THE H_∞ -OPTIMIZATION IN LOCALLY CONVEX SPACES¹

CHUAN-GAN HU

Nankai University

Department of Mathematics

Tianjin 300071, P.R. China

E-mail: chuganhu@public.tpt.tj.cn, chuangan_hu@hotmail.com

LI-XIN MA

Dezhou University

Department of Mathematics

Dezhou 253023, Shandong, P.R. China

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In this paper, the ordinary H_∞ -control theory is extended to locally convex spaces through the form of a parameter. The algorithms of computing the infimal model-matching error and the infimal controller are presented in a locally convex space. Two examples with the form of a parameter are enumerated for computing the infimal model-matching error and the infimal controller.

Key words: Inner-Outer Function, Minimal Realization, Infimal Model-Matching, Locally Convex Space.

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1. Introduction

The ordinary H_∞ -control theory was summarized by B.A. Francis, G. Zames and J.C. Doyle *et al.* in [1-3]. In 1993, the H_∞ -control theory was extended from finite dimensional spaces to infinite dimensional Hilbert spaces (see [4]). In this paper, we extend the H_∞ -control theory to a complete Hausdorff locally convex space containing Hilbert spaces in order to enlarge the scope of solutions of H_∞ -control theory.

Assume that $\mathbb{R}$ is the real field and $\mathbb{R}^n$ is the Cartesian product of n copies of $\mathbb{R}$, where n is any positive integer, and that $\mathcal{C}$ is a complex plane.

To solve the problem for simplicity, we apply the $G(s)$ in the model matching problem to $G(s, \xi)$, where s in $\mathcal{C}$ and ξ in $\mathbb{R}^n$, and $G(s, \xi)$ is in $C^\infty(\mathbb{R}^n)$ (locally convex space) for each fixed s in $\mathcal{C}$ and in H_∞ for each fixed ξ in $\mathbb{R}^n$, respectively.

First, we extend several concepts.

Definition 1.1: The locally convex space VH_∞ consists of all complex-valued parameter functions $F(s, \xi)$, of a complex variable s and a parameter ξ , which are analytic and bounded about s in $\mathcal{R}(s) > 0$ (for any fixed ξ in $\mathbb{R}^n$). The VH_∞ -norm of $F(s, \xi)$ is defined by

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$$\|F\|_\infty = \sum_{k=1}^{\infty} \frac{g_k}{2^k(1+g_k)}, \quad g_k = \sup_{-k < \xi < k} \|F(\cdot, \xi)\|_\infty.$$

The subset of VH_∞ consisting of all real-rational functions of s and ξ is denoted by VRH_∞ .

Let $\|F_j(\cdot, \xi)\|_\infty = \alpha_j(\xi)$, $\xi \in \mathbb{R}^n$, $j = 1, 2$. In VH_∞ , the order may be defined as follows:

Definition 1.2: For any $F_1, F_2 \in VH_\infty$, we call $\alpha_1 \leq \alpha_2$ if $\alpha_1(\xi) \leq \alpha_2(\xi)$, $\xi \in \mathbb{R}^n$.

Definition 1.3: We call $F(s, \xi)$ to be strong proper if $F(s, \xi) \in VRH_\infty$ and $\sup_{\xi \in \mathbb{R}^n} \|F(\infty, \xi)\| < \infty$, strictly strong proper if $F(\infty, \xi) \equiv 0$. We call $F(s, \xi)$ to be stable if $F(s, \xi) \in VRH_\infty$ and $F(s, \xi)$ has no poles in the closed right half-plane $\mathcal{R}(s) \geq 0$ (for any fixed ξ in $\mathbb{R}^n$).

If $F(s, \xi)$ is real-rational about s in $\mathcal{R}(s) > 0$, then $F(s, \xi) \in VRH_\infty$ if and only if F is strong proper and stable (for any fixed $x \in \mathbb{R}^n$).

Similarly, we define the transfer function matrix

$$G(x, \xi) = \begin{bmatrix} T_1(s, \xi) & T_2(s, \xi) \\ T_3(s, \xi) & 0 \end{bmatrix}, \quad K(s, \xi) = -Q(s, \xi),$$

Then the model-matching problem is $\|T_1(\cdot, \xi) - T_2(\cdot, \xi)Q(\cdot, \xi)T_3(\cdot, \xi)\|_\infty =$ minimum in the sense of Definition 1.2, where $T_j (j = 1, 2, 3) \in VRH_\infty$, and K represents a controller in VRH_∞ .

We shall give the algorithms of computing the model-matching error α and the optimal controller Q in the form of the parameter case.

2. The Minimal Realization

Definition 2.1: A scalar-valued parameter function $T(s, \xi)$ in VRH_∞ is inner if

$$T(-s, \xi)T(s, \xi) = 1 \quad (2.1)$$

and outer if it has no zeros in $\mathcal{R}(s) > 0$. The zeros of an inner-function all lie in $\mathcal{R}(s) > 0$, the number of its zeros is called degree.

Theorem 2.1: If scalar-valued parameter function $T(s, \xi) \in VRH_\infty$, then

- (i) there is a factorization $T(s, \xi) = T_i(s, \xi)T_o(s, \xi)$, where T_i is an inner function, T_o is an outer function;
- (ii) $|T_i(j\omega, \xi)| = 1$ for any fixed $\xi \in \mathbb{R}^n$;
- (iii) if $T(j\omega, \xi) \neq 0$ for all ω in $[0, \infty]$ and any fixed ξ in $\mathbb{R}^n$, then $T_o^{-1}(s, \xi)$ exists and $T_o^{-1}(s, \xi) \in VRH_\infty$.

Proof: (i) Let $T_i(s, \xi)$ be the product of all factors of the form $(a(\xi) - s)/(\overline{a(\xi)} + s)$, here $a(\xi)$ ranges over all zeros of $T(s, \xi)$ in $\mathcal{R}(s) > 0$, counting multiplicities, being a polynomial of ξ , and define $T_o = T(s, \xi)/T_i(s, \xi)$. Then $T_i(s, \xi)$ and $T_o(s, \xi)$ are inner and outer respectively, and $T(s, \xi) = T_i(s, \xi)T_o(s, \xi)$.

(ii) From (i) we derive $T_i(-s, \xi)T_i(s, \xi) = 1$. Particularly, if $s = j\omega$, then $T_i(-j\omega, \xi)T_i(j\omega, \xi) = 1$. Thus $\overline{T_i(j\omega, \xi)}T_i(j\omega, \xi) = 1$, i.e. $|T_i(j\omega, \xi)| = 1$.

(iii) If $T(\infty, \xi) \neq 0$ and $T(j\omega, \xi)$ has no zeros on the imaginary axis for any fixed $\xi \in \mathbb{R}^n$, then so is $T_o(s, \xi)$. Consequently, $T_o^{-1}(s, \xi) \in VRH_\infty$. Q.E.D.

Suppose that

$$K(\lambda) = \lambda^n + K_1\lambda^{n-1} + \cdots + K_n = 0 \quad (2.2)$$

is the characteristic equation of a matrix $A(\xi)$, and that $\lambda_1, \lambda_2, \dots, \lambda_n$ are the characteristic roots of $A(\xi)$, where λ is a polynomial of ξ .

The matrix

$$C(\xi) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -K_n & -K_{n-1} & -K_{n-2} & \cdots & -K_1 \end{bmatrix} \quad (2.3)$$

is the companion form matrix associated with $K(\lambda)$ in (2.2).

Definition 2.2: Let $A(\xi)(a_{ij}(\xi))(m \times n)$ and $B(\xi)(p \times q)$ be two matrices. The Kronecker product of $A(\xi)$ and $B(\xi)$ is defined by

$$A(\xi) \otimes B(\xi) = \begin{bmatrix} a_{11}(\xi)B(\xi) & \cdots & a_{1n}(\xi)B(\xi) \\ \vdots & \ddots & \vdots \\ a_{n1}(\xi)B(\xi) & \cdots & a_{nn}(\xi)B(\xi) \end{bmatrix}.$$

Definition 2.3: The linear time invariant system S_1 is defined by

$$\dot{X}(t, \xi) = A(\xi)X(t, \xi) + B(\xi)u(t, \xi) \quad (2.4)$$

$$y(t, \xi) = C(\xi)X(t, \xi) \quad (2.5)$$

where A is an $n \times n$ matrix, B is an $n \times m$ matrix and C is a constant $r \times n$ matrix. S_1 is said to be completely controllable (c.c.) if the $n \times nm$ controllability matrix

$$U(\xi) = [B(\xi) \quad A(\xi)B(\xi) \quad \cdots \quad A^{n-1}(\xi)B(\xi)] \quad (2.6)$$

has rank n , denoted by $(A(\xi), B(\xi))$. The system S_1 is completely observable (c.o) if the observability matrix

$$V^T(\xi) = [C(\xi) \quad C(\xi)A(\xi) \cdots C(\xi)A^{n-1}(\xi)]^T \quad (2.7)$$

has rank n , denoted by $(A(\xi), C(\xi))$.

Definition 2.4: Given an $r \times m$ matrix $G(s, \xi)$ whose elements are rational functions of s (for any fixed ξ in $\mathbb{R}^n$), we wish to find matrices $A(\xi)$, $B(\xi)$ and $C(\xi)$ depending on ξ , having dimensions $n \times n$, $n \times m$ and $r \times n$, respectively, such that

$$G(s, \xi) = C(\xi)(sI_n - A(\xi))^{-1}B(\xi)$$

where I_n is the unit matrix of order n . $[A(\xi), B(\xi), C(\xi), 0]$ is termed a realization of $G(s, \xi)$ of order n . All the above realizations will include matrices $G(s, \xi)$ having the least dimensions be called the minimal realizations.

Theorem 2.2: If $[A(\xi), B(\xi), C(\xi), 0]$ is c.c. or c.o., then so is $[\tilde{A}(\xi), \tilde{B}(\xi), \tilde{C}(\xi), 0]$, where $\tilde{A}(\xi)$, $\tilde{B}(\xi)$ and $\tilde{C}(\xi)$ are of algebraical equivalence via a square matrix, respectively.

Proof: Using the algebraic relation of equivalence among matrices, there is a square matrix $P(\xi)$ such that

$$\tilde{A}(\xi) = P(\xi)A(\xi)P^{-1}(\xi) \quad \tilde{B}(\xi) = P(\xi)B(\xi) \quad \tilde{C}(\xi) = C(\xi)P^{-1}(\xi).$$

It follows that

$$\begin{aligned} & \text{rank}[\tilde{B}(\xi) \quad \tilde{A}(\xi)\tilde{B}(\xi)\cdots\tilde{A}^{n-1}(\xi)\tilde{B}(\xi)] \\ &= \text{rank}(P(\xi)[B(\xi) \quad A(\xi)B(\xi)\cdots A^{n-1}(\xi)B(\xi)]) = n. \end{aligned}$$

So is $[\tilde{A}(\xi), \tilde{B}(\xi), \tilde{C}(\xi), 0]$ since $[A(\xi), B(\xi), C(\xi), 0]$ is c.c. by Definition 2.3. The observability condition follows similarly. Q.E.D.

Suppose $R(s, \xi) = [r_{ij}(s, \xi)]$ is a $p \times m$ strictly proper rational-fraction matrix of s (for any fixed ξ in $\mathbb{R}^n$).

Theorem 2.3: *Let*

$$r(s, \xi) = s^q + g_1(\xi)s^{q-1} + \cdots + g_q(\xi)$$

be the monic least common denominator of all element $r_{ij}(s, \xi)$, and let

$$r(s, \xi)R(s, \xi) = s^{q-1}R_0(\xi) + s^{q-2}R_1(\xi) + \cdots + R_{q-1}(\xi), \quad (2.8)$$

where $R_i(\xi)$ is a constant $r \times m$ matrix depending on a parameter ξ . Then a realization of $R(s, \xi)$ is

$$A(\xi) = \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 \\ 0 & 0 & I_m & \cdots & 0 \\ & & & \ddots & \\ -g_q(\xi) & -g_{q-1}(\xi) & & \cdots & -g_1(\xi) \end{bmatrix}, \quad B(\xi) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ I_m \end{bmatrix} \quad \begin{matrix} q \\ \text{blocks} \end{matrix}$$

$$C(\xi) = [R_{q-1}(\xi) \quad R_{q-2}(\xi) \cdots R_0(\xi)]. \quad (2.9)$$

Proof: Let $\Upsilon(\xi)$ denote the companion matrix of (2.3). Clearly $A(\xi) = \Upsilon(\xi) \otimes I_m$, so that

$$(sI_m - A(\xi))^{-1} = (sI_q \otimes I_m - \Upsilon(\xi) \otimes I_m)^{-1} = [(sI_q - \Upsilon(\xi))^{-1}] \otimes I_m. \quad (2.10)$$

It follows that using $B(\xi) = e_q \otimes I_m$ where e_q denotes the least column of I_q , relation (2.10) yields

$$\begin{aligned} & (sI_m - A(\xi))^{-1}B(\xi) \\ &= [(sI_q - \Upsilon(\xi))^{-1} \otimes I_m][e_q \otimes I_m] \\ &= [(sI_q - \Upsilon(\xi))^{-1}e_q] \otimes I_m \\ &= [1 \quad s \quad s^2 \cdots s^{q-1}]^T \otimes I_m / g(s, \xi). \end{aligned} \quad (2.11)$$

Finally, combining (2.11) and the expression for $C(\xi)$ in (2.9), we have

$$\begin{aligned} & C(\xi)(sI_m - A(\xi))^{-1}B(\xi) \\ &= [R_{q-1}(\xi) \quad R_{q-2}(\xi) \cdots R_0(\xi)][I_m \quad sI_m \cdots s^{q-1}I_m] / r(s, \xi) \\ &= R(s, \xi). \end{aligned}$$

By virtue of (2.8), we have that $[A(\xi), B(\xi), C(\xi), 0]$ is a realization of $R(s, \xi)$. Q.E.D.

We now give the central conclusion of this section.

Theorem 2.4: *A realization $[A(\xi), B(\xi), C(\xi), 0]$ of a given transfer matrix $G(s, \xi)$ is minimal if $(A(\xi), B(\xi))$ is c.c. and $(A(\xi), C(\xi))$ is c.o.*

Proof: Let $U(\xi)$ and $V(\xi)$ be the controllability and observability matrices in (2.6) and (2.7), respectively. We wish to show that if these both have rank n , then $R(s, \xi)$ has the least order n .

Suppose that there exists a realization $[\tilde{A}(\xi), \tilde{B}(\xi), \tilde{C}(\xi), 0]$ of $R(s, \xi)$, with $\tilde{A}(\xi)$ having order n_1 . Since

$$C(\xi)(sI_m - A(\xi))^{-1}B(\xi) = \tilde{C}(\xi)(sI_m - \tilde{A}(\xi))^{-1}\tilde{B}(\xi),$$

$$C(\xi)e^{A(\xi)t}B(\xi) = \tilde{C}(\xi)e^{\tilde{A}(\xi)t}\tilde{B}(\xi).$$

It follows that

$$C(\xi)A^i(\xi)B(\xi) = \tilde{C}(\xi)\tilde{A}^i(\xi)\tilde{B}(\xi), \text{ for } i = 0, 1, 2, \dots$$

using the series of $(e^{A(\xi)t})$.

Consider the product

$$V(\xi)U(\xi) = [C(\xi) \ C(\xi)A(\xi) \ \dots C(\xi)A^{n-1}(\xi)]^T [B(\xi) \ A(\xi)B(\xi) \ \dots A^{n-1}(\xi)B(\xi)]$$

$$= \begin{bmatrix} C(\xi)B(\xi) & C(\xi)A(\xi)B(\xi) & \dots & C(\xi)A^{n-1}(\xi)B(\xi) \\ \vdots & \vdots & \ddots & \vdots \\ C(\xi)A^{n-1}(\xi) & C(\xi)A^n(\xi)B(\xi) & \dots & C(\xi)A^{2n-2}(\xi)B(\xi) \end{bmatrix}$$

$$= [\tilde{C}(\xi) \ \tilde{C}(\xi)\tilde{A}(\xi) \ \dots \tilde{C}(\xi)\tilde{A}^{n-1}(\xi)]^T [\tilde{B}(\xi) \ \tilde{A}(\xi)\tilde{B}(\xi) \ \dots \tilde{A}^{n-1}(\xi)\tilde{B}(\xi)]$$

$$= V_1(\xi)U_1(\xi).$$

By assumption, $V(\xi)$ and $U(\xi)$ both have rank n , so the matrix $V_1(\xi)U_1(\xi)$ also has rank n . However, the dimensions of $V_1(\xi)$ and $U_1(\xi)$ are $r_1n \times n_1$ and $n_1 \times m_1n$, respectively, where r_1 and m_1 are positive integers, so that the rank of $V_1(\xi)U_1(\xi)$ can not be greater than n_1 . This is $n < n_1$, so there can be no realization of $G(s, \xi)$ having order less than n . Q.E.D.

3. Lyapunov Equations

The Lyapunov equations are

$$A(\xi)L_c(\xi) + L_c(\xi)A^T(\xi) = B(\xi)B^T(\xi) \tag{3.1}$$

$$A^T(\xi)L_o(\xi) + L_o(\xi)A(\xi) = C^T(\xi)C(\xi). \tag{3.2}$$

Define the two controllability and observability gramians:

$$L_c(\xi) = \int_0^\infty e^{-A(\xi)t}B(\xi)B^T(\xi)e^{-A^T(\xi)t}dt$$

$$L_o(\xi) = \int_0^\infty e^{-A^T(\xi)t} C^T(\xi) C(\xi) e^{-A(\xi)t} dt.$$

Definition 3.1: A matrix $A(\xi)$ is said to be antistable if all the eigenvalues of $A(\xi)$ are in $\mathcal{R}(s) > 0$.

Theorem 3.1: If $A(\xi)$ is antistable, then $L_c(\xi)$ and $L_o(\xi)$ are the unique solutions of (3.1) and (3.2), respectively.

Proof: From the definition of $L_c(\xi)$ we derive

$$\begin{aligned} & A(\xi)L_c(\xi) + L_c(\xi)A^T(\xi) \\ &= \int_0^\infty [A(\xi)e^{-A(\xi)t}B(\xi)B^T(\xi)e^{-A^T(\xi)t} + e^{-A(\xi)t}B(\xi)B^T(\xi)e^{-A^T(\xi)t}A^T(\xi)]dt \\ &= \int_0^\infty d[e^{-A(\xi)t}B(\xi)B^T(\xi)e^{-A^T(\xi)t}] \\ &= B(\xi)B^T(\xi) - \lim_{t \rightarrow \infty} [e^{-A(\xi)t}B(\xi)B^T(\xi)e^{-A^T(\xi)t}]. \end{aligned}$$

Since $A(\xi)$ is antistable,

$$\lim_{t \rightarrow \infty} [e^{-A(\xi)t}B(\xi)B^T(\xi)e^{-A^T(\xi)t}] = 0.$$

Then $L_c(\xi)$ is the solution of (3.1).

Proof the uniqueness of $L_c(\xi)$ is as follows.

If $A(\xi)$ and $B(\xi)$ are $n \times n$ and $m \times m$ matrices, having characteristic roots $\lambda_i(\xi)$, $u_i(\xi)$ and vectors $w_i(\xi)$, $y_i(\xi)$, respectively, then

$$\begin{aligned} & (A(\xi) \times B(\xi))(w_i(\xi) \otimes y_i(\xi)) \\ &= A(\xi)w_i(\xi) \otimes B(\xi)y_i(\xi) \\ &= \lambda_i(\xi)w_i(\xi) \otimes u_j(\xi)y_j(\xi) \\ &= \lambda_i(\xi)u_j(\xi)w_i(\xi) \otimes y_j(\xi). \end{aligned}$$

So, the characteristic roots of $A(\xi) \otimes B(\xi)$ are

$$\lambda_i(\xi)u_j(\xi), i = 1, 2, \dots, n \text{ and } j = 1, 2, \dots, m.$$

Let

$$A(\xi)X(\xi) + X(\xi)B(\xi) = C(\xi), \quad (3.3)$$

be a matrix equation, where $X(\xi)$ and $C(\xi)$ are $n \times m$ matrices.

If $A(\xi)$ is an $n \times n$ matrix and $X(\xi)$ is an $n \times m$ matrix, then the matrix equation $A(\xi)X(\xi) = C(\xi)$ can be written as the form

$$(A(\xi) \otimes I_m)X(\xi) = C(\xi), \quad (3.4)$$

where

$$X(\xi) = [X_{11}, \dots, X_{1m}, \dots, X_{n1}, \dots, X_{nm}] \quad (3.5)$$

is the column mn -vector formed from the roots of $X(\xi)$ taken in order.

Similarly $X(\xi)B(\xi) = C(\xi)$ can be written as

$$(I_n \otimes B^T(\xi))X(\xi) = C(\xi). \quad (3.6)$$

Using (3.4) and (3.6), equation (3.3) can be written as the form

$$(A(\xi) \otimes I_m + I_n \otimes B^T(\xi))X(\xi) = C(\xi). \quad (3.7)$$

Let $D(\xi) = A(\xi) \otimes I_m + I_n \otimes B^T(\xi)$. Thus

$$D(\xi)X(\xi) = C(\xi). \quad (3.8)$$

The solution of (3.8) is unique if and only if the $mn \times mn$ matrix $D(\xi)$ is nonsingular. To find the condition for this to hold, consider

$$(I_n + \epsilon A(\xi)) \otimes (I_m + \epsilon B^T(\xi)) = I_n \otimes I_m + \epsilon D(\xi) + \epsilon^2 A(\xi) \otimes B(\xi)$$

which has characteristic roots

$$(I + \epsilon \lambda_i(\xi))(1 + \epsilon u_j(\xi)) = 1 + \epsilon(\lambda_i(\xi) + u_j(\xi)) + \epsilon^2 \lambda_i(\xi) u_j(\xi).$$

It follows by comparing terms in ϵ that $D(\xi)$ has characteristic roots $\lambda_i(\xi) + u_j(\xi)$, $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$. Hence, $D(\xi)$ is nonsingular if and only if there are no characteristic roots of $A(\xi)$ and $B(\xi)$ such that $\lambda_i(\xi) + u_j(\xi) = 0$, and this is the condition for the solution $X(\xi)$ of matrix equation (3.3) to be unique. Because $A(\xi)$ is antistable, the characteristic roots $\lambda_i(\xi) + \overline{\lambda_i(\xi)}$ of $A(\xi) + A^T(\xi)$ are not zero. Consequently, the solution of the Lyapunov equation (3.1) is unique.

$L_o(\xi)$ is the unique solution of (3.2) with similar proof above.

Q.E.D.

4. Infimal Model-Matching Error

Define

$$f(s, \xi) = [A(\xi), w(\xi), C(\xi), 0], \quad (4.1)$$

$$g(s, \xi) = [-A^T(\xi), \lambda^{-1}(\xi)L_o(\xi)w(\xi), B^T(\xi), 0]$$

and

$$X(s, \xi) = R(s, \xi) - \lambda(\xi)f(s, \xi)/g(s, \xi). \quad (4.2)$$

So

$$f(s, \xi) = C(\xi)(sI - A(\xi))^{-1}w(\xi) \in VRH_2^\perp$$

and

$$g(s, \xi) = B^T(\xi)(sI + A^T(\xi))^{-1}\lambda^{-1}(\xi)L_o(\xi)w(\xi) \in VRH_2.$$

Definition 4.1: Let $\alpha(\xi)$ denote the infimal model-matching error:

$$\alpha(\xi) = \inf \{ \| T_1(\cdot, \xi) - T_2(\cdot, \xi)Q(\cdot, \xi)T_3(\cdot, \xi) \|_\infty : Q \in VRH_\infty \}. \quad (4.3)$$

A matrix Q in VRH_∞ satisfying $\alpha(\xi) = \|T_1(\cdot, \xi) - T_2(\cdot, \xi)Q(\cdot, \xi)T_3(\cdot, \xi)\|_\infty$ is called optimal.

If $T_i(s, \xi)$ are scalar-valued, then there is no need for both $T_2(s, \xi)$ and $T_3(s, \xi)$. So we may as well suppose $T_3(s, \xi) = 1$. It is also assumed that $T_2^{-1}(s, \xi) \in VRH_\infty$ to avoid the trivial instance of the problem.

Returning to the model-matching problem, bringing in an inner-outer factorization of $T_2(s, \xi): T_2(s, \xi) = T_{2i}(s, \xi)T_{2o}(s, \xi)$, we have

$$\begin{aligned} & \|T_1(\cdot, \xi) - T_2(\cdot, \xi)Q(\cdot, \xi)\|_\infty \\ &= \|T_{2i}(\cdot, \xi)(T_{2i}^{-1}(\cdot, \xi)T_1(\cdot, \xi) - T_{2o}(\cdot, \xi)Q(\cdot, \xi))\|_\infty \\ &= \|T_{2i}^{-1}(\cdot, \xi)T_1(\cdot, \xi) - T_{2o}(\cdot, \xi)Q(\cdot, \xi)\|_\infty \\ &= \|R(\cdot, \xi) - X(\cdot, \xi)\|_\infty. \end{aligned} \tag{4.4}$$

Theorem 4.1: *The infimal in (4.3) is achieved if $T_2(s, \xi)$ has no zeros on the extended imaginary axis. In this case, the optimal $Q(s, \xi)$ is determined by the following property: $T_1(s, \xi) - T_2(s, \xi)Q(s, \xi)$ is a scalar multiple of an inner function of degree less than the number of zeros of $T_2(s, \xi)$ in $\mathcal{R}(s) > 0$ (for any fixed ξ in $\mathbb{R}^n$).*

Proof: Suppose $B_1 = VHR_\infty$ and $B = VRL_\infty$, then from Proposition A 2.2 in [7], the infimum in (4.4) is achieved.

Assume $\tilde{X}(s, \xi) = T_1(s, \xi) - T_2(s, \xi)Q(s, \xi)$, then using (4.4) we have

$$\|\tilde{X}(\cdot, \xi)\|_\infty = \|X(\cdot, \xi)T_{2i}^{-1}(\cdot, \xi)\|_\infty.$$

Consequently, the minimization of $\tilde{X}(s, \xi)$ can be accomplished by minimizing $\tilde{X}(s, \xi)T_{2i}^{-1}(s, \xi)$ and multiplying the result by $T_{2i}^{-1}(s, \xi)$. Now as $\tilde{X}(s, \xi)T_{2i}^{-1}(s, \xi)$ is analytic in $\mathcal{R}(s) > 0$, except for the poles of $T_{2i}^{-1}(s, \xi)$, which are $b_j, j = 1, \dots, r$ (depending on ξ). So $\tilde{X}(s, \xi)T_{2i}^{-1}(s, \xi)$ must have a continuation to the entire plane, with poles at b_j and $-b_j$. Therefore, $\tilde{X}(s, \xi)T_{2i}^{-1}(s, \xi)$ is rational, and has the form

$$\tilde{X}(s, \xi)T_{2i}^{-1}(s, \xi) = C(\xi) \frac{\prod_{i=1}^m (s - c_i(\xi))(s + c_i(\xi))}{\prod_{j=1}^n (s - b_j(\xi))(s + b_j(\xi))} \tag{4.5}$$

where $m < r$, $\mathcal{R}(c_i(\xi)) > 0$, $C(\xi) > 0$.

As $T_{2i}(s, \xi)$ is an inner function, obviously rational, so $X(s, \xi)$ is also rational.

Since $|\tilde{X}(j\omega, \xi)| = \alpha$ a.e., $\tilde{X}(s, \xi)$ is inner and the zeros of $\tilde{X}(s, \xi)$ must be among c_i in (4.5). Q.E.D.

Definition 4.2: The VL_p space, $1 \leq p < \infty$, will be viewed as p th power integrable functions about s and ξ . When $p = \infty$, VL_∞ is the space of essentially bounded functions (for any fixed ξ in $\mathbb{R}^n$).

Definition 4.3: The $VR L_p$ space, $1 \leq p \leq \infty$, will be viewed as a subset of VL_p , which consists of all real-variational functions of s and ξ .

Definition 4.4: Let $F(s, \xi) \in VL_\infty$ and $g(s, \xi) \in VL_2$. Then the operator

$$\Lambda_{F(s, \xi)}: \Lambda_{F(s, \xi)}g(s, \xi) = F(s, \xi)g(s, \xi)$$

is called the Laurent operator.

A related operator is $\Lambda_{F(s, \xi)}|VH_2$, the restriction of $\Lambda_{F(s, \xi)}$ to VH_2 , which maps VH_2 to VL_2 where $F(s, \xi) \in VL_\infty$.

For $F(s, \xi)$ in VL_∞ , the Hankel operator with symbol $F(s, \xi)$, denoted by $\Gamma_{F(s, \xi)}$, maps VH_2 to $VH_2^\perp$ and is defined as

$$\Gamma_{F(s, \xi)} := \prod_1 \Lambda_{F(s, \xi)} \mid VH_2,$$

where $VL_2 = VH_2 \oplus VH_2^\perp$, and $\prod_1$ is the projection from VL_2 onto $VH_2^\perp$.

Using a similar method to the classical methods we have the following conclusion:

Theorem 4.2: *There exists a closest VRH_∞ -function $X(s, \xi)$ to a given VRH_∞ -function $R(s, \xi)$, and $\|R(\cdot, \xi) - X(\cdot, \xi)\|_\infty = \|\Gamma_{R(\cdot, \xi)}\|$.*

From Section 3, a factor $R(s, \xi)$ can be written as $R_1(s, \xi) + R_2(s, \xi)$ with $R_1(s, \xi)$ strictly proper and analytic in $\mathcal{R}(s) < 0$ and $R_2(s, \xi)$ in VRH_∞ . Then $R_1(s, \xi)$ has the minimal state-space realization

$$R_1(s, \xi) = [A(\xi), B(\xi), C(\xi), 0].$$

And from Section 3, with $v(\xi) = \lambda^{-1}(\xi)L_o(\xi)w(\xi)$ we derive

$$L_c(\xi)v(\xi) = \lambda(\xi)w(\xi) \quad (4.6)$$

and

$$L_o(\xi)w(\xi) = \lambda(\xi)v(\xi). \quad (4.7)$$

Theorem 4.3: *The infimal model-matching error $\alpha(\xi)$ equals $\|\Gamma_R(\cdot, \xi)\|$ and the unique optimal X equals $R(s, \xi) - \lambda(\xi)f(s, \xi)/g(s, \xi)$.*

Proof: From Theorem 4.2, we derive that there is a function $X(s, \xi)$ in VH_∞ such that

$$\|R(\cdot, \xi) - X(\cdot, \xi)\|_\infty = \|\Gamma_{R(\cdot, \xi)}\|. \quad (4.8)$$

It is claimed that every $X(s, \xi)$ in VH_∞ satisfying (4.8) also satisfies

$$R(s, \xi) - X(s, \xi)g(s, \xi) = \Gamma_R(s, \xi)g(s, \xi). \quad (4.9)$$

But (4.9) has a unique solution $X(s, \xi) = R(s, \xi) - \lambda(\xi)f(s, \xi)/g(s, \xi)$. We know that

$$\Gamma_{R(s, \xi)}g(s, \xi) = \lambda(\xi)f(s, \xi) \quad (4.10)$$

holds. In fact, add and subtract $sL_c(\xi)$ on the left-hand side in (3.1) to get

$$-(sI - A(\xi))L_c(\xi) + L_c(\xi)(sI + A^T(\xi)) = B(\xi)B^T(\xi).$$

Now pre-multiply by $C(\xi)(sI - A(\xi))^{-1}$ and post-multiply by $(sI + A^T(\xi))^{-1}v(\xi)$ to get

$$\begin{aligned} & -C(\xi)L_c(\xi)(sI + A^T(\xi))v(\xi) + C(\xi)(sI - A(\xi))^{-1}L_c(\xi)v(\xi) \\ & = C(\xi)(sI - A(\xi))^{-1}B(\xi)B^T(\xi)(sI + A^T(\xi))^{-1}v(\xi). \end{aligned} \quad (4.11)$$

The first function on the left-hand side belongs to VH_2 ; from (4.1) and (4.6), the second function equals $\lambda(\xi)f(s, \xi)$; and from (4.2) and (4.6), the function on the right-hand side equals $R_1(s, \xi)g(s, \xi)$. Project both sides of (4.12) onto $VRH_2^\perp$ to get

$$\lambda(\xi)f(s, \xi) = \prod_1 R_1(s, \xi)g(s, \xi) = \Gamma_{R_1(s, \xi)}g(s, \xi).$$

But $\Gamma_{R_1(s,\xi)} = \Gamma_{R(s,\xi)}$; hence (4.10) holds. It follows that (4.10) and Theorem 4.2 imply $\alpha(\xi) = \lambda(\xi)$. There is $X(s, \xi) = R(s, \xi) - \alpha(\xi)f(s, \xi)/g(s, \xi)$. Set $\alpha(\xi) = \lambda(\xi)$ and

$$Q(s, \xi) = T_2^{-1}(s, \xi)X(s, \xi). \quad (4.12)$$

Since $T_{2o}(s, \xi)$ and $T_{2o}^{-1}(s, \xi) \in VRH_\infty$, (4.12) sets up the one-to-one correspondence between functions $Q(s, \xi)$ in VRH_∞ and functions $X(s, \xi)$ in VRH_∞ . The optimal $X(s, \xi)$ yields the optimal $Q(s, \xi)$ via (4.9).

5. Steps of Computation

From Section 2 through Section 4, we derive that in the form of parameter valued case, the steps in the design procedure on the H_∞ -optimization in locally convex spaces are as follows:

Step 1: Do an inner-outer factorization

$$T_2(s, \xi) = T_{2i}(s, \xi)T_{2o}(s, \xi).$$

Step 2: Define

$$R(s, \xi) = T_{2i}^{-1}(s, \xi)T_1(s, \xi)$$

and find a minimal realization

$$R(s, \xi) = [a(\xi), B(\xi), C(\xi), 0] + (\text{a function in } VRH_\infty).$$

Step 3: Solve the equations

$$A(\xi)L_c(\xi) + L_c(\xi)A^T(\xi) = B(\xi)B^T(\xi)$$

and

$$A^T(\xi)L_o(\xi) + L_o(\xi)A(\xi) = C^T(\xi)C(\xi).$$

Step 4: Find the maximum eigenvalue λ^2 of $L_c(\xi)L_o(\xi)$ and a corresponding eigenvector $w(\xi)$.

Step 5: Define

$$f(s, \xi) = [A(\xi), w(\xi), C(\xi), 0]$$

$$g(s, \xi) = [-A^T(\xi), \lambda^{-1}(\xi)L_o(\xi)w(\xi), B^T(\xi), 0]$$

and

$$X(s, \xi) = R(s, \xi) - \lambda(\xi) \frac{f(s, \xi)}{g(s, \xi)}.$$

Step 6: Set $\alpha(\xi) = \lambda(\xi)$ and $Q(s, \xi) = T_{2o}^{-1}(s, \xi)X(s, \xi)$.

For a single-input and single-output design in the form of parameter valued case, we have a similar to ordinary computing method.

Example 1:

$$P(s, \xi) = [(s-1)(s-2)]/[(s+1)(s^2 + s + 1 + \xi^2)] \in VRH_\infty, \quad \omega_1 = 0.01, \quad \epsilon = 0.1.$$

Step 1: $-P(s, \xi) = N(s, \xi)/M(s, \xi),$

$$N(s, \xi) = -P(s, \xi), M(s, \xi) = 1 = X(s, \xi), Y(s, \xi) = 0.$$

Step 2: $W(s, \xi) = (s + 1)/(10s + 1).$

Step 3: $T_1(s, \xi) = (s + 1)^k/(10s + 1)^k,$

$$T_2(s, \xi) = -[(s + 1)^k(s - 1)(s - 2)]/[(10s + 1)^k(s + 1)(s^2 + s + 1 + \xi^2)],$$

$$V(s) = s + 1.$$

Step 4: When $k = 1$

Step (1) $T_{2i}(s, \xi) = [(s - 1)(s - 2)]/[(s + 1)(s + 2)],$

$$T_{2o}(s, \xi) = -[(s + 1)(s + 2)]/[(10s + 1)(s^2 + s + 1 + \xi^2)].$$

Step (2) $R(s, \xi) = [(s + 1)^2(s + 2)]/[(10s + 1)(s^2 + s + 1 + \xi^2)],$

and A, B, C in the minimal realization of $R(s, \xi)$ are

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} -12/11 \\ 12/7 \end{bmatrix}, C = [1 \quad 1],$$

respectively.

Step (3)

$$L_c = \begin{bmatrix} 72/121 & -48/77 \\ -48/77 & 36/49 \end{bmatrix}, L_o = \begin{bmatrix} 1/2 & 1/3 \\ 1/3 & 1/4 \end{bmatrix}.$$

Step (4)

$$L_c L_o = \begin{bmatrix} 0.0898 & 0.0425 \\ -0.0668 & -0.0853 \end{bmatrix},$$

then $\alpha_1 = 0.2299 > 0.1 = \epsilon$. Thus we take $k = 2$.

Step (1) $T_{2i}(s, \xi) = [(s - 1)(s - 2)]/[(s + 1)(s + 2)],$

$$T_{2o}(s, \xi) = -[(s + 1)(s + 2)]/[(10s + 1)(10s + 1)(s^2 + s + 1 + \xi^2)].$$

Step (2) $R(s, \xi) = [(s + 1)^3(s + 2)]/[(10s + 1)^2(s^2 + s + 1 + \xi^2)],$

and A, B, C in the minimal realization of $R(s, \xi)$ are

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} -24/121 \\ 12/49 \end{bmatrix}, C = [1 \quad 1],$$

respectively.

Step (3)

$$L_c = \begin{bmatrix} (24 \times 12)/(121 \times 121) & -(8 \times 12)/(121 \times 49) \\ -(8 \times 12)/(121 \times 49) & (12 \times 3)/(49 \times 49) \end{bmatrix}, L_o = \begin{bmatrix} 1/2 & 1/3 \\ 1/3 & 1/4 \end{bmatrix}.$$

Step (4)

$$L_c L_o = \begin{bmatrix} 0.0044 & -0.0025 \\ 0.0031 & -0.0017 \end{bmatrix}, \quad \lambda = 0.05113, \quad w = \begin{bmatrix} 1 \\ -0.7209 \end{bmatrix}.$$

Step (5) $f(s) = (0.2791s - 1.2791)/[(s-1)(s-2)],$

$$g(s) = \lambda^{-1}(-0.0141s - 0.0657)/[(s+1)(s+2)],$$

$$X(s) = 6.15[(s+1)(s+2)(s+0.4)]/[(10s+1)^2(s+4.66)].$$

Step (6) Set

$$\alpha = \lambda = 0.05113,$$

$$Q(s) = -6.15[(s+0.4)(s^2+s+1+\xi^2)]/[(s+1)(s+4.66)].$$

Step 5: $Q_1(s, \xi) = -6.15[(s+0.4)(s^2+s+1+\xi^2)]/[(s+1)^2(s+4.66)].$

Step 6:

$$K(s, \xi) = \frac{0.615(s+0.4)(s+1)(s^2+s+1+\xi^2)}{(s^4+6.145s^3+12.54s^2+13.53s+0.0232)}.$$

Note $K(s, \xi) \notin RH_\infty$, but $K(s, \xi) \in VRH_\infty$.

Example 2:

$$P(s, \xi) = \frac{(10s+1)(s-2-\xi^2)}{(s+1)(s^2+s+1+\xi^2)} \in VRH_\infty, \quad \omega_1 = 0.01, \epsilon = 0.15.$$

Step 1:

$$-P(s, \xi) = N(s, \xi)/M(s, \xi),$$

$$N(s, \xi) = -P(s, \xi), \quad M(s, \xi) = 1 = X(s, \xi), \quad Y(s, \xi) = 0.$$

Step 2:

$$W(s, \xi) = (s+1)/(10s+1).$$

Step 3:

$$T_1(s, \xi) = (s+1)^k/(10s+1)^k,$$

$$T_2(s, \xi) = -\frac{(s+1)^k(10s+1)(s-2-\xi^2)}{(10s+1)^k(s+1)(s^2+s+1+\xi^2)}, \quad V(s) = s+1.$$

Step 4: When $k = 1$,

$$\textbf{Step (1)} \quad T_{2i}(s, \xi) = -\frac{s-2-\xi^2}{s+2+\xi^2}, \quad T_{2o}(s, \xi) = -\frac{s+2+\xi^2}{s^2+s+1+\xi^2}.$$

Step (2) $R(s, \xi) = -[(s+1)(s+2+\xi^2)]/[(10s+1)(s-2-\xi^2)],$
then $A(\xi), B(\xi)$ and $C(\xi)$ in the minimal realization of $R(s, \xi)$ are $A(\xi) = 2 + \xi^2, B(\xi) = -2[(2 + \xi^2)(3 + \xi^2)]/(10\xi^2 + 21)$ and $C(\xi) = 1$, respectively.

Step (3)

$$L_c(\xi) = \frac{2(2+\xi^2)(3+\xi^2)^2}{(10\xi^2+21)^2}, \quad L_o(\xi) = \frac{1}{2(2+\xi^2)}.$$

Step (4) $L_c(\xi)L_o(\xi) = (2 + \xi^2)^2/(10\xi^2 + 21)^2,$
then

$$\alpha_1(\xi) = \frac{3+\xi^2}{10\xi^2+21} < 0.15.$$

Step (5)

$$f(s, \xi) = \frac{1}{s-2-\xi^2}, \quad g(s, \xi) = \frac{\lambda^{-1}(\xi)(\xi^2+3)}{(s+2+\xi^2)(10\xi^2+21)},$$

$$X(s, \xi) = \frac{(s+2+\xi^2)(9s+9\xi^2-18)}{(s-2-\xi^2)(10s+1)(10\xi^2+21)}.$$

Step (6) Set

$$\alpha(\xi) = \frac{3+\xi^2}{10\xi^2+21}, \quad Q(s, \xi) = -\frac{9(s^2+s+1+\xi^2)}{(10s+1)(10\xi^2+21)}.$$

Step 5:

$$Q_1(s, \xi) = -\frac{9(s^2+s+1+\xi^2)}{(10s+1)(10\xi^2+21)(s+1)}.$$

Step 6:

$$K(s, \xi) = \frac{9(s^2+s+1+\xi^2)(s+1)}{(10s+1)[(10\xi^2+21)s^2+(110\xi^2+22)s+(19\xi^2+39)]}.$$

Note $K(s, \xi) \notin RH_\infty$, but $K(s, \xi) \in VRH_\infty$.

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