ON THE STABILITY OF STATIONARY SOLUTIONS OF A LINEAR INTEGRO-DIFFERENTIAL EQUATION

A. YA. DOROGVTSEV

*Kiev Institute of Business and Technology
Blvd. T. Shevchenko, 4, 311
01033 Kiev-33 Ukraine*

O. YU. TROFIMCHUK

*Kiev University
Mechanics and Mathematics Department
Vladimirskay 64, 01033 Kiev-33 Ukraine*

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In this paper the following two connected problems are discussed. The problem of the existence of a stationary solution for the abstract equation

$$\epsilon x''(t) + x'(t) = Ax(t) + \int_{-\infty}^t E(t-s)x(s)ds + \xi(t), t \in \mathbf{R} \quad (1)$$

containing a small parameter ϵ in Banach space B is considered. Here $A \in \mathcal{L}(B)$ is a fixed operator, $E \in C([0, +\infty), \mathcal{L}(B))$ and ξ is a stationary process. The asymptotic expansion of the stationary solution for equation (1) in the series on degrees of ϵ is given.

We have proved also the existence of a stationary with respect to time solution of the boundary value problem in B for a telegraph equation (6) containing the small parameter ϵ . The asymptotic expansion of this solution is also obtained.

Key words: Stationary Solutions, Singular Perturbations, Telegraph Equation, Time-Stationary Solutions, Asymptotic Expansions.

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1. Introduction

Let $(B, \|\cdot\|)$ be a complex Banach space, $\bar{0}$ the zero element in B , and $\mathcal{L}(B)$ the Banach space of bounded linear operators on B with the operator norm, denoted also by the symbol $\|\cdot\|$. For a B -valued function, continuity and differentiability refer to continuity and differentiability in the B -norm. For an $\mathcal{L}(B)$ -valued function, continuity is the continuity in the operator norm. For operator A , the sets $\sigma(A)$ and $\rho(A)$ are its spectrum and resolvent set, respectively.

In the following, we will consider random element son the same complete probability space $(\Omega, \mathfrak{F}, P)$. The uniqueness of a random process that satisfies an equation, is its uniqueness up to stochastic equivalence. We consider only B -valued random functions which are continuous with a probability of one. All equalities with random elements in this article are always equalities with a probability one. For a given equation, we consider only solutions which are measurable with respect to the right-hand side random process.

It is well known that the stationary solutions of difference and differential equations are steady with respect to various perturbations of the right-hand side and perturbation of coefficients. For example, see [5]. In the present work, it is shown that stability has a place with respect to perturbations such as degeneracy of the equation.

In the first part of this paper, we consider the following equation

$$\epsilon x''(t) + x'(t) = Ax(t) + \int_{-\infty}^t E(t-s)x(s)ds + \xi(t), t \in \mathbf{R} \quad (1)$$

containing a parameter ϵ in B . Here $A \in \mathcal{L}(B)$ is fixed operator, ξ is a stationary process in B and $E \in C([0, +\infty), \mathcal{L}(B))$ is a function satisfying the condition

$$a := \int_0^{+\infty} \|E(s)\| ds < +\infty.$$

We suppose that the following condition

$$\sigma(A) \cap i\mathbf{R} = \emptyset \quad (2)$$

holds. Under condition (2) the function

$$G(t) := \begin{cases} -e^{At}P_+, & t < 0; \\ e^{At}P_-, & t > 0 \end{cases}$$

satisfies the inequality

$$b := \int_{\mathbf{R}} \|G(s)\| ds < +\infty.$$

Here P_- and P_+ are Riesz spectral projectors corresponding to the spectral sets $\sigma(A) \cap \{z \mid \operatorname{Re} z < 0\}$ and $\sigma(A) \cap \{z \mid \operatorname{Re} z > 0\}$, respectively.

Let S be the class of all stationary B -valued processes $\{\xi(t); t \in \mathbf{R}\}$ which possess continuous derivatives of all orders on $\mathbf{R}$ with a probability one and such that, for some numbers $L = L_\xi > 0$, $C = C_\xi > 0$, $\delta > 0$, the following inequalities

$$\forall n \geq 0: \mathbf{E} \left\{ \sup_{0 \leq s \leq \delta} \|\xi^{(n)}(s)\| \right\} \leq LC^n$$

hold. The notations $\xi \in S(L, C, \delta)$ and $\xi \in S$ will be used. Then we have the following result.

Theorem 1: Let $A \in \mathcal{L}(B)$ be an operator satisfying (2). Suppose that $\xi \in S$ and $ab < 1$. Then there exists $\epsilon_0 > 0$ such that for every ϵ with $|\epsilon| < \epsilon_0$, the equation (1) has a stationary solution $x_\epsilon \in S$, which for every bounded subset J of $\mathbf{R}$, satisfies

$$E\left(\sup_{s \in J} \|x_\epsilon(s) - y_0(s)\|\right) \rightarrow 0, \quad \epsilon \rightarrow 0,$$

where y_0 is a unique stationary solution of the equation

$$x'(t) = Ax(t) + \int_{-\infty}^t E(t-s)x(s)ds + \xi(t), \quad t \in \mathbf{R}. \quad (3)$$

The process x_ϵ is a unique solution of (1) in the class of all stationary connected processes in S .

This theorem is proved in Section 2. The method of proof uses a modification of the proof of Theorem 1 in [7] about the stability of stationary solutions for equation (1) with $E \equiv 0$.

Remark 1: The asymptotic expansion for a stationary solution of (1) is obtained.

Remark 2: The assumption (2) is equivalent to the existence of a unique stationary solution $\{x(t) \mid t \in \mathbf{R}\}$ with $E\|x(0)\| < +\infty$ of the equation

$$x'(t) = Ax(t) + \xi(t), \quad t \in \mathbf{R}$$

for every stationary process $\{\xi(t) \mid t \in \mathbf{R}\}$ with $E\|\xi(0)\| < +\infty$, see [3, pp. 201-202].

Remark 3: The general approach to the analysis of the Cauchy problem for deterministic differential equations containing a small parameter leads to the appearance of boundary layer summands in the asymptotic expansion of solution [10]. These summands are absent in the asymptotic expansion of the stationary solution in the considered problem.

Remark 4: The problem of the existence of stationary solutions for difference and differential stochastic equations has been investigated by many authors. See, for example, monograph [1], surveys [2, 4] and article [6].

Corollary 1: Let $A \in \mathcal{L}(B)$ be an operator satisfying (1). Suppose that $\xi \in S$. Then there exists $\epsilon_0 > 0$ such that for every ϵ with $|\epsilon| < \epsilon_0$, the equation

$$\epsilon x''(t) + x'(t) = Ax(t) + \xi(t), \quad t \in \mathbf{R} \quad (4)$$

has a unique stationary solution $x_\epsilon \in S$, which, for every bounded subset J of $\mathbf{R}$, satisfies

$$E\left(\sup_{s \in J} \|x_\epsilon(s) - x_0(s)\|\right) \rightarrow 0, \quad \epsilon \rightarrow 0,$$

where x_0 is a unique stationary solution of the equation

$$x'(t) = Ax(t) + \xi(t), \quad t \in \mathbf{R}.$$

The second part of this paper deals with the asymptotic expansion of the stationary with respect to time solution of a boundary value problem containing a small parameter. The following definition is necessary.

Definition 1: A B -valued random function u defined on $Q: = \mathbf{R} \times [0, \pi]$ is time-stationary if

$$\forall t \in \mathbf{R} \forall n \in \mathbf{N} \forall \{(t_1, x_1), \dots, (t_n, x_n)\} \subset Q \forall \{D_1, \dots, D_n\} \subset \mathfrak{B}(B):$$

$$P \left\{ \bigcap_{k=1}^n \{\omega: u(\omega; t_k + t, x_k) \in D_k\} \right\} = P \left\{ \bigcap_{k=1}^n \{\omega: u(\omega; t_k, x_k) \in D_k\} \right\},$$

where $\mathfrak{B}(B)$ is the Borel σ -algebra of B .

Let

$$C_0^3: = \{g: [0, \pi] \rightarrow C \mid g^{(k)}(0) = g^{(k)}(\pi) = 0, k = 0, 1, 2\} \cap C^3([0, \pi]).$$

Theorem 2: Let $A \in \mathcal{L}(B)$ be an operator satisfying the following condition

$$\{k^2 + i\alpha \mid k \in \mathbf{N}, \alpha \in \mathbf{R}\} \subset \rho(A). \quad (5)$$

Suppose that $g \in C_0^3$ and $\xi \in S$ with a number $\delta > 0$ and $ab < 1$. Then there exists $\epsilon_0 > 0$ such that for every ϵ with $|\epsilon| < \epsilon_0$, the boundary value problem

$$\begin{cases} \epsilon u''_{tt}(t, x; \epsilon) + u'_t(t, x; \epsilon) - u''_{xx}(t, x; \epsilon) \\ \quad = Au(t, x; \epsilon) + g(x)\xi(t), \quad t \in \mathbf{R}, x \in [0, \pi] \\ u(t, 0; \epsilon) = u(t, \pi; \epsilon) = \bar{0}, t \in \mathbf{R} \end{cases} \quad (6)$$

has a unique time-stationary solution $u(\cdot, \cdot; \epsilon)$ with

$$\mathbf{E} \left(\sup_{0 \leq s \leq \delta, 0 \leq x \leq \pi} \|u(s, x; \epsilon)\| \right) + \mathbf{E} \left(\sup_{0 \leq s \leq \delta, 0 \leq x \leq \pi} \|u'_t(s, x; \epsilon)\| \right) < +\infty,$$

which, for every $t \in \mathbf{R}$, satisfies

$$\mathbf{E} \left(\sup_{t \leq s \leq t + \delta, 0 \leq x \leq \pi} \|u(t, x; \epsilon) - v(t, x)\| \right) \rightarrow 0, \quad \epsilon \rightarrow 0,$$

where v is the unique time-stationary solution of the following boundary value problem for a heat equation

$$\begin{cases} v'_t(t, x) - v''_{xx}(t, x) = Av(t, x) + g(x)\xi(t), \quad t \in Q \\ v(t, 0) = v(t, \pi) = 0, \quad t \in \mathbf{R} \end{cases} \quad (7)$$

with

$$\sup_{0 \leq x \leq \pi} \mathbf{E} \|v(0, x)\| < +\infty.$$

This theorem is proved in Section 3.

Remark 5: Condition (5) is a necessary and sufficient condition of the existence of a time-stationary solution for boundary value problem (7) [8].

Remark 6: Note that, if $\epsilon > 0$, problem (6) is a boundary value problem for a hyperbolic equation and that, if $\epsilon = 0$, we have a boundary value problem for a parabolic equation.

Remark 7: The study of the asymptotic behavior of a solution $u(\cdot, \cdot; \epsilon)$ of the telegraph equation from (6) as $\epsilon \rightarrow 0+$ has also physical sense [9].

2. Asymptotic Expansion of the Stationary Solution of Equation (1)

In order to prove Theorem 1, a few lemmas will be needed.

Lemma 1: *Let $A \in \mathcal{L}(B)$ be an operator satisfying (2). Suppose that $\xi \in S$. Then the equation*

$$x'(t) = Ax(t) + \xi(t), t \in \mathbf{R}$$

has a unique stationary solution $x \in S$, which can be presented in the form

$$x(t) = \int_{\mathbf{R}} G(t-s)\xi(s)ds = \int_{\mathbf{R}} G(s)\xi(t-s)ds, t \in \mathbf{R}.$$

Proof: This is the corollary of Theorem 1 in [3, pp. 201-202].

Lemma 2: *Let $A \in \mathcal{L}(B)$ be an operator satisfying (2). Suppose that $\xi \in S$. The following two statements are equivalent:*

- (i) *A stationary process $x \in S$ is a unique stationary solution of the equation (3).*
- (ii) *A stationary process $x \in S$ is a unique stationary solution of the equation*

$$x(t) = \int_{\mathbf{R}} G(t-s) \int_{-\infty}^s E(s-u)x(u)duds + \int_{\mathbf{R}} G(t-s)\xi(s)ds, t \in \mathbf{R}. \quad (8)$$

Proof: The result is a consequence of Lemma 1.

Lemma 3: *Let $A \in \mathcal{L}(B)$ be an operator satisfying (2) and $ab < 1$. Suppose that ξ is a stationary process in B , which, for some $\delta > 0$, satisfies*

$$E\left(\sup_{0 \leq t \leq \delta} \|\xi(t)\|\right) < +\infty.$$

Then the equation (8) has a unique stationary solution x , which satisfies

$$E\left(\sup_{0 \leq t \leq \delta} \|x(t)\|\right) < +\infty. \quad (9)$$

Proof: Let S_0 be the class of all stationary connected B -valued processes x which are stationary connected with ξ and, for given $\delta > 0$, satisfy (9). Let us introduce the operator

$$(Tx)(t) = \int_{\mathbf{R}} G(t-s) \int_{-\infty}^s E(s-u)x(u)duds + \int_{\mathbf{R}} G(t-s)\xi(s)ds, t \in \mathbf{R}.$$

Then $Tx \in S_0$ and

$$E\left(\sup_{0 \leq t \leq \delta} \|(Tx)(t) - (Ty)(t)\|\right) \leq ab E\left(\sup_{0 \leq t \leq \delta} \|x(t) - y(t)\|\right),$$

therefore T is a continuous operator on S_0 . Set

$$x_0(t) = \int_{\mathbf{R}} G(t-s)\xi(s)ds, \quad t \in \mathbf{R},$$

then $x_0 \in S_0$ and

$$E\left(\sup_{0 \leq t \leq \delta} \|x_0(t)\|\right) \leq bE\left(\sup_{0 \leq t \leq \delta} \|\xi(t)\|\right).$$

Introduce the sequences of random processes

$$x_0, x_1: = Tx_0, x_2: = Tx_1, \dots, x_n: = Tx_{n-1}, \dots$$

It is clear that

$$x_n \in S_0, n \in \mathbf{N}; \quad x_{n+1} = Tx_n, n \geq 0$$

and for every $t \in \mathbf{R}$

$$E\|x_{n+1}(t) - x_n(t)\| \leq E\left(\sup_{t \leq s \leq t+\delta} \|x_{n+1}(s) - x_n(s)\|\right)$$

$$\leq a(ab)^{n+1}E\left(\sup_{0 \leq t \leq \delta} \|\xi(s)\|\right), \quad n \geq 0.$$

Hence, the series

$$x(t): = x_0(t) + [x_1(t) - x_0(t)] + \dots + [x_n(t) - x_{n-1}(t)] + \dots$$

converges with a probability one for every $t \in \mathbf{R}$ and this convergence is uniform over the bounded subset of $\mathbf{R}$ with a probability one. By continuity of T we have $x = Tx$. The solution x of (8) is unique.

Lemma 4: Let $A \in \mathcal{L}(B)$ be an operator satisfying (2) and $ab < 1$. Suppose that ξ is a stationary process in B , which, for some $\delta > 0$, satisfies

$$E\left(\sup_{0 \leq t \leq \delta} \|\xi(t)\|\right) < +\infty.$$

Then equation (3) has a unique stationary solution x , which satisfies (9).

Proof: The result is an immediate consequence of Lemma 2 and Lemma 3.

Set $c: = (1 - ab)^{-1}$.

Lemma 5: Let $A \in \mathcal{L}(B)$ be an operator satisfying (2) and $ab < 1$. Suppose that $\xi \in S(L, C, \delta)$. The equation (3) has a unique stationary solution $x \in S(bcL, C, \delta)$.

Proof: We return to the proof of Lemma 3 where the stationary solution x for equation (3) was given. From the inclusion $\xi \in S(L, C, \delta)$ and representation

$$x_0(t) = \int_{\mathbf{R}} G(s)\xi(t-s)ds, t \in \mathbf{R}$$

it follows that

$$x_0^{(k)}(t): = \int_{\mathbf{R}} G(s)\xi^{(k)}(t-s)ds, \quad t \in \mathbf{R}$$

for every $k \geq 0$ and $x_0 \in S(bL, C, \delta)$. For the process $x_1 - x_0$, we have

$$x_1(t) - x_0(t) = \int_{\mathbf{R}} G(u) \int_0^{+\infty} E(v)x_0(t-u-v)dudv, t \in \mathbf{R}.$$

Hence, for every $k \geq 0$, we have

$$x_1^{(k)}(t) - x_0^{(k)}(t) = \int_{\mathbf{R}} G(u) \int_0^{+\infty} E(v)x_0^{(k)}(t-u-v)dudv, t \in \mathbf{R},$$

and $(x_1 - x_0) \in S(ab^2L, C, \delta)$. By induction, we find

$$(x_n - x_{n-1}) \in S(b(ab)^nL, C, \delta), n \geq 1.$$

Therefore,

$$x \in S(bcL, C, \delta).$$

Lemma 5 is proved.

Proof of Theorem 1: Let $\xi \in S(L, C, \delta)$. We shall construct the asymptotic expansion for a solution of (1) in the following way. From Lemma 5, equation (3) has a unique stationary solution $y_0 \in S(bcL, C, \delta)$. Note that $y_0'' \in S(bcLC^2, C, \delta)$. Let y_1 be a unique stationary solution for equation

$$y_1'(t) = Ay_1(t) + \int_{-\infty}^t E(t-s)y_1(s)ds - y_0''(t), t \in \mathbf{R}.$$

This solution exists from Lemma 5 and

$$y_1 \in S(b^2c^2LC^2, C, \delta).$$

By analogy with y_1 , let y_2 be a unique stationary solution for equation

$$y_2'(t) = Ay_2(t) + \int_{-\infty}^t E(t-s)y_2(s)ds - y_1''(t), t \in \mathbf{R}.$$

For this solution, we have $y_2 \in S(b^3c^3LC^4, C, \delta)$.

If the processes $y_0, y_1, \dots, y_{n-1}$ for $n \geq 1$ are already constructed we will define process y_n as a unique stationary solution of the equation

$$y_n'(t) = Ay_n(t) + \int_{-\infty}^t E(t-s)y_n(s)ds - y_{n-1}''(t), t \in \mathbf{R},$$

which satisfies

$$y_n \in S(b^{n+1}c^{n+1}LC^{2n}, C, \delta).$$

It is clear that the processes y_n , $n \geq 0$ are stationary connected [3].

Set

$$y_\epsilon(t) = \sum_{n=0}^{\infty} \epsilon^n y_n(t), t \in \mathbf{R}. \quad (10)$$

Since

$$\sum_{n=0}^{\infty} |\epsilon|^n \mathbf{E} \left(\sup_{t \leq s \leq t+\delta} \|y_n(s)\| \right) \leq \sum_{n=0}^{\infty} |\epsilon|^n \frac{b^{n+1}LC^{2n}}{(1-ab)^{n+1}} \leq \frac{2bL}{1-ab}$$

for every $t \in \mathbf{R}$ and $|\epsilon| \leq \epsilon_0 = (1-ab)/(2bC^2)$, the series for y_ϵ converges uniformly on bounded subsets of $\mathbf{R}$ with a probability one. This shows that y_ϵ is continuous on $\mathbf{R}$ with a probability one stationary process.

By exactly the same arguments as those used above, we claim that the series for y'_ϵ , y''_ϵ are also absolutely and uniform convergent on bounded subsets of $\mathbf{R}$ with a probability one and we have

$$\begin{aligned}
 \epsilon y''_\epsilon(t) + y'_\epsilon(t) &= \sum_{n=0}^{\infty} (\epsilon^{n+1} y''_n(t) + \epsilon^n y'_n(t)) \\
 &= \sum_{n=0}^{\infty} \left[\epsilon^{n+1} \left(A y_{n+1}(t) + \int_{-\infty}^t E(t-s) y_{n+1}(s) ds - y'_{n+1}(t) \right) + \epsilon^n y'_n(t) \right] \\
 &= \sum_{n=0}^{\infty} \epsilon^{n+1} A y_{n+1} + \sum_{n=0}^{\infty} \epsilon^{n+1} \int_{-\infty}^t E(t-s) y_{n+1}(s) ds - \sum_{m=1}^{\infty} \epsilon^m y'_m(t) + \sum_{n=0}^{\infty} \epsilon^n y'_n(t) \\
 &= A \left(\sum_{m=1}^{\infty} \epsilon^m y_m(t) \right) + \int_{-\infty}^t E(t-s) \left(\sum_{m=1}^{\infty} \epsilon^m y_m(s) \right) ds + y_0(t) \\
 &= A y_\epsilon(t) + \int_{-\infty}^t E(t-s) y_\epsilon(s) ds - A y_0(t) - \int_{-\infty}^t E(t-s) y_0(s) ds + y'_0(t) \\
 &= A y_\epsilon(t) + \int_{-\infty}^t E(t-s) y_\epsilon(s) ds + \xi(t), t \in \mathbf{R}.
 \end{aligned}$$

Moreover, for every $t \in \mathbf{R}$, we have

$$\mathbf{E} \left(\sup_{t \leq s \leq t+\delta} \|y_\epsilon(s) - y_0(s)\| \right) \leq \sum_{m=1}^{\infty} |\epsilon|^{m-1} \frac{b^{m+1} L C^2 m}{(1-ab)^{m+1}} \leq \frac{2b^2 L C^2}{(1-ab)^2} \epsilon,$$

if $|\epsilon| \leq \epsilon_0$.

To complete the proof of Theorem 1 we need show only the uniqueness. It is sufficient to prove the following fact. If z is stationary connected with the process x_ϵ solution of (1), which satisfies

$$\mathbf{E} \left(\sup_{0 \leq t \leq \delta} \|z(t)\| \right) < +\infty, \quad \mathbf{E} \left(\sup_{0 \leq t \leq \delta} \|z'(t)\| \right) < +\infty,$$

then $z = x_\epsilon$. We apply Lemma 4 in the following way. The difference $u := x_\epsilon - z$ is a stationary process which satisfies the equation

$$\epsilon u''(t) + u'(t) = A x(t) + \int_{-\infty}^t E(t-s) u(s) ds, \quad t \in \mathbf{R} \quad (11)$$

and

$$E \left(\sup_{0 \leq t \leq \delta} \|u(t)\| \right) < +\infty, \quad E \left(\sup_{0 \leq t \leq \delta} \|u'(t)\| \right) < +\infty.$$

Let us consider a Banach space B^2 of two vectors equipped with term-by-term linear operations and with the norm which is equal to the sum of the norms of the coordinates. Let

$$\mathbf{u}(t) = \begin{pmatrix} u' \\ u \end{pmatrix}, \quad \mathbb{A} = \begin{pmatrix} -\epsilon^{-1} & \epsilon^{-1}A \\ I & \Theta \end{pmatrix}, \quad \mathbb{E} = \begin{pmatrix} \Theta & \epsilon^{-1}E \\ \Theta & \Theta \end{pmatrix},$$

where Θ and I are the zero operator and identity operator on B , respectively. Then the following equation in B^2

$$\mathbf{u}'(t) = \mathbb{A}\mathbf{u}(t) + \int_{-\infty}^t \mathbb{E}(t-s)\mathbf{u}ds, \quad t \in \mathbf{R} \quad (12)$$

is equivalent to the equation (11) in B . By direct computation we obtain that condition

$$\sigma(\mathbb{A}) \cap i\mathbf{R} = \emptyset$$

is fulfilled if, for every $\alpha \in \mathbf{R}$, an operator $A - (i\alpha + \alpha^2\epsilon)I$ has a bounded inverse. For the justification of this assertion for all small $|\epsilon|$ it suffices to make use of condition (2) and the boundedness of operator A . Then, by Lemma 4, the equation (12) has a unique stationary solution and hence $\mathbf{u}(t) = \bar{0}$, $t \in \mathbf{R}$ with the probability of one.

The proof is complete.

Remark 8: Let $B = \mathbf{R}$. It can be proven that the existence of expansion (10) for the solution of equation (4) leads to condition $\xi = C^\infty(\mathbf{R})$.

3. Time-Stationary Solutions of the Boundary Value Problem for PDE Containing a Parameter

Proof of Theorem 2: Let a process $\xi \in S(L, C, \delta)$ and a function $g \in C_0^3$ be given. Then, one can expand g as

$$g(x) = \sum_{k=1}^{\infty} g_k \sin kx, \quad x \in [0, \pi]; \quad \{g_k: k \geq 1\} \subset C,$$

where the series on the right-hand side is uniformly convergent. Note that

$$g_k = \frac{2}{k} \int_0^\pi g(x) \sin kx \, dx, \quad k \geq 1.$$

Let $k \geq 1$ be fixed. From assumption (5) and Corollary 1, it follows that there is $\xi_k > 0$ such that for every ϵ with $|\epsilon| < \epsilon_k$, the equation

$$\epsilon v_k''(t; \epsilon) + v_k'(t; \epsilon) + k^2 v_k(t; \epsilon) = A v_k(t; \epsilon) + g_k \xi(t), \quad t \in \mathbf{R} \quad (13)$$

has a unique stationary solution $v_k(\cdot; \epsilon)$ such that

$$E \left(\sup_{t \in J} \|v_k(t; \epsilon) - v_k(t)\| \right) \rightarrow 0, \quad \epsilon \rightarrow 0,$$

where v_k is a unique stationary solution of the equation

$$v'_k(t) + k^2 v_k(t) = A v_k(t) + g_k \xi(t), \quad t \in \mathbf{R},$$

and J is a bounded subset of $\mathbf{R}$. Moreover, for every $t \in \mathbf{R}$, we have

$$E \left(\sup_{t \leq s \leq t + \delta} \|v_k(s; \epsilon)\| \right) \leq 2 |g_k| LL_{1,k} \quad (14)$$

and

$$E \left(\sup_{t \leq s \leq t + \delta} \|v_k(s; \epsilon) - v_k(s)\| \right) \leq 2 |g_k| LL_{1,k}^2 C^2 |\epsilon|, \quad (15)$$

if $|\epsilon| \leq \epsilon_k$, where

$$L_{1,k} = \int_{\mathbf{R}} \|G_k(s)\| ds < +\infty$$

and G_k is Green's function for operator $A - k^2 I$; $k \geq 1$. It follows from the properties of G_k that

$$L_{1,k} \leq \frac{\tilde{L}}{k^2 - k_0^2}, \quad k > k_0, \quad (16)$$

where a number $\tilde{L}$ can be chosen to be independent of k .

Now we shall remark, that by virtue of boundedness of an operator A , the numbers ϵ_k , $k \geq 1$ are identifiable and not depending on k . Really, let k_0 be the least natural number such that a spectrum of an operator $A - (\alpha^2 \epsilon - k_0^2)I$ resides in the left half-plane. Then the spectrum of an operator $A - (\alpha^2 \epsilon - k^2)I$, $k \geq k_0$ also resides in the left half-plane and it is possible to put $\epsilon_0 = \min\{\epsilon_1, \epsilon_2, \dots, \epsilon_{k_0}\} > 0$. Thus, for every ϵ , $|\epsilon| < \epsilon_0$, all equations (13) have a unique stationary solution.

Let us consider the series

$$u(t, x; \epsilon) = \sum_{k=1}^{\infty} v_k(t; \epsilon) \sin kx, \quad (t, x) \in Q \quad (17)$$

for $|\epsilon| \leq \epsilon_0$. It follows from (14) and (16) that

$$\sum_{k=1}^{\infty} E \left(\sup_{t \leq s \leq t + \delta, 0 \leq x \leq \pi} \|v_k(t; \epsilon) \sin kx\| \right) \leq \sum_{k=1}^{\infty} 2 |g_k| LL_{1,k} < +\infty,$$

for every $t \in \mathbf{R}$ and $|\epsilon| \leq \epsilon_0$. This implies that the series (17) converges absolutely and uniformly on $[t, t + \delta] \times [0, \pi]$ with the probability one and the random function $u(\cdot, \cdot; \epsilon)$ is a continuous, time-stationary with respect of time variable, random functions. In addition,

$$E \left(\sup_{0 \leq s \leq \delta, 0 \leq x \leq \pi} \|u(s, x; \epsilon)\| \right) < +\infty.$$

Using the above-mentioned reasoning, the following equalities are installed

$$\begin{aligned}
 u'_t(t, x; \epsilon) &= \sum_{k=1}^{\infty} v'_k(t, \epsilon) \sin kx, \\
 u''_{tt}(t, x; \epsilon) &= \sum_{k=1}^{\infty} v''_k(t; \epsilon) \sin kx, \\
 u''_{xx}(t, x; \epsilon) &= \sum_{k=1}^{\infty} (-k^2) v_k(t; \epsilon) \sin kx,
 \end{aligned} \tag{18}$$

for $(t, x) \in Q$ and uniform on $[t, t + \delta] \times [0, \pi]$ convergence with the probability one of an appropriate series for any $t \in \mathbf{R}$ and $|\epsilon| \leq \epsilon_0$. We have also

$$E \left(\sup_{0 \leq s \leq \delta, 0 \leq x \leq \pi} \| u'_t(s, x; \epsilon) \| \right) < +\infty.$$

From (17), (18), and (13), it follows that

$$\begin{aligned}
 &\epsilon u''_{tt}(t, x; \epsilon) + u'_t(t, x; \epsilon) - u''_{xx}(t, x; \epsilon) \\
 &= \sum_{k=1}^{\infty} (\epsilon v''_k(t; \epsilon) + v'_k(t; \epsilon) + k^2 v_k(t; \epsilon)) \sin kx \\
 &= \sum_{k=1}^{\infty} (A v_k(t; \epsilon) + g_k \xi(t)) \sin kx \\
 &= A u(t, x; \epsilon) + g(x) \xi(t), (t, x) \in Q.
 \end{aligned}$$

Hence, the random function $u(\cdot, \cdot; \epsilon)$ for ϵ with $|\epsilon| < \epsilon_0$ is a time-stationary solution of (6).

This solution is unique. To see this, we observe that for any $t \in \mathbf{R}$, the elements $\{v_k(t; \epsilon)\}$ are Fourier coefficients of $u(t, \cdot; \epsilon) \in C^2([0, \pi], B)$ which determine $u(t, \cdot; \epsilon)$ uniquely with the probability one. See, for example [3] for details. By Corollary 1, the solutions of (13) are also determined uniquely with a probability one.

Similarly, by repeating the above arguments, we conclude that random function

$$v(t, x) = \sum_{k=1}^{\infty} v_k(t) \sin kx, \quad (t, x) \in Q$$

is a unique, stationary with respect to time variable, solution of (7) and

$$E \left(\sup_{t \leq s \leq t + \delta, 0 \leq x \leq \pi} \| v(s, x; \epsilon) \| \right) < +\infty$$

for every $t \in \mathbf{R}$. Note that the random functions $u(\cdot, \cdot; \epsilon)$, $|\epsilon| \leq \epsilon_0$ and v are time-stationary connected.

Finally, let us consider the difference $u(\cdot, \cdot; \epsilon) - v(\cdot, \cdot)$ for $|\epsilon| < \epsilon_0$. By Corollary 1, the following inequalities

$$E \left(\sup_{t \leq s \leq t + \delta, 0 \leq x \leq \pi} \| u(t, x; \epsilon) - v(t, x) \| \right)$$

$$\leq \sum_{k=1}^{\infty} \mathbf{E} \left(\sup_{t \leq s \leq t+\delta} \|v(t; \epsilon) - v_k(t)\| \right) \leq \sum_{k=1}^{\infty} 2L \|g_k\|_{L^2_{1,k}} C^2 \|\epsilon\|$$

hold.

Theorem 2 is proved.

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