A NON-NONSTANDARD PROOF OF REIMERS’ EXISTENCE RESULT FOR HEAT SPDES

HASSAN ALLOUBA
Duke University, Mathematics Department
Durham, NC 27708 USA
E-mail: allouba@math.duke.edu

(Received November, 1996; Revised July, 1997)

In 1989, Reimers gave a nonstandard proof of the existence of a solution to heat SPDEs, driven by space-time white noise, when the diffusion coefficient is continuous and satisfies a linear growth condition. Using the martingale problem approach, we give a non-nonstandard proof of this fact, and with the aid of Girsanov’s theorem for continuous orthogonal martingale measures (proved in a separate paper by the author), the result is extended to the case of a measurable drift.

Key words: Stochastic PDEs, Stochastic Heat Equation, Space-time White Noise, Martingale Problem.
AMS subject classifications: 60H15, 60G48, 60G46.

1. Introduction

We consider the SPDE

\[ \frac{\partial U}{\partial t} = \frac{1}{2} \frac{\partial^2 U}{\partial x^2} + a(U) \frac{\partial^2 W}{\partial t \partial x}, \quad (t, x) \in \mathbb{T} \times \mathbb{R}, \]

\[ U(0, x) = h(x), \]

where \( \mathbb{T} = [0, T] \) for some \( 0 < T < \infty \), \( W(t, x) \) is the Brownian sheet corresponding to the driving space-time white noise \( \partial^2 W / \partial t \partial x \), with intensity Lebesgue measure (see [15]). Our main result in this paper is

Theorem 1.1: Suppose that \( a \in C(\mathbb{R}; \mathbb{R}) \) (continuous real-valued function on \( \mathbb{R} \)) and \( h \) is a deterministic function in \( C_c(\mathbb{R}; \mathbb{R}) \) (continuous real-valued function on \( \mathbb{R} \) with compact support). Suppose further that there exists a constant \( K > 0 \) such that

\[ a^2(x) \leq K(1 + x^2), \]

for all \( x \in \mathbb{R} \). Then, there exists a solution to the heat SPDE (1.1).

Our approach will be as follows:

- We approximate the SPDE in (1.1) by a sequence of Stochastic Differential-Difference Equations (SDDEs) associated with interacting diffusion models and
solve the SDDEs;

- We then derive bounds on moments of spatial and temporal differences of the solutions obtained in the previous step;
- From those bounds, we conclude the tightness of the sequence of solutions;
- We then extract a subsequential limit, which solves a martingale problem that is equivalent to (1.1).

Girsanov’s theorem for space-time white noise may then be used (see [1] or [2]) to prove existence for

\[
\frac{\partial H}{\partial t} = \frac{1}{2} \frac{\partial^2 H}{\partial x^2} + b(H) + a(H) \frac{\partial^2 W}{\partial t \partial s}, \quad (t, x) \in T \times \mathbb{R},
\]

where \( a \) and \( h \) are as above, and the drift \( b \) is a Borel measurable real-valued function on \( \mathbb{R} \) such that the random field \( X(t, x) = b(U(t, x))/a(U(t, x)) \) satisfies the Novikov condition:

\[
E \left[ \exp \left( \frac{1}{2} \int_{[0, t] \times \mathbb{R}} X^2(s, x) ds dx \right) \right] < \infty; \quad t \in \mathbb{T},
\]

Theorem 1.2: Let \( \lambda \) be Lebesgue measure on the Borel \( \sigma \)-field \( \mathcal{B}(\mathbb{R}) \) and \( \mathcal{B}_e = \{ A \in \mathcal{B}(\mathbb{R}) \mid \lambda(A) < \infty \} \). Let \( W = \{ W_t(A), \mathcal{F}_t; 0 \leq t < \infty, A \in \mathcal{B}_e \} \) be a space-time white noise on a filtered probability space \( (\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \geq 0}, P) \) (see [2] or [1]). Define the process \( \tilde{W} = \{ \tilde{W}_t(A), \mathcal{F}_t; 0 \leq t < \infty, A \in \mathcal{B}_e \} \) by

\[
\tilde{W}_t(A) = W_t(A) - \int_{[0, t] \times A} Z(s, x) ds dx,
\]

where \( Z \) is some predictable random field (see [15]). Suppose that \( Z \) satisfies (1.4). Then, for each fixed \( T \in [0, \infty) \) and \( A \in \mathcal{B}_e \), the process \( \tilde{W} = \{ \tilde{W}_t(B), \mathcal{F}_t; 0 \leq t \leq T, B \in \mathcal{B}_e \mid A \} \) is a space-time white noise on the probability space \( (\Omega, \mathcal{F}_T, Q_T^A) \) having the intensity Lebesgue measure, where \( Q_T^A \) holds \( B \in \mathcal{B}_e \mid B \subseteq A \) and \( Q_T^A \) is the probability measure whose Radon-Nikodym derivative is given by

\[
\frac{dQ_T^A}{dP} = \exp \left[ -\int_{[0, T] \times A} Z(s, x) W(ds, dx) - \frac{1}{2} \int_{[0, T] \times A} Z^2(s, x) ds dx \right].
\]

Theorem 1.3: Suppose that (1.1) has a solution \( U \) and assume that \( X \), as defined above, satisfies Novikov condition (1.4). Then, there exists a solution to (1.3).

There are two rigorous formulations for (1.1): the test function formulation and the Green’s function formulation. The test function formulation is given by

\[
(U(T) - h, \varphi) = \frac{1}{2} \int_0^t (U(s), \varphi') ds + \int_0^t a(U(s, x)) \varphi(x) W(ds, dx);
\]
for all \( t \in \mathbb{T} \) and \( \varphi \in C_c^\infty(\mathbb{R}; \mathbb{R}) \), where \((\cdot, \cdot)\) denotes the scalar product on \( L^2(\mathbb{R}) \), and \( C_c^\infty(\mathbb{R}; \mathbb{R}) \) is the space of infinitely differentiable functions with compact support. The Green’s function formulation is the integral formulation

\[
U(t, x) = \int_{\mathbb{R}} G_t(x, y)h(y)dy + \int_0^t \int_{\mathbb{R}} G_{t-s}(x, y)a(U(s, y))W(ds, dy); \tag{1.7}
\]

for all \( t \in \mathbb{T}, x \in \mathbb{R} \), where \( G_t(x, y) \) is the fundamental solution of the heat equation in \( \mathbb{T} \times \mathbb{R} \). It is a well-known fact (see a discussion in [15] pp. 312-321), that the formulations in (1.6) and (1.7) are equivalent, provided the random field \( a(U) \) is locally bounded, which we will assume throughout this article.

**Remark 1.4:** We will sometimes place a superscript \( R_i \), \( i = 1, 2, \ldots \) above a mathematical relation; e.g., \( \leq \). This makes it easy to refer to the relation in question and renders our explanations more concise. Also, throughout this article, \( K \) will denote a constant that may change its value from line to line.

### 2. The Interacting Diffusion Models

Consider the sequence of sets \((X_n)_{n=1}^\infty\) defined by

\[
X_n = \{\ldots, -2\delta_n, -\delta_n, 0, \delta_n, 2\delta_n, \ldots\},
\]

where \( \delta_n > 0 \) for all \( n \) and \( \delta_n \to 0 \). Then the SPDE in (1.1) may be approximated by the following sequence of stochastic differential-difference equations (SDDEs):

\[
d\tilde{U}_n^x(t) = \frac{1}{2}\Delta_n \tilde{U}_n^x(t)dt + a(\tilde{U}_n^x(t))\frac{dW_n^x(t)}{\delta_n}, \tag{2.1}
\]

where \( t \in \mathbb{T} \) and \( x \in X_n \), and \( \Delta_n f(x) \) is the \( n \)-th approximate Laplacian given by

\[
\Delta_n f(x) = \frac{f(x + \delta_n) - 2f(x) + f(x - \delta_n)}{\delta_n^2}.
\]

We think of \( W_n^x(t) \) as a sequence of standard Brownian motions indexed by \( x \in X_n \), and we assume that, for each \( n = 1, 2, \ldots \), \( \tilde{U}_n^x(0) = h(x) \) for all \( x \in X_n \). It follows from the boundedness of \( h \) that

\[
\sup_{x \in X_n} |\tilde{U}_n^x(0)| \leq K. \tag{2.2}
\]

By a straightforward adaptation of Reimers’ observations ([11], pp. 325-326) we get

**Lemma 2.1:** There is a solution \( \tilde{U}_n^x(t) \) to (2.1) satisfying

\[
\tilde{U}_n^x(t) = \sum_{y \in X_n} \int_0^t Q_{\delta_n}^{x-y}v(t-s)a(\tilde{U}_n^y(s))\frac{dW_n^y(s)}{\sqrt{\delta_n}} + \sum_{y \in X_n} Q_{\delta_n}^{x-y}v(t)\tilde{U}_n^y(0), \tag{2.3}
\]

where \( Q_{\delta_n}^{x-y}(t) \) is the density of a random walk on the lattice \( X_n \), in which the times between transitions are exponentially distributed with mean \( 2p\delta_n^2 \), where \( p \) is the pro-
bability of a transition to the right (or to the left) and $1 - 2p$ is the probability of no transition at a transition epoch. The subscript $\delta_n$ in $Q_{\delta_n}^x(t)$ is to remind us that the size of each step is $\delta_n$. The second term on the r.h.s. of (2.3) is deterministic and will henceforth be denoted by $\hat{U}_n(t)$. From (2.2), it follows that
\[
|\hat{U}_n^x(t)| \leq K. \tag{2.4}
\]

Remark 2.2: Equation (2.3) may be thought of as the discrete-space-continuous time "Green’s function formulation" of the SDDE in (2.1).

3. Some Bounds

Here, we give bounds on the moments of spatial and temporal differences of the sequence $\{\hat{U}_{\delta_n}^x(t)\}_{n=1}^{\infty}$ that are used to conclude tightness for our approximating sequence, along with some inequalities related to $Q_{\delta_n}^x(t)$ and some bounds on the moments of $\hat{U}_{\delta_n}^x(t)$ that are useful in proving these spatial and temporal bounds. Since all the results in this section hold for all $n$, we will suppress the dependence on $n$ to simplify the notation. This section is a simple adaptation of Reimers’ corresponding results to our setting, and most of the proofs will be omitted.

3.1 Bounds related to $Q_{\delta_n}^x(t)$

Lemma 3.1: There is a constant $K$ such that
\[
\sum_{x \in \mathcal{X}} (Q_{\delta}^x(t))^2 \leq K\delta / \sqrt{t}.
\]

Lemma 3.2: There is a constant $K$ such that
\[
\int_0^t \sum_{x \in \mathcal{X}} (Q_{\delta}^x(s))^2 ds \leq K\delta \sqrt{t}.
\]

Lemma 3.3: There is a constant $K$ such that
\[
\int_0^t \sum_{x \in \mathcal{X}} (Q_{\delta}^x(s) - Q_{\delta}^x(s))^2 ds \leq K\delta \mid z \mid.
\]

Lemma 3.4: There is a constant $K$ such that
\[
\int_0^t \sum_{x \in \mathcal{X}} (Q_{\delta}^x(t) - Q_{\delta}^x(r - s))^2 ds \leq K\delta \sqrt{t-r},
\]
for $r > t$, and with the convention that $Q_{\delta}^x(t) = 0$ if $t < 0$. 

3.2 Bounds on moments of $\tilde{U}^x(t)$

**Lemma 3.5:** There exists a constant $K$ depending only on $p, q, \max_x |\tilde{U}^x(0)|$, and $T$ such that

$$F_q(t) \leq K \left(1 + \int_0^t \frac{F_q(s)}{\sqrt{t-s}} ds\right) \quad \forall t \in \mathbb{T},$$

where $F_q(t) = \sup_x E |\tilde{U}^x(t)|^{2q}$.

**Proof:** Fix $q > 1$, we then have:

$$E |\tilde{U}^x(t)|^{2q} = E \left| \sum_{y \in X} \int_0^t Q_{\delta} x - y(t-s)\frac{a(\tilde{U}^y(s))}{\sqrt{\delta}} dW^y(s) + \tilde{U}^x(t) \right|^{2q} \quad (3.1)$$

Now, as in [11] p. 327, applying Burkholder inequality to

$$V^x(t) = \sum_{y \in X} \int_0^t Q_{\delta} x - y(t-s)\frac{a(\tilde{U}^y(s))}{\sqrt{\delta}} dW^y(s),$$

we get

$$E |V^x(t)|^{2q} \leq KE \left| \sum_{y \in X} \int_0^t (Q_{\delta} x - y(t-s))^{2a(\tilde{U}^y(s))\delta} ds \right|^q \quad (3.2)$$

so that (3.2) reduces to

$$E |\tilde{U}^x(t)|^{2q} \leq KE \left| \sum_{y \in X} \int_0^t (Q_{\delta} x - y(t-s))^{2a^2(\tilde{U}^y(s))\delta} ds \right|^q + K |\tilde{U}^x(t)|^{2q}. \quad (3.3)$$

Now, for a fixed point $(t,x) \in \mathbb{T} \times X$, let $\mu_t^x$ be the measure on $[0,t] \times X$ defined by

$$d\mu_t^x(s,y) = ((Q_{\delta} x - y(t-s))^2/\delta) ds,$$

and let $|\mu_t^x| = \mu_t^x([0,t] \times X)$. Then, (3.3) can be rewritten as

$$E |\tilde{U}^x(t)|^{2q} \leq KE \left| \int_{[0,t] \times X} \frac{a^2(\tilde{U}^y(s))d\mu_t^x(s,y)}{|\mu_t^x|} \right|^q |\mu_t^x|^{q} + K |\tilde{U}^x(t)|^{2q}. \quad (3.4)$$

Observing that $\mu_t^x / |\mu_t^x|$ is a probability measure, we apply Jensen’s inequality to (3.4) to obtain

$$E |\tilde{U}^x(t)|^{2q} \leq KE \left[ \int_{[0,t] \times X} |a(\tilde{U}^y(s))|^{2q} \frac{d\mu_t^x(s,y)}{|\mu_t^x|} \right] |\mu_t^x|^{q} + K |\tilde{U}^x(t)|^{2q}.$$
\[
= K \left[ \int_{[0,t] \times X} E \left| a(\tilde{U}(s)) \left| \frac{2q}{d} \mu_t^\pi(s,y) \right| \right| \mu_t^\pi \right|^{-1} + K \left| \tilde{U}(t) \right|^{2q} \\
= K \left[ \sum_{y \in X} \int_0^t E \left| a(\tilde{U}(s)) \left| \frac{2q}{d} \left( Q_\delta^x - y(t-s) \right)^2 \right| ds \right| \mu_t^\pi \right|^{-1} + K \left| \tilde{U}(t) \right|^{2q}. \right]
\]

Now, by Lemma 3.2, \( |\mu_t^\pi|^{-1} \) is uniformly bounded for \( t \leq T \). This, together with (2.4) give us
\[
E \left| \tilde{U}(t) \right|^{2q} \leq K \left( 1 + \sum_{y \in X} \int_0^t E \left| a(\tilde{U}(s)) \left| \frac{2q}{d} \left( Q_\delta^x - y(t-s) \right)^2 \right| ds \right) \right.
\]
\[
\leq K \left( 1 + \sum_{y \in X} \int_0^t \left( 1 + \left( Q_\delta^x - y(t-s) \right)^2 \right) \left( Q_\delta^x - y(t-s) \right)^2 ds \right)
\]
\[
\leq K \left( 1 + \sum_{y \in X} \int_0^t E \left| a(\tilde{U}(s)) \left| \frac{2q}{d} \left( Q_\delta^x - y(t-s) \right)^2 \right| ds \right) \right.
\]
\[
= K \left( 1 + \sum_{y \in X} \int_0^t E \left| a(\tilde{U}(s)) \left| \frac{2q}{d} \left( Q_\delta^x - y(t-s) \right)^2 \right| ds \right) \right.
\]
\[
\leq K \left( 1 + \sum_{y \in X} \int_0^t E \left| a(\tilde{U}(s)) \left| \frac{2q}{d} \left( Q_\delta^x - y(t-s) \right)^2 \right| ds \right) \right.
\]
\[
\leq K \left( 1 + \int_0^t \left( 1 + \left( Q_\delta^x - y(t-s) \right)^2 \right) \left( Q_\delta^x - y(t-s) \right)^2 ds \right)
\]
Here \( R_1 \) follows from the linear growth condition on \( a \) and \( R_2 \) is a consequence of Lemma 3.2. Now, letting \( F_q(s) = \sup_{x} E \left| \tilde{U}(x) \right|^{2q} \), we get
\[
E \left| \tilde{U}(t) \right|^{2q} \leq K \left( 1 + \int_0^t F_q(s) \sum_{y \in X} \left( Q_\delta^x - y(t-s) \right)^2 ds \right)
\]
\[
\leq K \left( 1 + \int_0^t F_q(s) ds \right).
\]
Here, \( R_3 \) follows from Lemma 3.1. this implies that
\[
F_q(t) \leq K \left( 1 + \int_0^t F_q(s) ds \right).
\]

We easily obtain:
**Corollary 3.6:** There exists a constant \( K \) such that
\[ F_q(t) \leq K \left( 1 + \int_0^t F_q(s)ds \right), \]

for all \( t \in \mathbb{T} \).

**Proof:** Iterating the bound in Lemma 3.5 once, and changing the order of integration, we obtain

\[
F_q(t) \leq K \left\{ 1 + K \left[ \int_0^t \frac{1}{\sqrt{t-s}} ds + \int_0^t F_q(r) \left( \int_0^T \frac{1}{\sqrt{T-s}} \frac{1}{\sqrt{s-r}} ds \right) dr \right] \right\}
\]

\[
\leq K \left\{ 1 + K \left[ 2\sqrt{T} + \int_0^t F_q(r) \left( \int_0^T \frac{1}{\sqrt{T-s}} \frac{1}{\sqrt{s-r}} ds \right) dr \right] \right\}
\]

\[
\leq K \left\{ 1 + \int_0^t F_q(r) dr \right\}.
\]

By Gronwall's Lemma we have:

**Lemma 3.7:** There exists a constant \( K \) depending only on \( p, q, \max_x |\tilde{U}^x(0)|, \) and \( T \) such that

\[ E |\tilde{U}^x(t)|^{2q} \leq K \exp (Kt), \]

for all \( t \in \mathbb{T} \) and all \( x \in \mathbb{X} \).

### 3.3 Bounds on moments of spatial and temporal differences

Let \( \tilde{U}^x(t) = V^x(t) + \tilde{U}^x(t) \), where \( V \) denotes the first term on the r.h.s. of (2.3) (the random term). Using the inequalities of the previous two subsections, we obtain:

**Lemma 3.8:** (Spatial Differences) There exists a constant \( K \) depending only on \( p, q, \max_x |\tilde{U}^x(0)| \), and \( T \) such that

\[ E |V^x(t) - V^y(t)|^{2q} \leq K |x - y|^q, \]

for all \( x, y \in \mathbb{X} \) and for \( t \in \mathbb{T} \).

**Lemma 3.9:** (Temporal Differences) There exists a constant \( K \) depending only on \( p, q, \max_x |\tilde{U}^x(0)| \), and \( T \) such that

\[ E |V^x(t) - V^x(r)|^{2q} \leq K |t - r|^{q/2}, \]

for all \( x \in \mathbb{X} \) and for all \( t, r \in \mathbb{T} \).

### 4. The Sequence of SDDEs Solutions is Tight

Let \( \tilde{U}^x_n(t, x) \) be the extension of \( \tilde{U}^x_n(t) \) to \( \mathbb{T} \times \mathbb{R} \) obtained by linear interpolation of the \( \tilde{U}^x_n(t) \)'s between the lattice points of \( \mathbb{X}^n \). The following is Kolmogorov’s continuity criterion for random fields. (See [9] pp. 53-55 and p. 118; see also Corollary 1.2 in [15].)

**Lemma 4.1:** Suppose \( \{X(t); t \in [0, T]^d\}, d \geq 2, \), is a real-valued random field
satisfying
\[ E \left| X(t) - X(s) \right|^\alpha \leq C \| t - s \|^d + \beta \]
for some positive constants \( \alpha, \beta \) and \( C \), for some norm \( \| \cdot \| \) on \( \mathbb{R}^d \). Then there exists a continuous modification \( \tilde{X} = \{ \tilde{X}(t); t \in [0, T]^d \} \) of \( X \).

We also have the following tightness criterion:

**Lemma 4.2:** For each \( n = 1, 2, \ldots \), let \( \{X_n(t); t \in \mathbb{T} \times \mathbb{R}\} \) be a real-valued continuous random field. The sequence \( \{X_n\} \), regarded as a sequence of random variables taking values in \( C(\mathbb{T} \times \mathbb{R}) \), is tight in \( C(\mathbb{T} \times \mathbb{R}) \) if:

(i) \( \{X_n(0)\} \) is tight,

(ii) \( E \left| X_n(t) - X_n(s) \right|^\alpha \leq C \| t - s \|^2 + \beta \), for some positive constants \( \alpha, \beta, C \).

The proof of Lemma 4.2 follows very closely the proof of Theorem 12.3 in [3] and will not be repeated.

**Remark 4.3:** Since all norms on \( \mathbb{R}^d \) are equivalent, the norm \( \| \cdot \| \) in Lemma 4.1 and Lemma 4.2 may be chosen to be any norm on \( \mathbb{R}^d \).

**Lemma 4.4:** For each \( n \), the random field \( \tilde{U}_n = \{ \tilde{U}_n(t, x); t \in \mathbb{T}, x \in \mathbb{R}\} \) has a continuous modification, which we will also denote by \( \tilde{U}_n \), and the sequence \( \left( \tilde{U}_n \right)_{n=1}^\infty \) is tight in \( C(\mathbb{T} \times \mathbb{R}) \).

**Proof:** This follows from Lemma 3.8, Lemma 3.9, Lemma 4.1, and Lemma 4.2 (see also the discussion on p. 97 in [3]). This is a routine argument, so we omit the details.

## 5. The Martingale Problem

Since the sequence \( \left( \tilde{U}_n(t, x) \right)_{n=1}^\infty \) is tight, it follows that there exists a subsequence \( \tilde{U}_{n_k} \), which induces measures \( \tilde{P}_{n_k} \) on \( (C, \mathcal{C}) \) that converge weakly to a limit \( \tilde{P} \), where \( C = C(\mathbb{T} \times \mathbb{R}; \mathbb{C}) \) and \( \mathcal{C} = \mathcal{B}(C) \), where \( \mathcal{B}(C) \) is the Borel \( \sigma \)-field over \( C \). Now, following Skorokhod [13], we can construct processes \( Y_k \overset{d}{=} \tilde{U}_{n_k} \) on some probability space \( (\Omega^S, \mathcal{S}^S, \mathbb{P}^S) \) such that with probability 1, as \( k \to \infty \), \( Y_k(t, x) \) converges to a random field \( Y(t, x) \) uniformly on compact subsets of \( \mathbb{T} \times \mathbb{R} \) for any \( T < \infty \). We will show that \( Y(t, x) \) is a solution to the heat SPDE (1.1) by solving an equivalent martingale problem ([8]).

For every \( \varphi \in C_c^\infty(\mathbb{R}; \mathbb{R}) \), let
\begin{equation}
S^\varphi(Y_k, t) = \sum_{x \in X_k} Y_k(t, x) \varphi(x) \delta_{n_k} - \frac{1}{2} \int_0^t \sum_{x \in X_k} Y_k(s, x) \Delta_n \varphi(x) \delta_{n_k} \, ds \tag{5.1}
\end{equation}

and let \( \mathcal{G}_t \) be the filtration on \( (\Omega^S, \mathcal{S}^S, \mathbb{P}^S) \) generated by the process \( S^\varphi(Y_k, t) \) for all \( \varphi \) and all \( k \); i.e., \( \mathcal{G}_t = \sigma[S^\varphi(Y_k, s); 0 \leq s \leq t, \varphi \in C_c^\infty(\mathbb{R}; \mathbb{R}), k = 1, 2, \ldots] \). We prove Theorem 1.1 by proving:

**Theorem 5.1:** There exists an extension \( (\tilde{\Omega}^S, \tilde{\mathcal{S}}^S, \{\tilde{\mathcal{G}}_t\}_{t \geq 0}^0, \mathbb{P}^S) \) of the filtered probability space \( (\Omega^S, \mathcal{S}^S, \{\mathcal{G}_t\}_{t \geq 0}^0, \mathbb{P}^S) \) and a white noise \( W \) defined on it such that the pair \( (Y, W) \) solves SPDE (1.1) on \( (\tilde{\Omega}^S, \tilde{\mathcal{S}}^S, \{\tilde{\mathcal{G}}_t\}_{t \geq 0}^0, \mathbb{P}^S) \).
The proof of Theorem 5.1 follows from Theorem 3.3 in [8] in conjunction with Theorem 5.2 and Theorem 5.3 below.

**Theorem 5.2:** For \( \varphi \in C_c^\infty (\mathbb{R}; \mathbb{R}) \), we have

(i) \( \{S^\varphi (t), \mathcal{F}_t\} \) is a martingale, for every \( \varphi \in C_c^\infty (\mathbb{R}; \mathbb{R}) \), where

\[
S^\varphi (t) = (Y(t), \varphi) - \frac{1}{2} \int_0^t (Y(s), \varphi''(s))ds,
\]

where \((\cdot, \cdot)\) denotes the scalar product on \( L^2(\mathbb{R}) \).

(ii) \( \langle (Y, \varphi) \rangle_t = \int_0^t \int_{\mathbb{R}} a^2 (Y(s, x)) \varphi^2 (x) dx ds. \)

**Proof:** (i) Assume that the sequence of Brownian motions \( W_n^x(t) \) in (2.1) is defined on some probability space \((\Omega, \mathcal{F}, P)\) and adapted to a filtration \( \{\mathcal{F}_t\}_{t \geq 0} \). We first observe that for any \( k \),

\[
\tilde{U}_{n,k}^x (t) \varphi(x) \delta_{n,k} - \frac{1}{2} \int_0^t \Delta_{n,k} \tilde{U}_{n,k}^x (s) \varphi(x) \delta_{n,k} ds
\]

is a \( \mathcal{F}_t \)-martingale for each \( x \in X_{n,k} \). This is obvious from (2.1). Now since \( \varphi \) has a compact support, it follows that

\[
\sum_{x \in X_{n,k}} \tilde{U}_{n,k}^x (t) \varphi(x) \delta_{n,k} - \frac{1}{2} \int_0^t \Delta_{n,k} \tilde{U}_{n,k}^x (s) \varphi(x) \delta_{n,k} ds
\]

\[
= \sum_{x \in X_{n,k}} \tilde{U}_{n,k}^x (t) \varphi(x) \delta_{n,k} - \frac{1}{2} \int_0^t \sum_{x \in X_{n,k}} \tilde{U}_{n,k}^x (s) \Delta_{n,k} \varphi(x) \delta_{n,k} ds
\]

\[
= S^\varphi (\tilde{U}_{n,k}, t)
\]

is a finite sum, and hence an \( \mathcal{F}_t \)-martingale. Replacing the \( \tilde{U}_{n,k}^x (t) \)'s in (5.2) by the \( Y_k(t, x) \)'s and letting \( k \to \infty \) we get that \( S^\varphi (Y_k, t) \to S^\varphi (t) \) a.s. uniformly on \( \mathbb{T} \). In addition, the \( S^\varphi (Y_k, t) \) are uniformly integrable for each \( t \) and each \( \varphi \) (for each \( t \) and each \( \varphi \in C_c^\infty (\mathbb{R}; \mathbb{R}) \), \( E |S^\varphi (Y_k, t)|^p \leq M_p < \infty \forall k \), for some constant \( M_p \), for any \( p > 2 \)). So, if \( s < t \),

\[
E[(S^\varphi (t) - S^\varphi (s)) | \mathcal{G}_s] = \lim_{k \to \infty} E[(S^\varphi (Y_k, t) - S^\varphi (Y_k, s)) | \mathcal{G}_s] = 0.
\]

This proves (i).

(ii) From (2.1) it follows that

\[
d \left[ \sum_{x \in X_{n,k}} \tilde{U}_{n,k}^x (t) \varphi(x) \delta_{n,k} \right] = \frac{1}{2} \sum_{x \in X_{n,k}} \Delta_{n,k} \tilde{U}_{n,k}^x (t) \varphi(x) \delta_{n,k} dt
\]

\[
+ \sum_{x \in X_{n,k}} \frac{a (\tilde{U}_{n,k}^x (t))}{\sqrt{\delta_{n,k}}} \varphi(x) \delta_{n,k} dW_{n,k}^x (t)
\]
Observing that the first term on the right-hand side of the above equation is of bounded variation, and that the \( (W^x_{n_k}(t))_{x \in \mathcal{X}_{n_k}} \) is a sequence of independent Brownian motions, we obtain

\[
\mathbb{E} \left[ \sum_{x \in \mathcal{X}_{n_k}} \tilde{U}^x_{n_k}(t) \varphi(x) \delta_{n_k} \right] = \int_0^t \left[ \sum_{x \in \mathcal{X}_{n_k}} a^2 \left( \tilde{U}^x_{n_k}(s) \right) \varphi^2(x) \delta_{n_k} \right] ds.
\] (5.3)

Again, replacing the \( \tilde{U}^x_{n_k}(t) \)'s in (5.3) by the \( Y_k(t,x) \)'s, we get, for \( 0 \leq r \leq t < \infty \),

\[
\mathbb{E} \left[ \left( \sum_{x \in \mathcal{X}_{n_k}} Y_k(t,x) \varphi(x) \delta_{n_k} - \sum_{x \in \mathcal{X}_{n_k}} Y_k(r,x) \varphi(x) \delta_{n_k} \right)^2 \right] \leq \zeta_r.
\] (5.4)

We now observe that \( \sum_{x \in \mathcal{X}_{n_k}} Y_k(t,x) \varphi(x) \delta_{n_k} \) are uniformly integrable for \( t \in \mathbb{T} \) and for each \( \varphi \) (again, for \( t \in \mathbb{T} \) and for each \( \varphi \in C^\infty_c(\mathbb{R};\mathbb{R}) \), \( \mathbb{E} \left[ \sum_{x \in \mathcal{X}_{n_k}} Y_k(t,x) \varphi(x) \delta_{n_k} \right]^p \leq K_p < \infty \forall k \), for some constant \( K_p \), for any \( p > 2 \)). Consequently,

\[
\lim_{k \to \infty} \mathbb{E} \left[ \left( \sum_{x \in \mathcal{X}_{n_k}} Y_k(t,x) \varphi(x) \delta_{n_k} - \sum_{x \in \mathcal{X}_{n_k}} Y_k(r,x) \varphi(x) \delta_{n_k} \right)^2 \right] \leq \zeta_r.
\] (5.5)

Also, \( a \) is continuous, and hence locally bounded. We see by the Dominated Convergence theorem that

\[
\lim_{k \to \infty} \mathbb{E} \left[ \int_t^r a^2(Y_k(s,x)) \varphi^2(x) \delta_{n_k} ds \right] \leq \zeta_r.
\] (5.6)

Now, equations (5.4), (5.5) and (5.6) yield

\[
= \int_r^t \int a^2(Y(s,x)) \varphi^2(x) dx ds \right] \leq \zeta_r.
\]
and (ii) is proved. \ \square

Let $\mathcal{D} = \{ f : C(\mathbb{R}; \mathbb{R}) \to \mathbb{R} \mid f(X) = f((X, \varphi_1), \ldots, (X, \varphi_n)) \text{ for some } n \geq 1, \varphi_1, \ldots, \varphi_n \in C^{\infty}_K(\mathbb{R}; \mathbb{R}) \text{ and } \tilde{f} \in C^{\infty}_K(\mathbb{R}^n; \mathbb{R}) \}$, and define the operator $L$ with domain $\mathcal{D}$ as follows:

$$Lf(X) = \frac{1}{2} \sum_{i,j=1}^{n} \left\{ \int_{\mathbb{R}} a^2(x)\varphi_i(x)\varphi_j(x)dx \right\} D_{ij} \tilde{f}((X, \varphi_1), \ldots, (X, \varphi_n))$$

$$+ \frac{1}{2} \sum_{i} \int_{\mathbb{R}} \varphi_i''(x) D_i \tilde{f}((X, \varphi_1), \ldots, (X, \varphi_n)).$$

We now prove our key martingale theorem.

**Theorem 5.3:** If $f \in \mathcal{D}$, then

$$f(Y(t, \cdot)) - \int_{0}^{t} Lf(Y(s, \cdot))ds \text{ is a } \mathcal{G}_t\text{-martingale.}$$

**Proof:** Since $f \in \mathcal{D}$, we have that

$$f(Y(t, \cdot)) - f(Y(0, \cdot))$$

$$\overset{R_1}{=} \tilde{f}((Y(t), \varphi_1), \ldots, (Y(t), \varphi_n)) - \tilde{f}((Y(0), \varphi_1), \ldots, (Y(0), \varphi_n))$$

$$\overset{R_2}{=} \sum_{i=1}^{n} \int_{0}^{t} D_i \tilde{f}((Y(t), \varphi_1), \ldots, (Y(t), \varphi_n))d(Y(s), \varphi_i)$$

$$+ \frac{1}{2} \sum_{i,j=1}^{n} \int_{0}^{t} D_{ij} \tilde{f}((Y(t), \varphi_1), \ldots, (Y(t), \varphi_n))d((Y, \varphi_i), (Y, \varphi_j)),$$

where $R_1$ follows for some $n \geq 1$, $\varphi_1, \ldots, \varphi_n \in C^{\infty}_K(\mathbb{R}; \mathbb{R})$ and $\tilde{f} \in C^{\infty}_K(\mathbb{R}^n, \mathbb{R})$, by the definition of $\mathcal{D}$; and $R_2$ follows from Itô's rule for $n$-continuous semimartingales.

However,

$$\langle (Y, \varphi_i), (Y, \varphi_j) \rangle_s = \frac{1}{4}((Y, \varphi_i + \varphi_j)_s - ((Y, \varphi_i) - \varphi_j)_s)$$

$$\overset{R_1}{=} \int_{0}^{s} \int_{\mathbb{R}} a^2(Y(u, x))\varphi_i(x)\varphi_j(x)dx du$$

$$= \int_{0}^{s} (a^2(Y(u), \varphi_i)\varphi_j)du,$$

where $R_1$ follows from part (ii) of Theorem 5.2. Also, by part (i) of Theorem 5.2, we have $d(Y(s), \varphi) = dS^\varphi(s) + \frac{1}{2}(Y(s), \varphi'')ds$, so that we can rewrite the expression for
\( f(Y(t, \cdot)) - f(Y(0, \cdot)) \) above as:

\[
\begin{align*}
  f(Y(t, \cdot)) - f(Y(0, \cdot)) \\
  = & \frac{1}{2} \sum_{i=1}^{n} \int_{0}^{t} D_{i} \tilde{f} ((Y(s), \varphi_1), \ldots, (Y(s), \varphi_n)) (Y(s), \varphi_i) ds \\
  & + \frac{1}{2} \sum_{i,j=1}^{n} \int_{0}^{t} D_{ij} \tilde{f} ((Y(s), \varphi_1), \ldots, (Y(s), \varphi_n)) \{(a^2(Y(s)), \varphi_i \varphi_j)\} ds \\
  & + \frac{1}{2} \sum_{i=1}^{n} \int_{0}^{t} D_{i} \tilde{f} ((Y(s), \varphi_1), \ldots, (Y(s), \varphi_n)) dS_{i}(s).
\end{align*}
\]

(5.10)

The third term on the right-hand side of (5.10) is a finite sum of \( \mathcal{G}_t \)-martingales, by part (i) of Theorem 5.2 and the boundedness of \( D_{ij} \tilde{f} \), and hence is a \( \mathcal{G}_t \)-martingale. Now,

\[
\begin{align*}
  \int_{0}^{t} L f(Y(s, \cdot)) ds \\
  = & \frac{1}{2} \sum_{i=1}^{n} \int_{0}^{t} D_{i} \tilde{f} ((Y(s), \varphi_1), \ldots, (Y(s), \varphi_n)) (Y(s), \varphi_i) ds \\
  & + \frac{1}{2} \sum_{i,j=1}^{n} \int_{0}^{t} D_{ij} \tilde{f} ((Y(s), \varphi_1), \ldots, (Y(s), \varphi_n)) \{(a^2(Y(s)), \varphi_i \varphi_j)\} ds.
\end{align*}
\]

Consequently,

\[
\begin{align*}
  f(Y(t, \cdot)) - f(Y(0, \cdot)) - \int_{0}^{t} L f(Y(s, \cdot)) ds
\end{align*}
\]

is a \( \mathcal{G}_t \)-martingale. This completes the proof. \( \square \)

Acknowledgement

This paper is based on a part of my Ph.D. dissertation [1] under Rick Durrett. I would like to thank Rick for his constant encouragement and for the excellent scientific environment he provided for me while at Cornell. I wish to also thank an anonymous referee for his comments, which made this paper more readable.

References


Submit your manuscripts at http://www.hindawi.com