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# THE SOLUTION OF AN OPEN PROBLEM GIVEN BY H. HARUKI AND T.M. RASSIAS

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Haruki and Rassias [1] generalized the Poisson kernel in two dimensions and discussed integral formulas for each case. They presented an open problem for an integral formula. In this paper, we give a solution to that problem.

**Key words:** Poisson Kernel, Integral Formula.
**AMS subject classifications:** 31A05, 31A10.

## 1. Introduction

Haruki and Rassias [1] introduced two types of generalizations of the Poisson kernel. One of them is defined by

$$Q(\theta; a, b) \triangleq \frac{1-ab}{(1-ae^{i\theta})(1-be^{-i\theta})},$$

where $a, b$ are complex parameters satisfying $|a| < 1$ and $|b| < 1$.

They proved the integral formulas:

$$\frac{1}{2\pi}\int_0^{2\pi} Q(\theta; a, b)d\theta = 1, \tag{1}$$

$$\frac{1}{2\pi}\int_0^{2\pi} Q(\theta; a, b)^2 d\theta = \frac{1+ab}{1-ab}. \tag{2}$$

They set the open problem as follows:
"Let

$$I_n \triangleq \frac{1}{2\pi}\int_0^{2\pi} Q(\theta;a,b)^{n+1}d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1-ab}{(1-ae^{i\theta})(1-be^{-i\theta})}\right)^{n+1}d\theta, \quad (n=0,1,\ldots), \tag{3}$$

where $a,b$ are complex parameters satisfying $|a|<1$ and $|b|<1$. Compute $I_n$ for $n=2,3,4,\ldots$."

In the next section, we will give the solution to the problem.

## 2. Solution of the Problem

**Theorem 1:** *$I_n$, defined by (3), satisfies*

$$I_n = \sum_{j=0}^{n} \frac{(2n-j)!}{j!((n-j)!)^2}\left(\frac{ab}{1-ab}\right)^{n-j},$$

*for $n=0,1,2,\ldots$, and complex values $a,b$ are such that $|a|<1$ and $|b|<1$.*

**Proof:** By the change of variables, with $z=e^{i\theta}$, (3) becomes

$$I_n = \frac{1}{2\pi i}\oint_{|z|=1}\left(\frac{1-ab}{(1-az)(1-bz^{-1})}\right)^{n+1} z^{-1}dz$$

$$= \frac{1}{2\pi i}\oint_{|z|=1}\left(\frac{1-ab}{1-az}\right)^{n+1} z^n(z-b)^{-n-1}dz.$$

Let

$$f(z) \triangleq \left(\frac{1-ab}{1-az}\right)^{n+1} z^n(z-b)^{-n-1}.$$

Then $f(z)$ is analytic on $\{z\in\mathbb{C}\colon |z|\le 1, z\ne b\}$ and has a pole at $z=b$. Therefore, by the residue theorem, $I_n$ is the residue of $f(z)$ at $z=b$.

The Laurent series expansion of $f(z)$ at $z=b$ gives:

$$f(z) = \left(\frac{1}{1-\frac{a}{1-ab}(z-b)}\right)^{n+1}(b+(z-b))^n(z-b)^{-n-1}$$

$$= \sum_{k=0}^{\infty}\binom{n+k}{k}\left(\frac{a}{1-ab}\right)^k(z-b)^k\sum_{j=0}^{n}\binom{n}{j}b^{n-j}(z-b)^j(z-b)^{-n-1}$$

$$= \sum_{k=0}^{\infty}\sum_{j=0}^{n}\binom{n+k}{k}\binom{n}{j}\left(\frac{a}{1-ab}\right)^k b^{n-j}(z-b)^{k+j-n-1}.$$

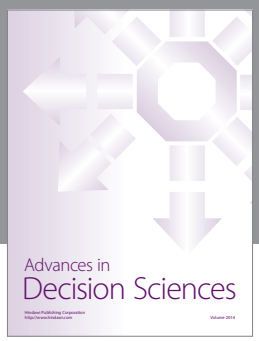
Therefore, the residue of $f(z)$ at $b$, which is $I_n$, is given by

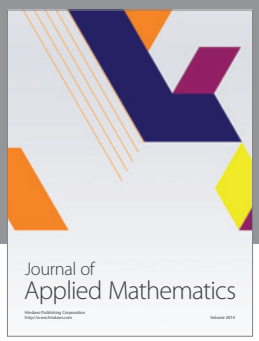
$$I_n = \sum_{j=0}^{n} \binom{2n-j}{n-j} \binom{n}{j} \left(\frac{ab}{1-ab}\right)^{n-j}$$
$$= \sum_{j=0}^{n} \frac{(2n-j)!}{j!((n-j)!)^2} \left(\frac{ab}{1-ab}\right)^{n-j}. \qquad \square$$

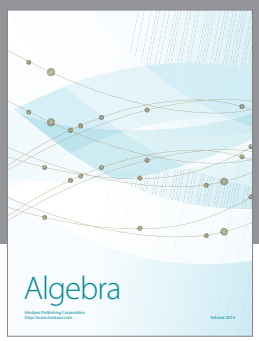
Note that we obtain (1) and (2) by substituting $n = 0$ and $n = 1$, respectively.

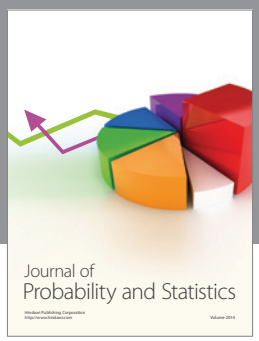
## References

[1] Haruki, H. and Rassias, T.M., New generalizations of the Poisson kernel, *J. Appl. Math. Stoch. Anal.* **10**:2 (1997), 191-196.







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