$$

if $1 \le j \le n+1-a$. Since $\omega_n(0)$ has the same distribution as $n+1-\omega_n(1)$, we have

$$\mathbf{P}\{\omega_n(0) = j\} = \begin{cases} \mathbf{P}\{\omega_n(1) = j\} & \text{if } 1 \le j \le n, \\ \mathbf{P}\{\omega_n(1) = 0\} & \text{if } j = n+1. \end{cases} \tag{76}$$

By using (74), it is easy to prove that (22) holds and that $\mathbf{P}\{\omega(\alpha) \le x\}$ is given explicitly by (12) and (13).

Our next aim is to determine the binomial moments of $\omega_n(a)$. We shall show that the rth binomial moment of $\omega_n(a)$ can be expressed as a linear combination of the $2r$th binomial moments of $\tau_n(a-3r+k)$ for $k = 1, 2, \ldots, 3r$.

By (74) we obtain that

$$(1-w)\sum_{n=0}^{\infty}\mathbf{E}\{z^{\omega_n(a)}\}w^n = 1 - [\gamma(w)]^a + \frac{[1-\gamma(w)][\gamma(w)]^{a-1}[1-\gamma(zw)]zw}{2(1-zw)} \tag{77}$$

if $|w| < 1$, $|zw| < 1$ and $a \ge 1$, where $\gamma(w)$ is defined by (33). If we form the rth derivative of (77) with respect to z at $z = 1$, we get

$$\Phi_r(a) = (1-w)\sum_{n=0}^{\infty}\mathbf{E}\left\{\binom{\omega_n(a)}{r}\right\}w^n = \frac{2^{r-1}\gamma^{a+r-1}(1+\gamma^2)}{(1-\gamma^2)^{2r}}L_r(\gamma) \tag{78}$$

where

$$L_r(x) = (1+x)^{2r} + (1-x)^2(1+x)g_r(x) - (1+x^2)\sum_{j=0}^{r}(1+x)^{2r-2j+1}g_j(x) \tag{79}$$

is a polynomial of degree $< 3r$ in x. In (78) we used the abbreviation $\gamma = \gamma(w)$, and in (79), $g_j(x)$ is defined by (49). If we use the abbreviation $\Psi_r(a)$ for (63), suppressing w, then (78) can be expressed in the following way:

$$2^r\Phi_r(a) = \Psi_{2r}(a-3r+1)L_r(\gamma). \tag{80}$$

Since $\Psi_r(a)\gamma = \Psi_r(a+1)$ for any $r = 1, 2, \ldots$ and $a = 1, 2, \ldots$, the right-hand side of (80) can be expressed as a linear combination of $\Psi_{2r}(a-3r+k)$ for $k = 1, 2, \ldots, 3r$.

In particular, we have

$$2\Phi_1(a) = \Psi_2(a-2) + \Psi_2(a-1) \tag{81}$$

and

$$8\Phi_2(a) = \Psi_4(a-5) + 5\Psi_4(a-4) + 5\Psi_4(a-3) + \Psi_4(a-2). \tag{82}$$

Hence

$$2\mathbf{E}\{\omega_n(a)\} = \mathbf{E}\left\{\binom{\tau_n(a-2)}{2}\right\} + \mathbf{E}\left\{\binom{\tau_n(a-1)}{2}\right\} \tag{83}$$

and

$$8\mathbf{E}\left\{\binom{\omega_n(a)}{2}\right\} = \mathbf{E}\left\{\binom{\tau_n(a-5)}{4}\right\} + 5\mathbf{E}\left\{\binom{\tau_n(a-4)}{4}\right\}$$
$$+ 5\mathbf{E}\left\{\binom{\tau_n(a-3)}{4}\right\} + \mathbf{E}\left\{\binom{\tau_n(a-2)}{4}\right\}. \tag{84}$$

Theorem 2: *If $\alpha > 0$ and $r \geq 1$, then*

$$\lim_{n\to\infty} \mathbf{E}\left\{\left(\frac{\omega_n([\alpha\sqrt{n}])}{n}\right)^r\right\} = \frac{m_{2r}(\alpha)}{2^r r!} \tag{85}$$

where $m_r(\alpha)$ is given by (16).

Proof: Since $g_j(1) = C_{j-1}/2$ for $j = 1,2,\ldots$ where C_{j-1} is defined by (36) we have

$$L_r(1) = 2^{2r} - 2\sum_{j=1}^{r} C_{j-1} 2^{2r-2j} = \binom{2r}{r} \tag{86}$$

if $r \geq 1$. If in (80), $a = [\alpha\sqrt{n}]$, $a > 0$, and $r \geq 1$, we obtain that

$$\mathbf{E}\left\{\binom{\omega_n(a)}{r}\right\} \sim \mathbf{E}\left\{\binom{\tau_n(a)}{2r}\right\}\binom{2r}{r}\frac{1}{2^r} \tag{87}$$

or

$$\mathbf{E}\{[\omega_n(a)]^r\} \sim \mathbf{E}\{[\tau_n(a)]^{2r}\}\frac{1}{2^r r!} \tag{88}$$

as $n\to\infty$. This proves (85), and (9) follows from (85). □

7. The Sojourn Time $\omega_n^*(a)$

We defined $\omega_n^*(a)$ as the number of subscripts $r = 1,2,\ldots,n$ for which $|\zeta_r| \geq a$ where $a = 1,2,\ldots$. Let us associate a stochastic process $\{\zeta(t), t \geq 0\}$ with the random walk $\{\zeta_r, r \geq 0\}$. We say that the process $\{\zeta(t), t \geq 0\}$ is in state B in the interval $[r, r+1)$ if $|\zeta_r| \geq a$, and in state A if $|\zeta_r| < a$ where $a = 1,2,\ldots$. Then the process $\{\zeta(t), t \geq 0\}$ alternately is in the states A and B, and we can interpret $\omega_n^*(a)$ as $\beta(n+1)$ where $\beta(n+1)$ is the total time that the process $\{\zeta(t), t \geq 0\}$ spends in state

B in the interval $(0,n+1)$. If $\alpha_1,\beta_1,\alpha_2,\beta_2,\ldots$ denote the lengths of the successive intervals spent in states A and B respectively, then $\{\alpha_i\}$ and $\{\beta_i\}$ are independent sequences of independent random variables. Now α_1 has the same distribution as $\Theta(a,a)$; α_i, $(i \geq 2)$, has the same distribution as $\Theta(1,2a-1)$, and β_i, $(i \geq 1)$, has the same distribution as $\rho(1)$. The random variables $\Theta(a,b)$ and $\rho(a)$ are defined in Section 2. If we use the notation

$$R_{a,b}(w) = \mathbf{E}\{w^{\Theta(a,b)}\} = \frac{[\gamma(w)]^a + [\gamma(w)]^b}{1+[\gamma(w)]^{a+b}} \tag{89}$$

where $a \geq 1$ and $b \geq 1$, then by (45) we can write that

$$(1-w)(1-zw)\sum_{n=0}^{\infty}\mathbf{E}\{z^{\omega_n^*(a)}\}w^n = 1-zw-(1-z)\frac{R_{a,a}(w)[1-\gamma(zw)]}{1-\gamma(zw)R_{1,2a-1}(w)} \tag{90}$$

or

$$(1-w)\sum_{n=0}^{\infty}\mathbf{E}\{z^{\omega_n^*(a)}\}w^n = 1-2[\gamma(w)]^a\Big(\frac{1-z}{1-zw}\Big)\Big(\frac{1-\gamma(zw)}{A-B\gamma(zw)}\Big) \tag{91}$$

if $|w| < 1$ and $|zw| < 1$ where $A = 1+[\gamma(w)]^{2a}$ and $B = \gamma(w)+[\gamma(w)]^{2a-1}$.

By forming the rth derivative of (91) with respect to z at $z=1$ we obtain that

$$\Phi_r^*(a) = (1-w)\sum_{n=0}^{\infty}\mathbf{E}\left\{\binom{\omega_n^*(a)}{r}\right\}w^n = \frac{2^r\gamma^{a+r-1}(1+\gamma^2)}{(1-\gamma^2)^{2r}}U_r(\gamma) \tag{92}$$

for $a \geq 1$, $r \geq 1$ and $|w| < 1$ where

$$\begin{aligned} U_r(\gamma) = (1+\gamma)^{2r-1} - \sum_{s=1}^{r-1}\sum_{\nu=1}^{s}(1-\gamma^{2a-1})(\gamma+\gamma^{2a-1})^{\nu-1} \\ \cdot(1+\gamma^2)^{\nu}(1+\gamma)^{2r-2s-1}\nu! Q_{s,\nu}(\gamma)/s! \end{aligned} \tag{93}$$

is a polynomial in γ. In (92) and (93), $\gamma = \gamma(w)$ is defined by (33) and $Q_{s,\nu}(\gamma)$ by (52).

If we use the notation (63), we can write that

$$2^{r-1}\Phi_r^*(a) = \Psi_{2r}(a-3r+1)U_r(\gamma). \tag{94}$$

If we take into consideration that $\Psi_r(a)\gamma = \Psi_r(a+1)$ for $a \geq 1$ and $r \geq 1$, then the right-hand side of (94) can be expressed as a linear combination of $\Psi_{2r}((2j-1)a-3r+k)$ for $j=1,2,\ldots,r$ and $k=1,2,\ldots,3r$. By forming the coefficient of w^n on both sides of (94) we can express the rth binomial moment of $\omega_n^*(a)$ as a linear combination of the $2r$th binomial moments of $\tau_n((2j-1)a-3r+k)$ for $j=1,2,\ldots,r$ and $k=1,2,\ldots,3r$.

Theorem 3: *If $\alpha > 0$ and $r \geq 1$, then*

$$\lim_{n\to\infty}\mathbf{E}\left\{\left(\frac{\omega_n^*([\alpha\sqrt{n}])}{n}\right)^r\right\} = M_r^*(\alpha) \tag{95}$$

exists and $M_r^*(\alpha)$ is given by (10).

Proof: By (39) and (54) we can prove that

$$2^r x\{1 - \sum_{s=1}^{r-1} \sum_{\nu=1}^{s} (1-x^2)(1+x^2)^{\nu-1} 2^{\nu} \nu! Q_{s,\nu}(1)/(2^{2s}s!)\}$$
$$= \frac{4}{2^r} \sum_{j=1}^{r} \binom{2r-1}{r-j} x^{2j-1} \tag{96}$$

for $r \geq 1$ and therefore if in (94) we put $a = [\alpha\sqrt{n}]$, $\alpha > 0$, we obtain that

$$\mathbf{E}\left\{\binom{\omega_n^*(a)}{r}\right\} \sim \frac{4}{2^r} \sum_{j=1}^{r} \binom{2r-1}{r-j} \mathbf{E}\left\{\binom{\tau_n((2j-1)a)}{2r}\right\} \tag{97}$$

as $n \to \infty$ or

$$\mathbf{E}\{[\omega_n^*(a)]^r\} \sim \frac{(r-1)!}{2^{r-1}} \sum_{j=1}^{r} \frac{1}{(r-j)!(r+j-1)!} \mathbf{E}\{[\tau_n((2j-1)a)]^{2r}\} \tag{98}$$

as $n \to \infty$. We obtain (10) from (98) by making use of (24) and (26). □

Clearly, (23) and (95) imply (8).

Theorem 4: *If $\alpha > 0$, then*

$$\lim_{n \to \infty} \mathbf{P}\left\{\frac{\omega_n^*([\alpha\sqrt{n}])}{n} \leq x\right\} = G_\alpha(x) \tag{99}$$

for $0 < x < 1$ where $G_\alpha(x)$ is given by (11) for $0 \leq x < 1$ and $G_\alpha(1) = 1$.

Proof: By (9) and (10) we can write that

$$M_r^*(\alpha) = 2r!(r-1)! \sum_{k=1}^{r} \frac{1}{(r-k)!(r+k-1)!} M_r((2k-1)\alpha) \tag{100}$$

for $r \geq 1$ and $\alpha > 0$, and

$$M_r(\alpha) = r \int_0^1 x^{r-1}[1 - F_\alpha(x)]dx \tag{101}$$

for $r \geq 1$ and $\alpha > 0$ where the distribution function $F_\alpha(x)$ is given by (12). We shall determine $G_\alpha(x)$ by using Laplace-Stieltjes transforms.

Let us define

$$\Psi_\alpha^*(s) = \int_{-0}^{1} e^{-sx} dG_\alpha(x) \tag{102}$$

for $Re(s) \geq 0$. Since

$$\Psi_\alpha^*(s) = 1 + \sum_{r=1}^{\infty} \frac{(-1)^r s^r}{r!} M_r^*(\alpha) \tag{103}$$

for $Re(s) \geq 0$, by (100), (101), (55) and (56) we obtain that

$$\Psi_\alpha^*(s) = 1 + 2\sum_{r=1}^{\infty} \sum_{k=1}^{r} \frac{(-1)^r s^r r!}{(r-k)!(r+k-1)!} \int_0^1 x^{r-1}[1 - F_{(2k-1)\alpha}(x)]dx$$

$$= 1 + 2\sum_{k=1}^{\infty} \sum_{r=k}^{\infty} \frac{(-1)^r s^r r!}{(r-k)!(r+k-1)!} \int_0^1 x^{r-1}[1 - F_{(2k-1)\alpha}(x)]dx$$

$$= 1 + 2\sum_{k=1}^{\infty} (-1)^k s^k \int_0^1 \left(\sum_{j=0}^{\infty} \frac{(-1)^j s^j (j+k)! x^{j+k-1}}{j!(j+2k-1)!} \right) [1 - F_{(2k-1)\alpha}(x)]dx \tag{104}$$

$$= 1 + 2\sum_{k=1}^{\infty} \frac{(-1)^k s^k k!}{(2k-1)!} \int_0^1 e^{-sx} x^{k-1} M(k-1, 2k, sx)[1 - F_{(2k-1)\alpha}(x)]dx.$$

Now by (55),

$$M(k-1, 2k, sx) = \sum_{j=0}^{\infty} Q_k(j)(sx)^j, \tag{105}$$

where $Q_k(0) = 1$, $Q_k(1) = (k-1)/(2k)$ if $k \geq 1$, $Q_1(j) = 0$ if $j \geq 1$, and

$$Q_k(j) = \frac{(2k-1)!(k+j-2)!}{(k-2)!(2k+j-1)!j!} \tag{106}$$

for $k \geq 2$ and $j \geq 1$. By using (105) we can write that

$$\Psi_\alpha^*(s)$$

$$= 1 + 2\sum_{k=1}^{\infty} \frac{(-1)^k k!}{(2k-1)!} \sum_{j=0}^{\infty} Q_k(j) s^{k+j} \int_0^1 e^{-sx} x^{k+j-1}[1 - F_{(2k-1)\alpha}(x)]dx$$

$$= 1 + 2\sum_{k=1}^{\infty} \frac{(-1)^k k!}{(2k-1)!} \sum_{j=0}^{\infty} (-1)^{k+j} Q_k(j) \int_0^1 \left(\frac{d^{k+j} e^{-sx}}{dx^{k+j}} \right)$$

$$x^{k+j-1}[1 - F_{(2k-1)\alpha}(x)]dx$$

$$= 1 + 2\sum_{k=1}^{\infty} \frac{(-1)^k k!}{(2k-1)!} \sum_{\ell=k}^{\infty} (-1)^{\ell} Q_k(\ell-k) \int_0^1 \left(\frac{d^{\ell} e^{-sx}}{dx^{\ell}} \right)$$

$$\cdot x^{\ell-1}[1 - F_{(2k-1)\alpha}(x)]dx \tag{107}$$

$$= 1 + 4\sum_{k=1}^{\infty} \frac{(-1)^k k!}{(2k-1)!} \sum_{\ell=k}^{\infty} Q_k(\ell-k)(\ell-1)![1 - \Phi((2k-1)\alpha)]$$

$$+ 2\sum_{k=1}^{\infty} \frac{(-1)^k k!}{(2k-1)!} \sum_{\ell=k}^{\infty} Q_k(\ell-k) \int_0^1 e^{-sx} \frac{d^{\ell} x^{\ell-1}[1 - F_{(2k-1)\alpha}(x)]}{dx^{\ell}} dx.$$

By (107) we can conclude that

$$G_\alpha(0) = 1 + 4\sum_{k=1}^{\infty} \frac{(-1)^k k!}{(2k-1)!} \left(\sum_{\ell=k}^{\infty} Q_k(\ell-k)(\ell-1)! \right) [1 - \Phi((2k-1)\alpha)] \tag{108}$$

for $\alpha > 0$. By using (57) we can prove that (108) is indeed equal to (14). Furthermore, we have

$$\frac{dG_\alpha(x)}{dx} = 2\sum_{k=1}^{\infty}\frac{(-1)^k k!}{(2k-1)!}\sum_{\ell=k}^{\infty} Q_k(\ell-k)\frac{d^\ell x^{\ell-1}[1-F_{(2k-1)\alpha}(x)]}{dx^\ell} \tag{109}$$

if $0 < x < 1$.

By (107) it follows also that

$$G_\alpha(x) = 1 + 2\sum_{k=1}^{\infty}\frac{(-1)^k k!}{(2k-1)!}\sum_{\ell=k}^{\infty} Q_k(\ell-k)\,\frac{d^{\ell-1} x^{\ell-1}[1-F_{(2k-1)\alpha}(x)]}{dx^{\ell-1}} \tag{110}$$

for $0 \le x < 1$ and $\alpha > 0$. This proves (99). □

Finally, by (23) and (99) we can conclude that $\mathbf{P}\{\omega^*(\alpha) \le x\} = G_\alpha(x)$ and $G_\alpha(x)$ is given by (11) for $\alpha > 0$ and $0 \le x < 1$.

8. The Brownian Meander

The distribution of the sojourn time for the Brownian meander can be obtained in the same way as we found the distribution of $\omega^*(\alpha)$ for the Brownian motion. Let $\{\xi^+(t), 0 \le t \le 1\}$ be a standard Brownian meander such that $\mathbf{P}\{\xi^+(0) = 0\} = 1$ and $\mathbf{P}\{\xi^+(t) \ge 0\} = 1$ for all $0 \le t \le 1$. Define

$$\omega^+(\alpha) = \int_0^1 \delta(\xi^+(t) > \alpha)dt \tag{111}$$

for $\alpha \ge 0$. We can prove that

$$\mathbf{E}\{[\omega^+(\alpha)]^r\} = \frac{4r!}{2^r}\sqrt{\frac{\pi}{2}}\sum_{j=1}^{r}\frac{(-1)^{j-1}}{(r-j)!(r+j)!}$$

$$\cdot[(r+j)m_{2r-1}((2j-1)\alpha) - jm_{2r-1}(2j\alpha)] \tag{112}$$

for $r \ge 1$ and $\alpha > 0$ where $m_r(\alpha)$ is given by (16).

The moments (112) uniquely determine the distribution of $\omega^+(\alpha)$ and we have

$$\begin{aligned}\mathbf{P}\{\omega^+(\alpha) \le x\} = {} & 2\sum_{\ell=1}^{\infty}\sum_{k=1}^{\ell}\frac{k(k+1)!}{(2k)!}A_k(\ell-k)\frac{d^{\ell-1}x^{\ell-1}[1-F_{k\alpha}(x)]}{dx^{\ell-1}} \\ & -4\sum_{\ell=1}^{\infty}\sum_{k=1}^{\ell}\frac{k(k+1)!}{(2k)!}B_k(\ell-k)\frac{d^{\ell-1}x^{\ell-1}[1-F_{(2k-1)\alpha/2}(x)]}{dx^{\ell-1}}\end{aligned} \tag{113}$$

for $0 \le x < 1$, and $\mathbf{P}\{\omega^+(\alpha) \le 1\} = 1$ where $F_\alpha(x)$ is defined by (12) and (13). The coefficients $A_k(j)$ and $B_k(j)$ are defined by the series

$$M(k-1, 2k+1, x) = \sum_{j=0}^{\infty} A_k(j)x^j \tag{114}$$

and

$$M(k-2,2k,x)=\sum_{j=0}^{\infty}B_k(j)x^j, \tag{115}$$

and determined by (55).

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