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ON THE STABILIZATION OF THE ENERGY OF A HARMONIC OSCILLATOR DISTURBED BY RANDOM PROCESSES OF THE "WHITE AND SHOT NOISES" TYPES

GRIGORI L. KULINICH
and SVITLANA V. KUSHNIRENKO
Kyiv University
Mechanics and Mathematics Faculty
64 Volodymyrska St.
Kyiv 252033, Ukraine

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In this paper the behavior of the instantaneous energy of a harmonic oscillator is investigated in the case when at a certain angle to the vector of the phase velocity of the oscillator, random disturbances of the "white and shot noises" types are acting.

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1. Introduction

By harmonic oscillator without friction we mean an oscillating system for which motion is described by the following linear differential equation of the second order

$$\ddot{u}(t) + k^2 u(t) = 0, \quad u(0) = u_0, \quad \dot{u}(0) = \dot{u}_0, \tag{1}$$

where u_0 is the initial position and $\dot{u}_0$ is the initial velocity of the oscillator ($u_0^2 + \dot{u}_0^2 > 0$); $k > 0$ is a parameter of the oscillator; $u(t)$ is the position and $\dot{u}(t)$ is the velocity of the oscillator at the moment of time $t > 0$, and $\varepsilon(t) = \frac{1}{2}[k^2 u^2(t) + \dot{u}^2(t)]$ is the instantaneous energy of the oscillator.

Equation (1) is equivalent to the system of first order differential equations

$$\begin{cases} \dot{x}_1(t) = k x_2(t), \\ \dot{x}_2(t) = -k x_1(t), \end{cases} \tag{2}$$

where $x_1(t) = ku(t), x_2(t) = \dot{u}(t)$. In addition, $2\varepsilon(t) = |x(t)|^2$, where $x(t) = (x_1(t), x_2(t))$.

In the present paper we investigate the behavior of instantaneous energy $\varepsilon(t)$ in the case when, at a certain angle to $b = (kx_2(t), -kx_1(t))$, where b is the vector of the phase velocity of system (2), fluctuations of the "white noise" type ($\dot{w}(t)$ is a "derivative" of a Wiener process $w(t)$) and fluctuations of the "shot noise" type ($\dot{\nu}([0,t), R)$ is a "derivative" of a Poisson measure $\nu([0,t), R)$) are acting. In this case system (2) is considered as the following system of stochastic differential equations without aftereffect (see [2]):

$$dx(t) = a(t, x(t))dt + b(t, x(t))dw(t) + \int_R c(t, x(t), u)\nu(dt, du), \tag{3}$$

where $a(t,x) = (q_1(t,x)x_1 + q_2(t,x)x_2,\ -q_2(t,x)x_1 + q_1(t,x)x_2)$,

$$x = (x_1, x_2) \in R \times R,\ x_1(0) = ku_0, x_2(0) = \dot{u}_0,$$

$$b(t,x) = (g_1(t,x)x_1 + g_2(t,x)x_2,\ -g_2(t,x)x_1 + g_1(t,x)x_2),$$

$$c(t,x,u) = (\gamma_1(t,x,u)x_1 + \gamma_2(t,x,u)x_2) - \gamma_2(t,x,u)x_1 + \gamma_1(t,x,u)x_2),$$

$u \in R$ is a non-random vector function, $w(t)$ is a one-dimensional Wiener process, $\nu([0,t), A))$ is a Poisson measure with parameter $t\Pi(A)$, such that $\Pi(R) < \infty$. The process $w(t)$ and the measure $\nu([0,t), A)$, are defined on the probability space (Ω, F, P). They are jointly independent and F_t-measurable for any $t \geq 0$ and A, where $F_t \subset F$ is a nondecreasing family of σ-algebras.

Qualitative analysis of the behavior of the harmonic oscillator without friction under the random perturbation along the vector of the phase velocity by stochastic process of the "white noise" type is made in paper [5] and qualitative analysis of the behavior of the harmonic oscillator with friction is made in paper [6]. Book [8] gives a formula for the fundamental matrix for linear equations of type (3) with varying coefficients. For equations with constant coefficients, conditions are given under which $|x(t)| \to 0$ with probability 1 as $t \to \infty$ as well as conditions under which $E|x(t)|^2 \to 0$ as $t \to \infty$. The behavior of the instantaneous energy of the harmonic oscillator under the random perturbation only of the second component of the vector of the phase velocity was investigated by many authors (see, for example [3, 4, 7, 9]).

In the present paper, we investigate the sufficient conditions under which the instantaneous energy does not change: $\varepsilon(t) = \varepsilon(0)$ (Corollary 1 of Theorem 1), the sufficient conditions under which the instantaneous energy $\varepsilon(t)$ changes only step-wise (Theorem 2), as well as the sufficient conditions of stability $\varepsilon(t)$ (Theorems 3-5) are established for equation (3) in terms of functions $q_i(t,x)$, $g_i(t,x)$, $\gamma_i(t,x,u)$. It is shown that it is possible to control the behavior of $\varepsilon(t)$ by the choice of function $q_1(t,x)$ (determined disturbance).

We will assume that functions $q_i(t,x)$, $g_i(t,x)$, $\gamma_i(t,x,u)$ are such that coefficients of equation (3) satisfy the conditions:

1. $\exists C > 0$: $|a(t,x)|^2 + |b(t,x)|^2 + \int_R |c(t,x,u)|^2 \Pi(du) \leq C[1 + |x|^2]$;

2. $\forall N > 0 \exists C_N$: $|a(t,x) - a(t,y)|^2 + |b(t,x) - b(t,y)|^2 + \int_R |c(t,x,u) - c(t,y,u)|^2 \Pi(du) \leq C_N |x-y|^2$ with $|x| \leq N$, $|y| \leq N$;
3. $\Pi\{u: |x + c(t,x,u)| = 0\} = 0$ for all $t \geq 0$, $|x| \neq 0$.

It is known (see [2]) that conditions 1, 2 guarantee existence of the unique continuation from the right strong solution $x(t) = (x_1(t), x_2(t))$ of equation (3).

In addition, we will use the following designations:

$$\tilde{\nu}(t,A) = \nu(t,A) - t\Pi(A);$$

$$I(t,x) = 2q_1(t,x) + g_2^2(t,x);$$

$$I_1(t,x) = I(t,x) + g_1^2(t,x);$$

$$I_2(t,x) = I(t,x) - g_1^2(t,x);$$

$$\psi(t,x,u) = (1 + \gamma_1(t,x,u))^2 + \gamma_2^2(t,x,u).$$

2. Stabilization of $\varepsilon(t)$

According to the generalized Ito's formula (see [2]):

$$d|x(t)|^2 = [2(x(t), a(t,x(t))) + |b(t,x(t))|^2]dt + 2(x(t), b(t,x(t)))dw(t) + \int_R [|x(t) + c(t,x(t),u)|^2 - |x(t)|^2]\nu(dt,du),$$

where $(\cdot,\cdot)$ is the inner product.

Thus,

$$d|x(t)|^2 = |x(t)|^2 \left\{ I_1(t,x(t))dt + 2g_1(t,x(t))dw(t) + \int_R [\psi(t,x(t),u) - 1]\nu(dt,du) \right\}. \tag{4}$$

Condition 3 implies that $\psi(t,x,u) > 0$ in measure $\Pi(du)$ for all $t \geq 0$, x. Therefore (see [8]):

$$|x(t)|^2 = |x(0)|^2 \exp\left\{ \int_0^t I_2(s,x(s))ds + 2\int_0^t g_1(s,x(s))dw(s) \int_0^t \int_R \ln\psi(s,x(s),u)\nu(ds,du) \right\}. \tag{5}$$

Relations (4) and (5) imply the following statements.

Theorem 1: *If for all $t \geq 0$ and x*

(1) $g_1(t,x) = 0$;
(2) $\Pi\{u: \psi(t,x,u) \neq 1\} = 0$,
then with probability 1 *for all* $t \geq 0$ *the following inequality holds true:*

$$|x(0)|^2 e^{\int_0^t m(s)ds} \leq |x(t)|^2 \leq |x(0)|^2 e^{\int_0^t M(s)ds},$$

where

$$m(t) = \inf_x I(t,x), \quad M(t) = \sup_x I(t,x).$$

Proof: Therefore, in this case relation (5) takes the form

$$|x(t)|^2 = |x(0)|^2 \exp\left\{\int_0^t I(s,x(s))ds\right\},$$

which implies the statement of Theorem 1.

Remark 1: Condition (1) means that system (2) is perturbed by "white noise" only along the vector of the phase velocity. It follows from condition (2) that

$$2(x, c(t,x,u)) = -|c(t,x,u)|^2 < 0$$

in measure $\Pi(du)$ for all $t \geq 0$, $|x| \neq 0$. Thus, condition (2) means that system (2) is perturbed by "shot noise" at an obtuse angle to the radius-vector.

Corollary 1: *Under conditions* (1) *and* (2) *of Theorem* 1 *it is possible to control the behavior of* $\varepsilon(t)$ *of a perturbed system by the choice of function* $q_1(t,x)$ *(determined disturbance). For example:*

(1) *if* $2q_1(t,x) = -g_2^2(t,x)$, *then* $\varepsilon(t) = \varepsilon(0)$ *with probability* 1 *for all* $t \geq 0$;

(2) *if* $|\int_0^t M(s)ds| \leq C$, *then* $\varepsilon(t) \leq \varepsilon(0)e^C$ *with probability* 1 *for all* $t \geq 0$;

(3) *if* $2q_1(t,x) = -g_2^2(t,x) + c_0$, *then* $\varepsilon(t) = \varepsilon(0)e^{c_0 t}$; *etc.*

Theorem 2: *If* $\gamma_i(t,x,u) = \gamma_i(t,u)$, $i = 1,2$ *and* $g_1(t,x) = 0$, $2q_1(t,x) + g_2^2(t,x) = 0$ *for all* $t \geq 0$ *and* x, *then*

$$|x(t)|^2 = \begin{cases} |x(0)|^2, & \text{if } t < \tau_1, \\ |x(0)|^2 \prod_{\tau_k < t} \psi(\tau_k, u_k), & \text{if } \tau_k \leq t < \tau_{k+1}, \end{cases} \tag{6}$$

where $0 < \tau_1 < \tau_2 < \ldots$ *are shock-points of a Poisson process* $\nu([0,t),R)$ *and* $\nu(\{\tau_k\}, \{u_k\}) = 1$, $k = 1,2\ldots$.

Proof: Therefore, under the conditions of Theorem 2, relation (5) takes the following form:

$$|x(t)|^2 = |x(0)|^2 \exp\left\{\int_0^t \int_R \ln\psi(s,u)\nu(ds,du)\right\},$$

which implies equality (6) (see [2]).

Corollary 2: *Under the conditions of Theorem* 2, $\varepsilon(t)$ *changes only step-wise: moreover, shocks take place only in the moments of impulse disturbance, that is, in the moments of jumps of a Poisson process* $\nu([0,t),R)$. *In particular, if*

$(1+\gamma_1)^2+\gamma_2^2$ *is a constant magnitude, then*

$$|\,x(t)\,|^2 = |\,x(0)\,|^2[(1+\gamma_1)^2+\gamma_2^2]^{\nu((0,t],R)}. \tag{7}$$

Remark 2: If $\Pi\{u\colon(1+\gamma_1(t,u))^2+\gamma_2^2(t,u)\neq 0\}=0$ for all $t\geq 0$, then

$$|\,x(t)\,|^2 = \begin{cases} |\,x(0)\,|^2, & \text{if } t<\tau_1, \\ 0, & \text{if } \tau_1\leq t. \end{cases} \tag{8}$$

This means that under the first impulse disturbance, the considered system moves into the equilibrium state and does not leave it with probability 1. Thus in this case with small disturbances of coefficients $\gamma_i(t,u)$, it is possible to achieve equality (7) and then obtain (8) by passing to the limit.

Theorem 3: *If for all* $t\geq 0$

$$\sup_x\left[I_1(t,x)+\int_R[\psi(t,x,u)-1]\Pi(du)\right]\leq 0,$$

then

$$P\left\{\sup_{t\geq 0}|\,x(t)\,|^2<\varepsilon_1\right\}\geq 1-\varepsilon_2$$

for any $\varepsilon_1>0$, $\varepsilon_2>0$ *as soon as* $|\,x(0)\,|<\delta$; $\delta>0$.

Proof: Formula (4) implies the following equality:

$$|\,x(t)\,|^2 = |\,x(0)\,|^2+\int_0^t|\,x(s)\,|^2\left\{I_1(s,x(s))+\int_R[\psi(s,x(s),u)-1]\Pi(du)\right\}ds+\eta(t),$$

where

$$\eta(t)=2\int_0^t g_1(s,x(s))dw(s)+\int_0^t[\psi(s,x(s),u)-1]\widetilde{\nu}\,(ds,du).$$

Therefore, with probability 1 for all $t\geq 0$

$$|\,x(t)\,|^2\leq|\,x(0)\,|^2+\eta(t). \tag{9}$$

Since $\eta(t)$ is a square integrable martingale, then from the inequality (9) we have (see [1]):

$$P\left\{\sup_{t\geq 0}\eta(t)>\varepsilon_1\right\}\leq\frac{|\,x(0)\,|^2}{\varepsilon_1}. \tag{10}$$

The statement of Theorem 3 follows from (9) and (10).

Theorem 4: *If*

(1)
$$\overline{\lim_{t\to\infty}}\frac{1}{t^\alpha}\int_0^t\sup_x\left[I_2(s,x)+\int_R\ln\psi(s,x,u)\Pi(du)\right]ds<0$$

for some $\alpha > \frac{1}{2}$; *and*

(2)

$$\int_0^t \sup_x \left[4g_1^2(s,x) + \int_R \ln^2\psi(s,x,u)\Pi(du) \right] \leq C_0 t,$$

then

$$P\left\{ \lim_{t\to\infty} |\, x(t)\,|^2 = 0 \right\} = 1.$$

Proof: Hence, we can rewrite equality (5) as

$$|\, x(t)\,|^2 = |\, x(0)\,|^2 \exp\left\{ t^\alpha \left[\frac{1}{t^\alpha} \int_0^t \left(I_2(s,x(s)) + \right.\right.\right.$$
$$\left.\left.\left. + \int_R \ln\psi(s,x(s),u)\Pi(du) \right) ds + \frac{1}{t^\alpha}\zeta(t) \right] \right\}, \tag{11}$$

where

$$\zeta(t) = 2\int_0^t g_1(s,x(s))dw(s) + \int_0^t \int_R \ln\psi(s,x(s),u)\widetilde{\nu}\,(ds,du).$$

According to Condition (1) of Theorem 4, we will find $\delta > 0$ and $\theta_\delta > 0$ such that with probability 1

$$\frac{1}{t^\alpha}\int_0^t \left[I_2(s,x(s)) + \int_R \ln\psi(s,x(s),u)\Pi(du) \right] ds \leq -\delta,$$

as $t > T_\delta$. Furthermore, since $\zeta(t)$ is a square integrable martingale with characteristics which satisfy the following inequality:

$$\langle \zeta(t) \rangle = \int_0^t \left[4g_1^2(s,x(s)) + \int_R \ln^2\psi(s,x(s),u)\Pi(du) \right] ds \leq C_0 t.$$

Reasoning similarly to [4, Lemma 7.1], it can be proved that

$$P\left\{ \lim_{t\to\infty} t^{-\alpha}\zeta(t) = 0 \right\} = 1.$$

Therefore, taking into account (11), we obtain the statement of Theorem 4.

Theorem 5: *If for all* $t \geq 0$ *and* x:

$$I_1(t,x) + \int_R [\psi(t,x,u) - 1]\Pi(du) = Q(t)$$

and

$$\overline{\lim_{t\to\infty}} \int_0^t Q(s)ds = -\infty,$$

then

$$\lim_{t\to\infty} E \mid x(t) \mid^2 = 0.$$

Proof: Hence, from relation (4) we have

$$E \mid x(t) \mid^2 = \mid x(0) \mid^2 + \int_0^t Q(s) E \mid x(s) \mid^2 ds.$$

Therefore,

$$E \mid x(t) \mid^2 = \mid x(0) \mid^2 e^{\int_0^t Q(s) ds}$$

This equality implies the statement of Theorem 5.

Remark 3: If the system is perturbed by "centralized shot noise" ($\dot{\tilde{\nu}}([0,t), A)$ is a "derivative" of a Poisson nature) instead of "shot noise" and other perturbations are fixed then only the orientation of $a(t,x)$ changes in equation (3), that is,

$$a(t,x) = (\widehat{q}_1(t,x)x_1 + \widehat{q}_2(t,x)x_2, -\widehat{q}_2(t,x)x_1 + \widehat{q}_1(t,x)x_2),$$

where

$$\widehat{q}_i(t,x) = q_i(t,x) - \int_R \gamma_i(t,x,u)\Pi(du).$$

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