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LINEAR-IMPLICIT STRONG SCHEMES FOR ITÔ-GALKERIN APPROXIMATIONS OF STOCHASTIC PDES[1]

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Linear-implicit versions of strong Taylor numerical schemes for finite dimensional Itô stochastic differential equations (SDEs) are shown to have the same order as the original scheme. The combined truncation and global discretization error of an γ strong linear-implicit Taylor scheme with time-step Δ applied to the N dimensional Itô-Galerkin SDE for a class of parabolic stochastic partial differential equation (SPDE) with a strongly monotone linear operator with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots$ in its drift term is then estimated by

$$K\left(\lambda_{N+1}^{-1/2} + \Delta^{\gamma}\right)$$

where the constant K depends on the initial value, bounds on the other coefficients in the SPDE and the length of the time interval under consideration.

Key words: Parabolic SPDE, Galerkin Approximation, Strong Taylor Scheme, Linear-Implicit Scheme.

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1. Introduction

A numerical method for parabolic stochastic partial differential equations (SPDE) [1, 3] of the form

$$dU_t = \{AU_t + f(U_t)\}dt + g(U_t)dW_t, \tag{1}$$

where $\{W_t, t \geq 0\}$ is a standard scalar Wiener process, based on the application of an order γ strong Taylor scheme [4] with constant time-step Δ to the N-dimensional Itô stochastic differential equations (SDE)

$$dU_t^N = \left\{A_N U_t^N + f_N(U_t^N)\right\}dt + g_N(U_t^N)dW_t, \tag{2}$$

obtained from (1) by a Galerkin approximation was shown by Grecksch and Kloeden [2] to have a combined truncation and global discretization error of the form

$$E\,|\,U_{k\Delta} - Y_k^N\,| \leq K\left(\lambda_{N+1}^{-1/2} + \lambda_N^{[\gamma+\frac{1}{2}]+1}\Delta^\gamma\right). \tag{3}$$

Here $[x]$ denotes the integer part of the real number x and K is constant depending on the initial value and bounds on the coefficient functions f, g of the SPDE (1) as well as on the length of the time interval $0 \leq k\Delta \leq T$ under consideration, and λ_j is the jth eigenvalue of the operator $-A$ (whose eigenfunctions provide the bases for the Galerkin approximations).

We refer to [1-3] for the functional analytical terminology and formalism of a stochastic PDE (1) with a Dirichlet boundary condition for a bounded domain $\mathfrak{D}$ in $\mathbb{R}^d$ with sufficiently smooth boundary $\partial\mathfrak{D}$, noting here that it has a unique strong solution

$$U \in L_2([0,T], \mathfrak{H}_0^{1,2}) \cap C([0,T], \mathfrak{L}_2)$$

for each finite $T > 0$ and initial condition $U_0 \in \mathfrak{H}_0^{1,2}$ under the assumption that f and g are uniformly Lipschitz continuous from $L_2(\mathfrak{D})$ into itself. The Itô-Galerkin SDE (2) then also has a unique solution U_t^N with the initial value $U_0^N = P_N U_0$, where P_N is the projection of $L_2(\mathfrak{D})$ or $H_0^{1,2}(\mathfrak{D})$ onto the N-dimensional subspace $\mathfrak{G}_N$ of $H_0^{1,2}(\mathfrak{D})$ spanned by $\{\phi_1, \ldots, \phi_N\}$, the first N eigenfunctions of the operator $-A$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_N$. For convenience we write U^N synonomously for $(U^{N,1}, \ldots, U^{N,N}) \in \mathbb{R}^N$ and $\sum_{j=1}^N U^{N,j}\phi_j \in \mathfrak{G}_N$ according to context and define $A_N = P_N A\,|_{\mathfrak{G}_N}, f_N = P_N f\,|_{\mathfrak{G}_N}$ and $g_N = P_N g\,|_{\mathfrak{G}_N}$. Finally, $|\cdot|^2$ and $\|\cdot\|^2$ denote the norms of $L_2(\mathfrak{D})$ and $H_0^{1,2}(\mathfrak{D})$, respectively.

For this setup, $\lambda_N \to \infty$ as $N \to \infty$, so a very small time-step Δ will be needed to give a reasonable bound in (3) for high dimensional Itô-Galerkin SDEs, which may give rise to numerical instabilities. These can be avoided by application of an implicit strong Taylor scheme [4], but at the expense of having to solve numerically a nonlinear algebraic equation for each time-step. However, the stiffness in the Itô-Galerkin SDEs with such a structure are ideally suited to linear-implicit versions of strong Taylor schemes, which neutralize this source of stiffness and are solvable explicitly for the next iterate by inversion of a diagonal matrix.

In the next section we introduce linear-implicit versions of a strong Taylor schemes and show that they retain the strong order of their original scheme. Then in Section 3 we consider their application to Itô-Galerkin SDE (2) obtained from a stochastic PDE (1) and show that the Δ term in their error estimate is no longer premultiplied by a power of λ_{N+1} as in (3).

2. Linear-Implicit Strong Taylor Schemes

Following Kloeden and Platen [4], an order γ strong Taylor scheme with constant time-step Δ for the SDE (2) has the form

$$Y_{k+1}^N = Y_k^N + \sum_{\alpha \in \mathcal{A}_\gamma \backslash \{v\}} F_\alpha^N(Y_k^N) I_{\alpha,k,\Delta}, \tag{4}$$

with coefficient functions F_α^N and multiple stochastic integrals $I_{\alpha,k,\Delta}$, where γ takes possible values 0.5 (Euler scheme), 1.0 (Milstein scheme), 1.5, 2.0,.... The admissible multi-indices $\alpha = (j_1, \ldots, j_l) \in \mathcal{A}_\gamma \backslash \{v\}$ here have components $j_i = 0$ or 1 corresponding to integration with respect to 'dt' or 'dW_t', respectively, while the jth component of F_α^N is defined by

$$F_\alpha^{N,j}(Y^N) = L_N^{j_1} \ldots L_N^{j_l} F_j(Y^N), \tag{5}$$

where the operators

$$L_N^0 = \sum_{j=1}^N (-\lambda_j U^{N,j} + f^{N,j}(U^N)) \frac{\partial}{\partial U^{N,j}}$$
$$+ \frac{1}{2} \sum_{i,j=1}^N g^{N,i}(U^N) g^{N,j}(U^N) \frac{\partial^2}{\partial U^{N,i} \partial U^{N,j}}$$

and

$$L_N^1 = \sum_{j=1}^N g^{N,j}(U^N) \frac{\partial}{\partial U^{N,j}}$$

are applied successively to $F_j(Y^N) \equiv Y^{N,j}$, the jth component of Y^N, and the result evaluated at Y^N. For example, the Euler scheme with constant time-step Δ for the SDE (2) has the form

$$Y_{k+1}^N = Y_k^N + (A_N Y_k^N + f^N(Y_k^N))\Delta + g^N(Y_k^N)\Delta W_k, \tag{6}$$

since $I_{(0),k,\Delta} = \Delta$ and $I_{(1),k,\Delta} = \Delta W_k$, the Wiener process increment for the kth time-step. See [4] for further details.

For a common initial value $U_0^N = Y_0^N$, the global strong discretization error of the numerical scheme (4) has the expectation

$$E\left| U_{k\Delta}^N - Y_k^N \right| \leq K_N \Delta^\gamma, \tag{7}$$

where the constant K_N depends on the dimension N as well as on the time interval

$0 \le k\Delta \le T$ under consideration and on uniform bounds on the functions f and g and their derivatives.

Linear-implicit schemes, which are sometimes called Rosenbrock schemes in the context of deterministic differential equations [7], are obtained from the explicit scheme by making implicit only the linear part of the coefficient functions. In order to ensure the consistency of Itô stochastic calculus, only the coefficients of the purely deterministic integral terms are made implicit, that is, only those for multi-indices α that have no nonzero components, in constructing implicit strong Taylor schemes for SDEs [4]. In defining their linear-implicit counterparts here we shall make only the linear part of the highest order purely deterministic integral term implicit (as this avoids having to adjust some other terms as happens in the usual implicit schemes [4]). For example, the linear-implicit version of the Euler scheme with constant time-step Δ for the SDE (2) has the form

$$Y_{k+1}^N = (I_N - \Delta A_N)^{-1}(Y_k^N + f^N(Y_k^N)\Delta + g^N(Y_k^N)\Delta W_k), \tag{8}$$

where I_N is the $N \times N$ identity matrix and $I_N - \Delta A_N$ is a diagonal matrix with jth diagonal component $1 + \Delta\lambda_j$.

Let $l(\alpha)$ be the length of a multi-index α and $n(\alpha)$ the number of components of α that are equal to zero. Define

$$L(\gamma) = \max\{l(\alpha): l(\alpha) = n(\alpha), \alpha \in \mathcal{A}_\gamma\}.$$

Let $\alpha^* = (0,\ldots,0)$ ($L(\gamma)$ times) be the multi-index with $n(\alpha^*) = l(\alpha^*) = L(\gamma)$. Then $I_{\alpha^*,k,\Delta} = \frac{1}{L(\gamma)!}\Delta^{L(\gamma)}$ and it can be shown [4] that

$$L(\gamma) = \left[\gamma + \frac{1}{2}\right].$$

We define the linear-implicit version of the order γ strong Taylor scheme for the Itô-Galerkin SDE (2) by

$$Y_{k+1}^N = \left(I_N - \frac{1}{L(\gamma)!}\Delta^{L(\gamma)}A_N^{L(\gamma)}\right)^{-1}\Bigg(Y_k^N + \sum_{\alpha \in \mathcal{A}_\gamma \setminus \{v,\alpha^*\}} F_\alpha^N(Y_k^N) I_{\alpha,k,\Delta}$$

$$+ \frac{1}{L(\gamma)!}\Delta^{L(\gamma)}\left[F_{\alpha^*}^N(Y_k^N) - A_N^{L(\gamma)}Y_k^N\right]\Bigg). \tag{9}$$

Here the $N \times N$ matrices

$$\frac{1}{L(\gamma)!}\Gamma^{L(\gamma)}A_N^{L(\gamma)}, \quad I_N - \frac{1}{L(\gamma)!}\Delta^{L(\gamma)}A_N^{L(\gamma)}$$

are diagonal matrices with jth diagonal component equal, respectively, to

$$\frac{1}{L(\gamma)!}(-1)^{L(\gamma)}\Delta^{L(\gamma)}\lambda_j^{L(\gamma)},\ 1 - \frac{1}{L(\gamma)!}(-1)^{L(\gamma)}\Delta^{L(\gamma)}\lambda_j^{L(\gamma)},$$

although the definition (9) of a linear-implicit scheme is also valid when the matrix A_N is not a diagonal matrix and the following theorem also holds in this more general case. The assumptions required on the coefficients of the SDE are the same

as those given in Theorem 10.6.3 in [4], essentially that they are sufficiently often continuous differentiable with bounded derivatives.

Theorem 1: *The linear-implicit version* (9) *of an order* γ *strong Taylor scheme also has strong order* γ.

Proof: In order to apply Theorem 11.5.1 of [4] we first rearrange the linear-implicit scheme (9) into the form

$$Y_{k+1}^N = Y_k^N + \sum_{\alpha \in \mathcal{A}_\gamma \backslash \{v\}} \widetilde{F}_\alpha^N(Y_k^N) I_{\alpha,k,\Delta}. \tag{10}$$

where

$$\widetilde{F}_\alpha^N(Y^N) \equiv \left(I_N - \frac{1}{L(\gamma)!} \Delta^{L(\gamma)} A_N^{L(\gamma)} \right)^{-1} F_\alpha^N(Y^N)$$

for all multi-indices α including α^*, which is possible since

$$\left(I_N - \frac{1}{L(\gamma)!} \Delta^{L(\gamma)} A_N^{L(\gamma)} \right)^{-1} \left(Y_k^N + \frac{1}{L(\gamma) \sim} \Delta^{L(\gamma)} \left[F_{\alpha^*}^N(Y_k^N) - A_N^{L(\gamma)} Y_k^N \right] \right)$$

$$= Y_k^N + \left(I_N - \frac{1}{L(\gamma)!} \Delta^{L(\gamma)} A_N^{L(\gamma)} \right)^{-1} F_{\alpha^*}^N(Y_k^N) I_{\alpha,k,\Delta}.$$

Then for all multi-indices $\alpha \in \mathcal{A}_\gamma \backslash \{v\}$, we have

$$\left| \widetilde{F}_\alpha^N(Y^N) - F_\alpha^N(Y^N) \right| = \left| \left(I_N - \frac{1}{L(\gamma)!} \Delta^{L(\gamma)} A_N^{L(\gamma)} \right)^{-1} F_\alpha^N(Y^N) - F_\alpha^N(Y^N) \right|$$

$$= \frac{1}{L(\gamma)!} \Delta^{L(\gamma)} \left| \left(I_N - \frac{1}{L(\gamma)!} \Delta^{L(\gamma)} A_N^{L(\gamma)} \right)^{-1} \right| \left| A_N^{L(\gamma)} F_\alpha^N(Y^N) \right|$$

$$\leq K \Delta^{L(\gamma)}.$$

The squared expectation of all of these terms is thus bounded by a constant multiplied by $\Delta^{2\gamma}$, since $\gamma \leq L(\gamma)$ and, without undue restrictiveness, $\Delta \leq 1$ Theorem 5.11.1 of [4] then gives the desired result. □

The analogous result for linear-implicit weak schemes can be found in [5] and an alternative proof for the linear-implicit strong Euler scheme in [6].

3. The Combined Error Bound

The next theorem shows that the use of a linear-implicit strong Taylor scheme (9) overcomes the stiffness in the structure of an Itô-Galerkin SDE (2).

Theorem 2: *The combined truncation and global discretization error of the order* γ *linear-implicit strong Taylor scheme* (9) *with constant time-step* Δ *applied to the* N*-dimensional Itô-Galerkin approximation* (2) *of the SPDE* (1) *has the form*

$$E \left| U_{k\Delta} - Y_k^N \right| \leq K \left(\lambda_{N+1}^{-1/2} + \Delta^\gamma \right), \tag{11}$$

where the constant K depends on $E\,\|U_0\|^2$, bounds on the f, g coefficients of the SPDE and the length of the time interval $0 \le k\Delta \le T$ under consideration.

Proof: The proof is similar to that for the error bound (3) in [2] for the order γ strong stochastic Taylor scheme. To determine the nature of the dependence of K_N on N in (7) here it is necessary to examine the terms in the remainder of the stochastic Taylor expansion used to derive the strong Taylor scheme (4). Details are given in the proof of Theorem 10.6.3 in [4] and will not be repeated in full here. In particular, the remainder consists of multiple stochastic integrals with nonconstant integrands $I_{\alpha,k,\Delta}(F_\alpha^N(U_\cdot^N))$ for multi-indices α in the remainder set $\mathfrak{B}_\gamma$. These are estimated in inequality (6.23) on page 364 of [4] under the assumption that the coefficient functions satisfy a linear growth bound. This is true here under the assumptions on the coefficients of the SPDE (1) and it is only the dependence of the coefficient K_N on N. Specifically, each application of L_N^0 contributes a single dominating power of λ_N. The bound in the squared inequality (6.23) on page 364 of [4] thus takes the form

$$K_T(1 + E\,|\,U_0^N\,|^2)\lambda_N^{2n(\alpha)}\Delta^{\phi(\alpha)}$$

where $\phi(\alpha)$ is equal to $2(l(\alpha)-1)$ if $l(\alpha) = n(\alpha)$ and $l(\alpha)+n(\alpha)-1$ if $l(\alpha) \neq n(\alpha)$. If γ is an integer, the dominant value occurs for $l(\alpha) = n(\alpha) = \gamma + 1$ and the required bound is of the form

$$K_{T,U_0}\lambda_N^{2\gamma+2}\Delta^{2\gamma},$$

while if 2γ but not γ is an integer, the dominant value occurs for $l(\alpha) = n(\alpha) = \gamma + \frac{3}{2}$ and the required bound is of the form

$$K_{T,U_0}\lambda_N^{2\gamma+3}\Delta^{2\gamma}.$$

Taking the square root and renaming the constants gives the estimate

$$E\,|\,U_{k\Delta} - Y_{k\Delta}^N\,| \le K_{T,U_0}\left(\lambda_{N+1}^{-1/2} + \lambda_N^{[\gamma+\frac{1}{2}]+1}\Delta^\gamma\right), \tag{12}$$

where $[x]$ is the integer part of the real number x, for the global discretization error.

There are two basic differences for the order γ linear-implicit strong Taylor scheme (9). The first is that all coefficients in the scheme and the remainder are multiplied by the inverse of the diagonal matrix $I_N - \frac{1}{L(\gamma)!}\Delta^{L(\gamma)}A_N^{L(\gamma)}$. The discretization error estimate will thus be divided by the norm of this matrix, which is dominated by the term $\lambda_N^{L(\gamma)} = \lambda_N^{[\gamma+\frac{1}{2}]}$. The second difference is that the integrands in the remainder terms coming from the highest order purely deterministic multiple integral in the scheme are obtained by applying the L^0 and the L^1 operators to the coefficient function of this highest order term after the linear part has been made implicit and shifted to the other side. In particular, the highest order power $\lambda_N^{[\gamma+\frac{1}{2}]+1}$ coming from this linear part will no longer be present, though lower order powers may be, that is at most $\lambda_N^{[\gamma+\frac{1}{2}]}$, but we divide these terms by norm of the diagonal matrix which is dominated by the power $\lambda_N^{[\gamma+\frac{1}{2}]}$. The global discretization error bound will

thus be of the form $K_{T,U_0}\Delta^\gamma$ and hence the combined truncation and global discretization error bound will be as asserted in Theorem 2. □

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